Chapter 1

Introduction

In its broadest meaning, the "Theory of Summability" deals with the theory of assignment of limits which is the foundation not only for analysis but also in almost all the areas of mathematics. In the present doctorial thesis, we have made efforts to introduce and develop certain generalized limits in some spaces out of which we will consider only Normed spaces, Banach spaces, Double sequence spaces, Probabilistic Normed spaces and Random $2-$ normed spaces. Accordingly, we start this chapter with some standard notations, definitions and results which are given without proof generally.

1.1 Statistical Convergence

The notion of statistical convergence as a generalized summability method was introduced in order to assign limits to those sequences which are not convergent in usual sense.

For any set K with $K \subset \mathbb{N}$ (the set of positive integers), the natural density or the asymptotic density of K is denoted by $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_K(k) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in K\}|;$$

provided the limit on the right side exists. Here χ_K denotes the characteristic function of K and the sum on the right side of the above expression denotes the cardinality of the set $\{k \le n : k \in K\}$. If $\delta(K) = 0$, then K is said to be a thin subset, otherwise K is called a non-thin subset of $\mathbb{N}$. Since $\delta(K)$ does not exist for all subsets of $\mathbb{N}$, so it is sometimes convenient to use the upper asymptotic density denoted by $\overline{\delta}(K)$ and is defined by

$$\overline{\delta}(K) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_K(k).$$

Similarly, the lower asymptotic density of $K \subset \mathbb{N}$ is defined by

$$\underline{\delta}(K) = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_K(k).$$

For convenience, we list below some properties of natural density. For arbitrary subsets A, B of $\mathbb{N}$,

(i) If $\delta(A)$ exists, then $\delta(A) = \overline{\delta}(A) = \underline{\delta}(A)$;

(ii) If $A \subset B$, then $\overline{\delta}(A) \le \overline{\delta}(B)$;

(iii) If A is finite, then $\delta(A) = 0$;

(iv) If $A = a_1 < a_2 < \cdots < a_n < \cdots$ is an infinite set, then $\delta(A) = \lim_{n\to\infty} \frac{n}{a_n}$, provided the limit exists;

(v) $\delta(A^c) = 1 - \delta(A)$, where $A^c = \mathbb{N} - A$, provided $\delta(A)$ exists.

In the middle of the twentieth century, Fast (1) and Steinhaus (5) independently used the notion of asymptotic density to introduce a novel generalization of usual convergence under the name of statistical convergence. The idea was actually of Zygmund (6), who proved in 1935 some theorems on the statistical convergence of Fourier series in the first edition of his book "Trigonometric Series" where he used the term "almost convergence" in place of statistical convergence.

DEFINITION 1.1.1 (1) *A sequence $x = (x_k)$ is said to be statistically convergent to L provided that, for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S - \lim_{k\to\infty} x_k = L$.

Let S denote the set of all statistically convergent sequences of scalars.

In the following example, the reader will be able to see that the sequence (x_k) defined by

$$x_k = \begin{cases} \sqrt{k}, & \text{if } k \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

is statistically convergence to zero but not convergent in the usual sense.

Since for every $\epsilon > 0$,

$$\begin{aligned} \{k \leq n : |x_k - 0| \geq \epsilon\} &= \{k \leq n : x_k = \sqrt{k}\} \\ &= \{k \leq n : k \text{ is a square}\} \subset \{1^2, 2^2, 3^2, \cdots\}, \end{aligned}$$

it follows that $S - \lim_{k\to\infty} x_k = 0$ while (x_k) is not convergent in usual sense. This shows that statistical convergence is a generalization of ordinary convergence. Moreover, the set S form a linear subspace of w, the linear space of all scalar valued sequences and the statistical limit act as a linear functional on w.

Although, statistical convergence was introduced in 1951 and further explored by Schoenberg (2) in 1959, but rapid investigations and developments on statistical convergence as a sequential limit started after the works of Šalát (7), Fridy (8), Connor

(9), Kolk (10) etc.

Šalát (7) obtained an important characterization of statistical convergence that " *A sequence* $x = (x_k)$ *is statistically convergent to a number* L*, if and only if, there exists a set* $K = \{k_1, k_2, \cdots\} \subset \mathbb{N}$ *such that* $\delta(K) = 1$ *and* $\lim_n x_{k_n} = L$*.*" He also proved that the set of all bounded statistically convergent sequences of real numbers is a nowhere dense subset of l_∞, the linear normed space of all bounded sequences of real numbers. He also showed that the set of all statistically convergent sequences of real numbers is a dense subset of the first Baire category in the Fréchet space.

Fridy (8) analyzed statistical convergence from sequence space point of view and linked it with summability theory. He also gave an equivalent condition of Cauchy type for the statistical convergence and defined the statistical analogue of Cauchy sequences.

DEFINITION 1.1.2 (8) *A sequence* $x = (x_k)$ *is said to be statistically Cauchy provided that for every* $\epsilon > 0$*, there exists a positive integer* $m = m(\epsilon)$ *such that*

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - x_m| \geq \epsilon\}| = 0.$$

Connor (9) obtained a natural relationship between statistical convergence and strong $p-$Cesàro summability. He has shown that if a sequence is strong $p-$Cesàro summable, for $0 < p < \infty$, then the sequence must be statistically convergent and a bounded statistically convergent sequence must be strong $p-$Cesàro summable. A sequence $x = (x_k)$ is said to be strongly Cesàro summable or $(C, 1)-$ summable to L provided that $\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$.
Let

$$|\sigma_1| = \left\{ x : \text{for some L}, \lim_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k - L| \right) = 0 \right\},$$

be the collection of strongly Cesàro summable sequences.

A sequence $x = (x_k)$ is said to be strongly $p-$Cesàro summable to L (where p is a positive real number) provided that $\lim_n \frac{1}{n}\sum_{k=1}^{n}|x_k - L|^p = 0$.
Let

$$w_p = \left\{ x : \text{for some L}, \lim_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k - L|^p \right) = 0 \right\},$$

be the space of all strongly $p-$Cesàro summable sequences.

Kolk (10) has shown that "*A sequence is statistically convergent if and only if it is statistically Cauchy sequence, where the entries of the sequence come from a Banach space instead of being scalars.*"

Following the concept of a statistically convergent sequence $x = (x_k)$, Fridy (11), introduced the notions of statistical limit point and statistical cluster point which are not equivalent when compared to the concept of ordinary limit point of the sequence. The notion of statistical limit point is defined by considering density of subsequences and using particular notations and terminology.

If x is a sequence, then $\{x_k : k \in \mathbb{N}\}$ represents the range of x. If $\left\{x_{k(j)}\right\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we denote $\left\{x_{k(j)}\right\}$ by $\{x\}_K$. If $\delta(K) = 0$, then $\{x\}_K$ is called a subsequence of density zero, or a thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if either $\delta(K)$ is a positive real number or K fails to have natural density.

DEFINITION 1.1.3 (11) *A number ξ is said to be a statistical limit point of the sequence $x = (x_k)$ provided that there is a non-thin subsequence of x that converges to ξ.*

Let Λ_x denotes the set of all statistical limit points of the sequence $x = (x_k)$.

DEFINITION 1.1.4 (11) *A number η is called a statistical cluster point of the sequence $x = (x_k)$ provided that for each $\epsilon > 0$, the set $\delta(\{k \in \mathbb{N} : |x_k - \eta| < \epsilon\})$ does not have density zero.*

Let Γ_x denotes the set of all statistical limit points of the sequence $x = (x_k)$.

Subsequently, Fridy and Orhan (12) defined statistical limit inferior, statistical limit superior and statistical core of sequences of numbers. They also gave the statistical analogue of Knopp's Core Theorem.

For a number sequence $x = (x_k)$, let

$$B_x = \{b \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k > b\}) \neq 0\};$$

and

$$A_x = \{a \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k < a\}) \neq 0\}.$$

The statistical limit inferior and superior of $x = (x_k)$ are defined as given below.

DEFINITION 1.1.5 (12) *If $x = (x_k)$ is a real number sequence, then the statistical limit inferior is given by*

$$S - \liminf x = \begin{cases} \inf A_x, & \textit{if } A_x \neq \emptyset, \\ +\infty, & \textit{if } A_x = \emptyset. \end{cases}$$

Also, the statistical limit superior of x is defined by

$$S - \limsup x = \begin{cases} \sup B_x, & \textit{if } B_x \neq \emptyset, \\ -\infty, & \textit{if } B_x = \emptyset. \end{cases}$$

It is remarkable that "*A statistically bounded sequence x is statistically convergent, if and only if, $S - \limsup x = S - \liminf x$*" (*A number sequence x is said to be statistically bounded if there is a number B such that $\delta\{k : |x_k| > B\} = 0$*).

Recently, Çolak (13) introduced an interesting concept of statistical convergence of order α and strongly $p-$Cesàro summability of order α of sequences of complex or real numbers, where $\alpha \in (0, 1]$. For $K \subset \mathbb{N}$, the $\alpha-$density of K is given as follows.

DEFINITION 1.1.6(13) *Let $\alpha \in (0, 1]$. The $\alpha-$density of the set K is defined by*

$$\delta_\alpha(K) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : k \in K\}|$$

provided the limit exists (finite or infinite).

It is trivial that any finite subset of $\mathbb{N}$ has zero $\alpha-$density and $\delta_\alpha(K^c) = 1 - \delta_\alpha(K)$ does not hold for $0 < \alpha < 1$ in general, the equality holds only if $\alpha = 1$. It is remarkable that the $\alpha-$density of any set reduces to the natural density of the set in case $\alpha = 1$. Also for $K \subset \mathbb{N}$, $\delta_\beta(K) \le \delta_\alpha(K)$ if $0 < \alpha \le \beta \le 1$.

DEFINITION 1.1.7(13) *A sequence $x = (x_k)$ of scalars is said to be statistically convergent of order α provided that for every $\epsilon > 0$ there is a number L such that*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : |x_k - L| \ge \epsilon\}| = 0.$$

In this case, we write $S^\alpha - \lim x_k = L$.

Let S^α denotes the collection of all statistically convergent sequences of order α.

After the initial developments of statistical convergence, the notion is further used to resolve many interesting problems in almost all the areas of mathematical analysis. For instance, in number theory (14, 15), measure theory (16, 17), trigonometric series (6), summability theory (18, 19, 20, 21), turnpike theory (22, 23), tauberian theory (24, 25, 26), Fourier series (27), locally convex spaces (28), difference sequence spaces(29, 29, 30, 31, 32, 33, 34), topological space (35) and intuitionistic fuzzy normed spaces (36, 37, 38).

1.2 Generalizations of Statistical Convergence

In this section, a brief review of some interesting generalizations of statistical convergence is given. We start with the generalization given by Fridy and Orhan (39) using lacunary sequences.

By a lacunary sequence $\theta = (k_r)$, *we mean an increasing sequence of non-negative integers with* $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ *as* $r \to \infty$. *The interval determined by* θ *will be denoted by* $I_r = (k_{r-1}, k_r]$ *and the ratio* $\frac{k_r}{k_{r-1}}$ *will be denoted by* q_r *i.e.,* $q_r = \frac{k_r}{k_{r-1}}$. *Also, the lacunary sequence* $\theta' = (k_r')$ *is said to be a lacunary refinement of lacunary sequence* $\theta = (k_r)$ *if* $(k_r) \subset (k_r')$.

Freedman *et al*. (40) generalized Cesàro summability by using lacunary sequences and defined the notion of lacunary summable sequences.

A sequence $x = (x_k)$ *of numbers is said to be lacunary summable or* $N_\theta-$*summable to a number* L *provided that*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0,$$

and it is denoted by $N_\theta - \lim x_k = L$ *or* $x_k \to L(N_\theta)$.

The authors obtained a strong connection between the sequence spaces $|\sigma_1|$ and N_θ, where

$$N_\theta = \left\{ x : \text{for some L}, \lim_r \left(\frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \right) = 0 \right\}.$$

For the particular case $\theta = (2^r)$, we have $N_\theta = |\sigma_1|$.

Apart from this, lacunary sequences were further used by Fridy and Orhan (39) to define a new generalized summability method known as lacunary statistical convergence. While doing so, the authors replaced the set $\{k : k \leq n\}$ in the definition of statistical convergence by the set $\{k : k_{r-1} \leq k \leq k_r\}$, for some lacunary sequence $\theta = (k_r)$. Subsequently, they compared the resulting summability method with statistical convergence and other summability methods.

DEFINITION 1.2.1 (39) *A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent or $S_\theta-$ convergent to L provided that, for every $\epsilon > 0$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_\theta - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S_\theta)$.

Let S_θ denotes the set of all lacunary statistically convergent sequences of scalars.

The limit in above DEFINITION can be expressed using matrix transformations of the characteristic function χ_K of the set $K = K(x, L, \epsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ and written as $\lim_n (C_\theta \chi_K)_n = 0$, where C_θ is the matrix given by

$$C_\theta[n,k] := \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r, \\ 0, & \text{if } k \notin I_r. \end{cases}$$

A subset K of $\mathbb{N}$ has $\theta-$density if $\delta_\theta(K) = \lim_r \frac{K \cap I_r}{h_r}$ exists.

Some interesting relations among the sequence spaces l_∞, N_θ, S and S_θ are listed below.

(i) $x_k \to L(N_\theta)$ implies $x_k \to L(S_\theta)$,

(ii) N_θ is a proper subspace of S_θ,

(iii) $S_\theta \cap l_\infty = N_\theta \cap l_\infty$, where l_∞ denotes the space of bounded sequences.

Also, the authors studied the inclusions $S \subseteq S_\theta$ and $S_\theta \subseteq S$ under certain restrictions on $\theta = (k_r)$ given below. It has been noted that,

$$S = S_\theta \text{ if and only if } 1 < \liminf_r q_r \leq \limsup_r q_r < \infty;$$

Also, $S - \lim_{k\to\infty} x_k = L$ implies $S_\theta - \lim_{k\to\infty} x_k = L$.

As an example of a lacunary sequence satisfying the above conditions, we can take $(k_r) = 2^r$ for $r > 0$, hence $S_{(2^r)} = S$.

The work was further carried out by the same authors in (41), where they defined the $S_\theta-$ analogue of the Cauchy sequences and presented Cauchy convergence criterion for $S_\theta-$ convergence. In addition, the authors showed that $S_\theta-$ summability can not be included by any matrix summability.

DEFINITION 1.2.2 (41) *A sequence $x = (x_k)$ is said to be lacunary statistically Cauchy or $S_\theta-$ Cauchy if there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $k^{'}(r) \in I_r$, for each r, $\lim_{r\to\infty} x_{k'(r)} = L$ and for each $\epsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : |x_k - x_{k'(r)}| \geq \epsilon \right\} \right| = 0.$$

Using above DEFINITION, the authors stated the equivalent condition that
"A sequence $x = (x_k)$ is $S_\theta-$convergent if and only if it is a $S_\theta-$Cauchy sequence."

Jinlu Li, K. Demirci and Pehlivan etc. carried out this idea in different directions. Out of which, Jinlu Li (42) studied the inclusion properties of different lacunary methods, whereas, K. Demirci (43) studied the concepts of lacunary statistical limit points and lacunary statistical cluster points as well as the concept of lacunary statistical core for a bounded complex number sequence. Further, Pehliven *et al*. (44) introduced the

$S_\theta-$analogue of ordinary cluster point in finite dimensional spaces and also discussed the closeness and compactness of the set of lacunary statistical cluster points.

DEFINITION 1.2.3 (43) *A number ξ is a lacunary statistical limit point of the sequence $x = (x_k)$, if there is a set $\{k_1 < k_2 < \cdots < k_r < \cdots\} \subseteq \mathbb{N}$, $\theta-$density of which is not zero, such that* $\lim_{r\to\infty} x_{k_r} = \xi$.

Let Λ_x^θ denote the set of all lacunary statistical limit points of $x = (x_k)$.

DEFINITION 1.2.4 (43) *A point γ is a lacunary statistical cluster point of the sequence $x = (x_k)$, if for every $\epsilon > 0$,*

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : |x_k - \gamma| < \epsilon\}| > 0.$$

Let Γ_x^θ denote the set of all cluster points of $x = (x_k)$.

Subsequently, many aspects of lacunary statistical convergence has been studied by several authors and connected in many areas of pure and applied mathematics. For instance, Çakalli (45) introduced lacunary statistical convergence in metrizable topological groups and proved some inclusion relations between S and S_θ. Nuray (46) extended this notion of convergence for sequences of fuzzy numbers and proved some inclusion relations. Çolak (47) studied lacunary strong convergence of difference sequences with the help of modulus function, whereas, Dutta (48) extended this notion for n-norms. This idea was extended in intuitionistic fuzzy normed spaces by Mursaleen and Mohiuddine (49) and Karakaya *et al*. (50).

Another interesting generalization of statistical convergence was given by Mursaleen (51), where he used the notion of $(V, \lambda)-$ summability of Leindler (52).
Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ with

$$\lambda_{n+1} \le \lambda_n + 1, \qquad \lambda_1 = 1.$$

The generalized de la Vallèe-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

DEFINITION 1.2.5 (52) *A sequence $x = (x_k)$ is said to be $(V, \lambda)-$ summable to L provided that $t_n(x) \to L$ as $n \to \infty$.*

For particular choice of $\lambda_n = n$, $(V, \lambda)-$ summability reduces to $(C, 1)-$ summability. Let

$$[V, \lambda] = \left\{ x = (x_k) : \text{for some } L,\ \lim_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| \right) = 0 \right\},$$

denotes the set of sequences $x = (x_k)$ which are strongly $(V, \lambda)-$ summable to L.

For $K \subseteq \mathbb{N}$, the set of positive integers, the $\lambda-$density of K is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} \left| \{ n - \lambda_n + 1 \leq k \leq n : k \in K \} \right|.$$

In particular, if we take $\lambda_n = n$, then $\lambda-$density reduces to natural density.

DEFINITION 1.2.6 (51) *A number sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to a number L if for each $\epsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} |\{ k \in I_n : |x_k - L| \geq \epsilon \}| = 0.$$

In this case, we write $S_\lambda - \lim_{k \to \infty} x_k = L$ or $x_k \to L(S_\lambda)$.

Let S_λ be the set of all $\lambda-$ statistically convergent sequences of scalars.
Observe that for the choice $\lambda_n = n$, $S_\lambda-$convergence coincides with statistical convergence.

Analogous to (39), Mursaleen (51) found the relationship of the sequence spaces l_∞, $[V, \lambda]$, S and S_λ, which are given as follows.

(i) $x_k \to L(V, \lambda)$ implies $x_k \to L(S_\lambda)$ and the inclusion $[V, \lambda] \subseteq S_\lambda$ is proper.

(ii) If $x \in l_\infty$ and $x_k \to L(S_\lambda)$, then $x_k \to L(V, \lambda)$ and $x_k \to L(C, 1)$, provided $x = (x_k)$ is not eventually constant.

(iii) $S_\lambda \cap l_\infty = [V, \lambda] \cap l_\infty$, where l_∞ denotes the space of bounded sequences.

The author also investigated the inclusion relation between S and S_λ,

$$S \subseteq S_\lambda \text{ if and only if } \liminf_n \frac{\lambda_n}{n} > 0.$$

This idea has inspired many authors to work in this direction. Savaş (53) extended this notion for the sequences of fuzzy numbers, whereas, Tuncer and Benli (54) defined the notion of $\lambda-$ statistically Cauchy for sequences of fuzzy numbers. Moreover, Benli (55) has defined the notions of $\lambda-$ statistical limit inferior and superior for sequences of fuzzy numbers. Mursaleen *et al.* (56) generalized the notion of almost convergence by using the concept of invariant mean and the generalized de la vallèe-poussin mean. Mursaleen and Alotaibi (57) defined a new type of summability method with the help of $(V, \lambda)-$summability and used this new summability method to prove a Korovkin type approximation Theorem. Mohiuddine and Lohani (38) extended this concept to intuitionistic fuzzy normed space and provided a better tool to study a more general class of sequences.

Also, an interesting generalization of $\lambda-$statistical convergence was introduced by Çolak and Bektaş (4) under the name of "$\lambda-$statistical convergence of order α" for some $\alpha \in (0, 1]$.

DEFINITION 1.2.7 (4) *Let $\lambda = (\lambda_n)$ be a sequence of real numbers as defined above and $\alpha \in (0, 1]$ be given. A sequence $x = (x_k)$ of numbers is said to* *be $\lambda-$statistically convergent of order α if there is a number L such that*

$$\lim_n \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : |x_k - L| \geq \epsilon\}\right| = 0.$$

In this case, we write $S_\lambda^\alpha - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S_\lambda^\alpha)$.

Let $S_\lambda^\alpha(x)$ represents the set of all $\lambda-$ statistically convergent sequences of order α.

The notion of $\lambda-$ statistical convergence has also been applied for sequences of functions. Karakaya *et al.* (58) defined $\lambda-$ statistical convergence for sequences of functions in intutionistic fuzzy normed space. Mikali *et al.* (59) defined $\lambda-$statistically convergence of order α for the sequences of functions. Srivastava and Ojha (60) has defined $\lambda-$ statistical convergence of fuzzy numbers and fuzzy functions of order θ.

Another type of statistical convergence can be defined with the help of an infinite summability matrix A. An infinite matrix $A = (a_{nk})$ of numbers transforms a given sequence $x = (x_k)$ into the sequence

$$Ax = \sigma_n = \lim_n \sum_{k=1}^{n} a_{nk}\ x_k$$

and defines a method of summability A which assigns the value $\lim_n \sigma_n$ to the sequence (x_k), whenever the limit exists. The method A is said to be regular if it assigns to each convergent sequence the value to which it converges *i.e.*, $\lim_n \sigma_n$ exists and equal to $\lim_k x_k$ (61). The notion of natural density can be generalized by using a non-negative regular summability matrix A in place of C_1.
Following Freedman and Sember (19), we say that a set $K \subset \mathbb{N}$ has $A-$density if

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} = \lim_n \sum_{k=1}^{\infty} a_{nk}\ \chi_K(k) = \lim_n (A\chi_K)_n$$

exists where A is a non-negative regular summability matrix.

DEFINITION 1.2.8 (18) *A sequence $x = (x_k)$ of numbers is said to be $A-$ statistically convergent to a number L provided that for each $\epsilon > 0$,*

$$\delta_A(\{k \leq n : |x_k - L| \geq \epsilon\}) = 0.$$

In this case, we write $S_A - \lim_{k\to\infty} x_k = L$.

For particular choice of the matrix $A = I$, the identity matrix, $A-$statistical convergence coincides with ordinary convergence, $A = C_1$, the Cesàro matrix, $A-$statistical convergence coincides with statistical convergence, for the choice $A = C_\theta$, as defined earlier $A-$ statistical convergence coincides with lacunary statistical convergence (62) and for the choice $A = (a_{nk})$, where,

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n, \\ 0, & \text{if } k \notin I_n, \end{cases}$$

$A-$ statistical convergence coincides with $\lambda-$statistical convergence.

Many concepts parallel to the concepts of statistical convergence such as $A-$statistical boundedness, $A-$statistical limit point, $A-$statistical cluster point, $A-$statistical limit inferior and $A-$statistical limit superior have been introduced in (63, 64). Demirci (65) extended the definition of strong $A-$summability to the definition of strong $A-$summability with respect to an orlicz function where A is a non-negative regular summability method via the ideal in l_∞. The author proved that strong $A-$summability with respect to an orlicz function which satisfies Δ_2-condition and strong $A-$ summability and $A-$statistical convergence are equivalent on bounded sequences. Duman *et al.* (66) introduced different approximation results related to the classical Korovkin theorem with the help of $A-$statistical convergence. They also studied the rates of $A-$statistical convergence of approximating positive linear operators. Belen and Yilidrim (67) introduced the idea of generalized $A-$statistical convergence and generalized $A-$summability with the help of ideals. They also proved a Korovkin type approximation theorem for double sequences of positive linear operators by using generalized $A-$statistical convergence. For more developments in this direction, we have referred (68, 69, 70, 71, 72, 73).

1.3 $\mathcal{I}-$ Convergence

In this section, we highlight a more generalized convergence namely $\mathcal{I}-$ convergence due to Kostyorko, Salàt and Wilcznski (74).

For a non-empty set X, let $\mathcal{P}(X)$ denotes the power set of X.

DEFINITION 1.3.1 (74) *A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal in X if and only if*

(i) $\emptyset \in \mathcal{I}$,

(ii) For each $A, B \in \mathcal{I}$ *we have,* $A \cup B \in \mathcal{I}$,

(iii) For each $A \in \mathcal{I}$ *and* $B \subseteq A$ *we have,* $B \in \mathcal{I}$.

DEFINITION 1.3.2 (74) *A nonempty family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is called a filter on X if and only if*

(i) $\emptyset \notin \mathcal{F}$,

(ii) For each $A, B \in \mathcal{F}$ *we have,* $A \cap B \in \mathcal{F}$,

(iii) For $A \in \mathcal{F}$ *and* $B \supseteq A$ *we have,* $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. If $\mathcal{I} \subset \mathcal{P}(X)$ is a non-trivial ideal in X, then the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$ is a filter on X. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

DEFINITION 1.3.3 (74) *A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e., if it contains $\{\{x\} : x \in X\}$.*

DEFINITION 1.3.4 (74) *An admissible ideal $\mathcal{I} \subset \mathcal{P}(X)$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $A_1, A_2, \cdots$ belonging to $\mathcal{I}$ there exists a countable family $B_1, B_2, \cdots$ belonging to $\mathcal{I}$ such that $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}$.*

Kostyrko *et al*. (74) used ideals to define $\mathcal{I}-$convergence of a sequence in a fixed metric space (X, d), but to make it convenient in regards of the work given in present thesis, we shall present definitions in $\mathbb{R}$ with respect to usual metric in place of (X, d).

DEFINITION 1.3.5 (75) *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$convergent to a number L if for each $\epsilon > 0$, we have*

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}.$$

The number L is called $\mathcal{I}-$limit of (x_k) and we can write it as $\mathcal{I} - \lim x_k = L$.

For $\mathcal{I} = \mathcal{I}_f$, the set of all finite subsets of $\mathbb{N}$, $\mathcal{I}_f$ is a non trivial admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f-$convergence coincides with the usual convergence of numbers. If we take $\mathcal{I} = \mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$, then $\mathcal{I}_\delta$ is a non-trivial admissible ideal and $\mathcal{I}_\delta-$convergence coincides with statistical convergence.

It is remarkable that if $\mathcal{I}$ is an admissible ideal, then usual convergence of scalars implies $\mathcal{I}-$convergence of scalars. Thus for an admissible ideal $\mathcal{I}$, $\mathcal{I}-$convergence is a more generalized form of ordinary convergence and depends on the choice of the ideal $\mathcal{I}$.

Another idea given by Kostyrko *et al*. (74) that is closely related to $\mathcal{I}-$ convergence is $\mathcal{I}^*-$ convergence.

DEFINITION 1.3.6 (74, 75) *A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}^*-$convergent to a number L if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1, m_2, \cdots\}$ such that* $\lim_{k\to\infty} x_{m_k} = L$.

It is observed that for an admissible ideal $\mathcal{I}$, $\mathcal{I}^*-$ convergence implies $\mathcal{I}-$convergence, however, the converse implication is valid only if the ideal $\mathcal{I}$ satisfies the condition (AP). Infact the converse implication between $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence depends completely on the structure of the metric space. If we allow the metric space more general (X, d) having no accumulation point then $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence coincide for each admissible ideal $\mathcal{I}$.

Dems (76) defined $\mathcal{I}-$ Cauchy sequences and proved that in a complete metric space (X, d), a Cauchy type condition is necessary and sufficient for $\mathcal{I}-$convergence of a given sequence.

DEFINITION 1.3.7 (76) *For an admissible ideal $\mathcal{I}$, a sequence, $x = (x_k)$ of numbers is said to be $\mathcal{I}-$Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer m such that the set*

$$\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in \mathcal{I}.$$

Following $\mathcal{I}-$convergence, Kostyrko *et al*. (74) naturally generalized the concepts of statistical limit points and statistical cluster points in terms of $\mathcal{I}-$limit points and $\mathcal{I}-$cluster points of a sequence as follows.

DEFINITION 1.3.8(74, 75) *A number ξ is said to be an $\mathcal{I}-$limit point of the sequence $x = (x_k)$ provided that there is a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$.*

DEFINITION 1.3.9 (74, 75) *A number γ is said to be an $\mathcal{I}-$cluster point of the sequence $x = (x_k)$ if for each $\epsilon > 0$, we have $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\} \notin \mathcal{I}$.*

Let $\Lambda_x(\mathcal{I})$ and $\Gamma_x(\mathcal{I})$ respectively denotes the set of $\mathcal{I}-$limit points and $\mathcal{I}-$cluster points of the sequence $x = (x_k)$.

The concepts of $\mathcal{I}-$ limit points and $\mathcal{I}-$ cluster points were further extended to the concepts of $\mathcal{I}-$limit inferior and $\mathcal{I}-$limit superior by K. Demirci (77), where certain properties of these notions were obtained.

For any real number sequence $x = (x_k)$, let

$$D_x = \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b\}\} \notin \mathcal{I};$$

and

$$C_x = \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a\}\} \notin \mathcal{I};$$

DEFINITION 1.3.10 (77) For an admissible ideal $\mathcal{I}$ and a number sequence $x = (x_k)$, the $\mathcal{I}-$limit inferior of x is given by

$$\mathcal{I} - \liminf x = \begin{cases} \inf C_x, & \text{if } C_x \neq \emptyset, \\ +\infty, & \text{if } C_x = \emptyset. \end{cases}$$

Also, $\mathcal{I}-$limit superior of x is given by

$$\mathcal{I} - \limsup x = \begin{cases} \sup D_x, & \text{if } D_x \neq \emptyset, \\ -\infty, & \text{if } D_x = \emptyset. \end{cases}$$

It is remarkable that if we take $\mathcal{I} = \{K \subset \mathbb{N} : \delta_A(K) = 0\}$ and $\mathcal{I} = \{K \subset \mathbb{N} : \delta(K) = 0\}$ in the above DEFINITION, then we get Demirci's (63) and Fridy & Connor's (12) DEFINITIONS of statistical limit inferior and superior respectively.
Also, for any number sequence $x = (x_k)$, we have

$$\liminf x_k \leq \mathcal{I} - \liminf x_k \leq \mathcal{I} - \limsup x \leq \limsup x_k.$$

Lahiri and Das (78) obtained some other properties of $\mathcal{I}-$limit inferior and $\mathcal{I}-$limit superior of a sequence of real numbers. For any $\mathcal{I}-$bounded sequences $x = x_k$ and $y = y_k$, they proved that

(i) $\mathcal{I} - \limsup(x_k + y_k) \leq \mathcal{I} - \limsup x_k + \mathcal{I} - \limsup y_k$;

(ii) $\mathcal{I} - \liminf(x_k + y_k) \geq \mathcal{I} - \liminf x_k + \mathcal{I} - \liminf y_k$.

Also, for any sequence $x = (x_k)$ if $\mathcal{I} - \limsup x = \xi$, then there exists a subsequence of x that is $\mathcal{I}-$convergent to ξ.

Ideal convergence has become an active area of research for many authors. Nabiev *et al*. (79) studied the notion of $\mathcal{I}^*-$Cauchy sequences and its relation with $\mathcal{I}-$convergence and $\mathcal{I}-$Cauchy sequences. Fillipów *et al*. (80) presented a generalization of the Bolzano-Weierstrass theorem for ideal convergence. They show examples of ideals with and without the Bolzano-Weierstrass property and gave characterizations of Bolzano-Weierstrass property in terms of sub-measures and extendibility to a maximal P-ideal. Further, the authors presented some applications to Rudin-Keisler and Rudin-Blass orderings of ideals and quotient boolean algebras. In particular they show that an ideal does not have Bolzano-Weierstrass property if and only if its quotient boolean algebra has a countably splitting family.

Later on, $\mathcal{I}-$convergence is connected with many important fields of summability theory. For instance, Duman (81) connected it with approximation theory and provided a Korovkin type approximation theorem by means of positive linear operators defined on an appropriate weighted space given with any interval of real line. He also studied rates of convergence by means of the modulus of continuity and the elements of Lipschitz class. Gezer and Karkus (82) studied this notion for sequences of functions and introduced many interesting convergence concepts with the help of ideals. They defined the concepts of $\mathcal{I}-$pointwise convergence, $\mathcal{I}-$uniform convergence, $\mathcal{I}^*-$pointwise convergence and $\mathcal{I}^*-$uniform convergence for sequences of functions. Balcerzak *et al.* (83) studied different types of statistical convergence and ideal convergence for sequences of functions.

Tripathy and Hazarika (84) studied the notions of $\mathcal{I}-$ monotonic sequences and proved the decomposition theorem to generalize some of the results on monotonic sequences. Also in (85), both these authors developed some $\mathcal{I}-$ convergent sequence spaces with the help of orlicz function.

Recently, another interesting generalization is introduced by Şavas and Das (86) by combining both ideal and statistical convergence under the name of $\mathcal{I}-$statistical convergence.

DEFINITION 1.3.11 (86) *A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$statistically convergent to a number L provided that for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \epsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}-$ statistical limit of the sequence $x = (x_k)$ and we write $S(\mathcal{I}) - \lim_k x_k = L$.

The authors also presented some $\lambda-$variant of $\mathcal{I}-$statistical convergence as follows.

DEFINITION 1.3.12 (86) *For a non-decreasing sequence $\lambda = (\lambda_n)$ of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1,$ a sequence $x = (x_k)$ of numbers is* *said to be* $\mathcal{I}-(V,\lambda)-$*summable to* $L \in X$ *if for any* $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| \geq \delta \right\} \in \mathcal{I},$$

In this case, L is called $(V,\lambda)(\mathcal{I})-$ *limit of the sequence* $x = (x_k)$.

DEFINITION 1.3.13 (86) *A sequence* $x = (x_k)$ *of numbers is said to be* $\mathcal{I}-\lambda-$*statistically convergent to L, if for every* $\epsilon > 0$ *and* $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \geq \epsilon\}\right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}-\lambda-$ *statistical limit of the sequence* $x = (x_k)$ *and we write* $S_\lambda(\mathcal{I}) - \lim_k x_k = L$.

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}-$ statistical convergence coincides with statistical convergence, $\mathcal{I}-(V,\lambda)-$ summability coincides with $(V,\lambda)-$ summability and $\mathcal{I}-\lambda-$ statistical convergence coincides with $\lambda-$ statistical convergence.

Let $S(\mathcal{I}), S_\lambda(\mathcal{I})$ and $[V,\lambda](\mathcal{I})$ be the collections of all $\mathcal{I}-$ statistically convergent, $\mathcal{I}-\lambda-$ statistically convergent and $\mathcal{I}-(V,\lambda)-$ summable sequences respectively.

Analogous to the paper (86), Das *et al.* (87) introduced $\mathcal{I}-$ lacunary statistical summability with the help of ideals and obtained the relationship of $\mathcal{I}-$ lacunary statistical summability and $\mathcal{I}-$ statistical summability of sequences as follows.

DEFINITION 1.3.14 (87) *Let* $\theta = (k_r)$ *be a lacunary sequence as defined earlier. A sequence* $x = (x_k)$ *of numbers is said to be* $\mathcal{I}-$ *lacunary statistically convergent to a number L or* $S_\theta(\mathcal{I})-$ *convergent to L provided that, for every* $\epsilon > 0$ *and each* $\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left|\{k \in I_r : |x_k - L| \geq \epsilon\}\right| \geq \gamma \right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}-$ *lacunary statistical limit of the sequence* $x = (x_k)$ *and we write* $S_\theta(\mathcal{I}) - \lim_k x_k = L$.

DEFINITION 1.3.15 (87) *A sequence $x = (x_k)$ of numbers is said to be $N_\theta(\mathcal{I})-$ convergent to L provided that, for every $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $x_k \to L(N_\theta(\mathcal{I}))$.

Let $S_\theta(\mathcal{I})$ and $N_\theta(\mathcal{I})$ be the collections of all $\mathcal{I}-$ lacunary statistically convergent and $N_\theta(\mathcal{I})-$ convergent sequences respectively.

In a different work, Das and Şavas (88) further generalized $\mathcal{I}-$ statistical convergence and $\mathcal{I}-$ lacunary statistical convergence and introduced new notions $\mathcal{I}-$ statistical convergence of order α and $\mathcal{I}-$ lacunary statistical convergence of order α, where $0 < \alpha \leq 1$.

DEFINITION 1.3.16 (88) *A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$ statistically convergent of order $\alpha, (\alpha \in (0, 1])$ to a number L provided that for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \ |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, the number L is called $\mathcal{I}-$ statistical limit of the sequence $x = (x_k)$ of order α and we write $S^\alpha(\mathcal{I}) - \lim_k x_k = L$.

Let $S^\alpha(\mathcal{I})$ denotes the set of all $\mathcal{I}-$ statistically convergent sequences of order α.

DEFINITION 1.3.17 (88) *A sequence $x = (x_k) \in X$ is said to be $\mathcal{I}-$ lacunary statistically convergent of order α to $L \in X$ provided that, for every $\epsilon > 0$ and each*

$\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \{ k \in I_r : |x_k - L| \geq \epsilon \} \right| \geq \gamma \right\} \in \mathcal{I}.$$

In this case, the number L is called $\mathcal{I}-$ lacunary statistical limit of the sequence $x = (x_k)$ of order α and we write $S_\theta^\alpha(\mathcal{I}) - \lim_k x_k = L$.

Let $S_\theta^\alpha(\mathcal{I})$ denotes the set of all $\mathcal{I}-$ lacunary statistically convergent sequences of order α.

Apart from these, ideal convergence has been discussed and developed in many research articles. For instance, $\mathcal{I}-$convergence has been developed for fuzzy numbers (89), in 2-normed spaces (90, 91); in n-normed spaces; in intuitionistic fuzzy normed spaces (92, 93). $\mathcal{I}-$convergence has also been discussed for sequences of functions. M. Balcerzak *et al*. (83) has introduced the notions of statistical and ideal convergence for sequences of functions whereas Karakaya *et al*. (94) extended this notion of convergence for the sequences of functions in intuitionistic fuzzy normed spaces.

1.4 Double Sequences

In this section, we explore how certain ideas have been extended from single sequences to double sequences. The concept of convergence of double sequences was initially discussed by Preingsheim (95) in 1900. In present thesis, by the convergence of a double sequence we mean the convergence in Priengsheim's sense.

DEFINITION 1.4.1 (95) *A double sequence $x = (x_{ij})$ of real numbers is said to be convergent in Priengsheim's sense or $P-$convergent if for every $\epsilon > 0$, there exists $n = n(\epsilon) \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j \geq n$. The number L is called Priengsheim limit of $x = (x_{ij})$ and we write $P - \lim_{i,j\to\infty} x_{ij} = L$.*

DEFINITION 1.4.2 (95) *A double sequence $x = (x_{ij})$ of real numbers is said to be bounded if there exists a positive integer M such that $|x_{ij}| < M$ for all i, j if, $\|x\|_{\infty,2} = \sup_{i,j} |x_{ij}| < \infty$.*

Let l_∞^2 denote the set of all real bounded double sequences.

In contrast to the case of single sequences, a convergent double sequence need not be bounded. For example, consider the sequence $x = (x_{ij})$ where x_{ij} are defined as follows.

$$x_{ij} = \begin{cases} \max\{i, j\} & \text{if } \min\{i, j\} = 0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $P - \lim_{i,j\to\infty} x_{ij} = 0$ and $\sup_{i,j} |x_{ij}| = \infty$.

DEFINITION 1.4.3 (95) *A double sequence $x = (x_{ij})$ of real numbers is said to be Cauchy in Priengsheim's sense if for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $|x_{ij} - x_{mn}| < \epsilon$, whenever $i \geq m \geq n_0$ and $j \geq n \geq n_0$.*

In past years, the above concept has been generalized in various forms such as Cesàro summability, almost convergence, generalization through orlicz function and modulus function etc. One of the most important generalization is presented by Mursaleen and Osama (96) using two dimensional analogue of natural density, which is given as follows.

For $K \subset \mathbb{N} \times \mathbb{N}$, let $K(n, m)$ denotes the number of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two dimensional analogue of natural density is defined as follows. The lower asymptotic density of $K \subset \mathbb{N}\times\mathbb{N}$ is defined as $\underline{\delta_2}(K) = \liminf_{m,n\to\infty} \frac{K(n,m)}{nm}$. In case the sequence $(\frac{K(n,m)}{nm})$ has a limit in Priengsheim's sense, then we say that K

has a double natural density and we write,

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K(n,m)}{nm}.$$

For Example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$, then

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K(n,m)}{nm} \leq \lim_{m,n\to\infty} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density $1/2$.

The authors used this idea of double natural density to define strong Cesàro summability and statistical convergence for double sequences as follows.

DEFINITION 1.4.4 (96) *A double sequence $x = (x_{ij})$ of real numbers is said to be Cesàro summable to L if*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=1,j=1}^{n,m} x_{ij} = L.$$

Let $(C, 1, 1)(x)$ denotes the set of all Cesàro summable double sequences of numbers.

DEFINITION 1.4.5 (96) *A double sequence $x = (x_{ij})$ of real numbers is said to be statistically convergent to L if for every $\epsilon > 0$,*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} |\{(i,j) \in \mathbb{N}\times\mathbb{N}, i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_2 - \lim_{i,j\to\infty} x_{ij} = L$.

Let $S_2(x)$ denotes the set of all statistically convergent double sequences.

It is remarkable that if $x = (x_{ij})$ is a convergent double sequence, then it is also statistically convergent to the same number, if $x = (x_{ij})$ is statistically convergent to a number L, then L is determined uniquely and if $x = (x_{ij})$ is statistically convergent, then $x = (x_{ij})$ need not be convergent, it is not necessarily bounded. For example, let $x = (x_{ij})$ be defined as,

$$x_{ij} = \begin{cases} \max\{i,j\} & \text{if } \min\{i,j\} = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that $S_2 - \lim_{i,j\to\infty} x_{ij} = 1$, since the cardinality of the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}, i \leq n, j \leq m : |x_{ij} - 1| \geq \epsilon\} \leq \sqrt{i}\sqrt{j},$$

for every $\epsilon > 0$. But x is neither convergent nor bounded.

DEFINITION 1.4.6 (96) *A double sequence $x = (x_{ij})$ of real numbers is said to be statistically Cauchy if for every $\epsilon > 0$, there exists $N = N(\epsilon)$ and $M = M(\epsilon)$ such that*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} |\{(i,j) \in \mathbb{N} \times \mathbb{N}, i \leq n, j \leq m : |x_{ij} - x_{NM}| \geq \epsilon\}| = 0.$$

Patterson and Savaş(97) extended the idea of statistical convergence of double sequences with the help of lacunary sequences and defined two dimensional analog of lacunary statistical convergence. Also, they introduced the idea of $S_2^\theta-$Cauchy sequences and established Cauchy convergence criterion for double sequences with the help of lacunary sequences.

DEFINITION 1.4.7 (97) *The double sequence $\theta = (k_r, l_s)$ is called double lacunary if there exists two increasing sequences of integers such that*

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

The intervals determined by θ *are* $I_r = (k_{r-1}, k_r]$, $I_s = (l_{s-1}, l_s]$ *and* $I_{r,s} = \{(k,l) : k \in I_r, l \in I_s\}$.

DEFINITION 1.4.8 (97) *Let* θ *be a double lacunary sequence. A double sequence* $x = (x_{ij})$ *of real numbers is said to be lacunary statistically convergent to L if for every* $\epsilon > 0$,

$$P - \lim_{r,s\to\infty} \frac{1}{h_r \bar{h}_s} |\{(i,j) \in I_{r,s} : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_2^\theta - \lim_{i,j\to\infty} x_{ij} = L$.

DEFINITION 1.4.9 (97) *A double indexed sequence* $\rho = \{(\bar{k_r}, \bar{l_s})\}$ *is said to be a double lacunary refinement of the double lacunary sequence* $\theta = \{k_r, l_s\}$ *if* $\{k_r, l_s\} \subseteq \{(\bar{k_r}, \bar{l_s})\}$.

DEFINITION 1.4.10 (97) *Let* θ *be a double lacunary sequence. A double sequence* $x = (x_{ij})$ *of real numbers is said to be lacunary statistically Cauchy if there exists a double subsequence* $\left(x_{\bar{k_r},\bar{l_s}}\right)$ *of* x *such that* $(\bar{k_r}, \bar{l_s}) \in I_{r,s}$ *for each* (r,s) $P - \lim_{r,s} x_{\bar{k_r},\bar{l_s}} = L$ *and for every* $\epsilon > 0$

$$P - \lim_{r,s\to\infty} \frac{1}{h_r \bar{h}_s} |\{(i,j) \in I_{r,s} : |x_{ij} - x_{\bar{k_r},\bar{l_s}}| \geq \epsilon\}| = 0.$$

The authors introduced the Cauchy convergence criterion as follows. *"A double sequence* $x = (x_{ij})$ *of real numbers is lacunary statistically convergent if and only if it is lacunary statistically Cauchy."*

Tripathy and Tripathy (98) extended the idea of $\mathcal{I}-$Convergence and $\mathcal{I}-$Cauchy from single sequences to double sequences. Also, they studied the properties like solidity, symmetricity, completeness and denseness etc. Further, Kumar (99) introduced the idea of $\mathcal{I}^*-$convergence for double sequences and obtained some relations between $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence for double sequences. Double sequences have also been connected with the notion of $\lambda-$ statistical convergence by Mursaleen *et al.* (100). They extended this notion from single sequences and called it $(\lambda, \mu)-$ statistical convergence for double sequences $x = (x_{ij})$. They also determined matrix transformations and core theorems for this new summability method.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to ∞ with

$$\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1 \text{ and } \mu_{m+1} \leq \mu_m + 1, \ \mu_1 = 1,$$

and $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. Then the $(\lambda, \mu)-$density of K is defined as

$$\delta_{(\lambda,\mu)}(K) = P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} |\{n - \lambda_n + 1 < i < n, m - \mu_m + 1 < j < m : (i, j) \in K\}|,$$

provided the limit on right hand side exists.

For $\lambda_n = n, \mu_m = m$ the $(\lambda, \mu)-$density reduces to the natural double density. Also, since $(\lambda_n/n) \leq 1$ and $(\mu_m/m) \leq 1$, $\delta_2(K) \leq \delta_{(\lambda,\mu)}(K)$ for every $K \subset \mathbb{N} \times \mathbb{N}$.
The generalized de la Vallée-Poussin mean of $x = (x_{ij})$ is defined by

$$t_{mn}(x) = \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} x_{ij},$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$.

DEFINITION 1.4.11 (100) *A double sequence $x = (x_{ij})$ is said to be $(V, \lambda, \mu)-$ summable to a number L provided that $t_{mn}(x) \to L$ as $m, n \to \infty$.*

DEFINITION 1.4.12 (100) *A double sequence $x = (x_{ij})$ of numbers is said to be*

$(\lambda,\mu)-$ statistically convergent to a number L provided for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called $(\lambda,\mu)-$statistical limit of the sequence $x = (x_{ij})$ and we write $S_{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = L$.

Let $[V,\lambda,\mu](x)$ and $S_{(\lambda,\mu)}(x)$ denotes the set of all $(V,\lambda,\mu)-$ summable and $(\lambda,\mu)-$ statistically convergent double sequences of numbers.

For $\lambda_n = n$ and $\mu_m = m$, the strong $(V,\lambda,\mu)-$ summability reduces to strong Cesàro summability for double sequences.

Recently, Çolak and Altin (101) introduced statistical convergence of order $\tilde{\alpha}$ for double sequences, where $\tilde{\alpha} = (a,b) : a,b \in (0,1]$. The $\tilde{\alpha}$ double density of $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_2^{\tilde{\alpha}}(K) = \lim_{n,m\to\infty} K(n,m)/n^a m^b.$$

It is worth attention that for any $K \subset \mathbb{N} \times \mathbb{N}$, $\delta_2^{\tilde{\alpha}}(K)$ may be greater than 1, even equal to ∞, but $\delta_2(K) \leq 1$. Also $\delta_2(K^c) = 1 - \delta_2(K)$ holds but $\delta_2^{\tilde{\alpha}}(K^c) = 1 - \delta_2^{\tilde{\alpha}}(K)$ does not hold in general.

DEFINITION 1.4.13 (101) *A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent of order $\tilde{\alpha}$ if there exists a number L such that for every $\epsilon > 0$,*

$$\lim_{n,m\to\infty} \frac{1}{n^a m^b} |\{(i,j) \in \mathbb{N} \times \mathbb{N}, i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_2^{\tilde{\alpha}} - lim_{i,j\to\infty} x_{ij} = L$.

Let $S_2^{\tilde{\alpha}}(x)$ denotes the set of all statistically convergent double sequences of order $\tilde{\alpha}$. For $\tilde{\alpha} = 1$, the statistical convergence of double sequences of order $\tilde{\alpha}$ reduces to statistical convergence of double sequences. The statistical convergence of double sequences of order $\tilde{\alpha}$ is well defined only for $\tilde{\alpha} \in (0, 1]$.

DEFINITION 1.4.14 (101) *A double sequence $x = (x_{ij})$ of real numbers is said to be Cesàro summable of order $\tilde{\alpha}$ to L if*

$$P - \lim_{n,m\to\infty} \frac{1}{n^a m^b} \sum_{i=1,j=1}^{n,m} x_{ij} = L.$$

Many authors including Bromwich (102), Robison (103), Hardy (104), Mòricz (105, 106, 107), Çakan and Altay (108), Patterson and Savaş (109), Tripathy and Sarma (110, 111, 112), Kumar and Mursaleen (37), Orhan and Dirik (113) and Savaş (114) etc. have shown their keen interest in different directions to study double sequences and related convergence problems.

1.5 Strong and Weak Convergence and its Generalizations

In this section, we wish to introduce strong and weak convergence of a sequence of vectors in a normed linear space X. We also present some generalized weak convergence in normed linear spaces. Let X be a normed linear space equipped with $\|.\|$.

A sequence (x_k) in a normed space X is said to be strongly convergent if for any number $\epsilon > 0$ there exists an element $x \in X$ and a positive integer N such that $\|x_k - x\| < \epsilon$ whenever $k \geq N$. In this case, we write L a limit for (x_k) and write $\lim_{k\to\infty} x_k = L$, or $x_k \to L$ as $k \to \infty$. A sequence $(x_k) \in X$ is said to be strongly Cauchy if for each $\epsilon > 0$ there corresponds $n \in \mathbb{N}$ for which $\|x_k - x_m\| < \epsilon$ for all $k, m \geq n$.

Another interesting and important concept that arises obviously upon the introduction of the dual space is that of weak convergence (115). Infact, weak convergence plays a prominent role to resolve many optimization problems.

A sequence (x_k) in a normed space X is said to be weakly convergent to $x \in X$ provided that $\lim_k \varphi(x_k - x) = 0$ for each $\varphi \in X^*$, the continuous dual of X(the set of all continous linear functionals on X). In this case, we write $W - \lim_k x_k = x$. (116)

Connor *et al*. (117) proceed on the same lines towards the case of number sequences and introduced weak statistical convergence in Banach spaces to describe these spaces with separable duals.

DEFINITION 1.5.1 (117) *Let X be a Banach space, a sequence $(x_k) \in X$ is said to be weak statistically convergent to $x \in X$ provided that, for each $\epsilon > 0$,*

$$\delta\left(\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right) = 0,$$

for each $\varphi \in X^*$, the continuous dual of X. In this case, we write $WS-\lim_{k\to\infty} x_k = x$.

The set of all weak statistically convergent sequences is denoted by $WS(X)$.

DEFINITION 1.5.2 (117) *A sequence $(x_k) \in X$ is said to be weakly statistically null provided that, for each $\epsilon > 0$,*

$$\delta\left(\{k \leq n : |\varphi(x_k)| \geq \epsilon\}\right) = 0,$$

for each $\varphi \in X^*$.

The authors proved some interesting properties of weak statistical convergence in Banach spaces.

(i) A Banach space is finite dimensional if and only if every weakly statistically null sequence in X has a bounded subsequence in X.

(ii) If X is a separable Banach space. Then X has a separable dual if and only if every bounded weakly statistically null sequence in X agrees with a weakly null sequence on almost all indices.

(iii) It has been shown that the space l_2 (space of square summable scalar sequences) has a statistical M-basis which is not a Schauder basis.

Bhardwaj and Bala (118) introduced weak statistically Cauchy sequences in a normed space X and presented an important characterization which is given as follows.

If the normed space X is reflexive, then every weak statistically Cauchy sequence is weakly statistically convergent.
They studied weak statistical convergence in l_p spaces as well.

DEFINITION 1.5.3 (118) *A sequence (x_k) in a normed space X is said to be weak statistically Cauchy if $(\varphi(x_k))$ is statistically Cauchy for each $\varphi \in X^*$.*

As every subsequence of a weakly convergent sequence is again weakly convergent, but this is not true in case of weak statistical convergence. The authors characterized an important result that every statistically dense subsequence of a weakly statistically convergent sequence is weakly statistically convergent, but the converse is not true in general.

After their work, Nuray (119) generalized the notion of weak statistical convergence to introduce lacunary weak statistical convergence with the help of a lacunary sequence as follows.

DEFINITION 1.5.4 (119) *Let $\theta = (k_r)$ be a lacunary sequence and (x_k) be a sequence in a Banach space X. Then (x_k) is said to be weak lacunary statistically* *convergent to* $x \in X$ *provided that, for each* $\varphi \in X^*$, *each* $\epsilon > 0$,

$$\delta_\theta\left(\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}\right) = 0.$$

In this case, we write $WS_\theta - \lim_{k\to\infty} x_k = x$.

DEFINITION 1.5.5 (119) *A sequence* (x_k) *in a Banach space* X *is weakly* $N_\theta-$ *convergent to* $x \in X$ *provided that for each* $\varphi \in X^*$, *the sequence* $(\varphi(x_k - x))$ *is* $N_\theta-$*convergent to* 0.

Let $WS_\theta(X)$ and $WN_\theta(X)$ be the sets of all weak lacunary statistically convergent and weak $N_\theta-$ convergent sequences in X.

For some further works in this direction, we have referred (120, 121, 122, 123, 124, 125, 126).

1.6 Probabilistic Normed Spaces

In this section, we give a brief review of probabilistic normed spaces ($PN-$ spaces). In 1942, Menger (127) introduced the notion of probabilistic metric space ($PM-$ space) to resolve the interpretative issue of quantum mechanics. He proposed transferring the probabilistic notions of quantum mechanics from physics to the underlying geometry. He replaced the distance between points p and q by a distribution function F_{pq} whose value $F_{pq}(x)$ at the real number x is interpreted as the probability that the distance between p and q is less than x. Studies of such spaces by numerous authors followed (Constantin and Istrătescu (128), Schweizer and Sklar (129), (130), (131), (132), (133), Sempi (134), Šerstnev (135), Tardiff (136), Throp (137), etc.).

In 1962, Šerstnev (138) generalized the notion of probabilistic metric spaces and introduced the notion of probabilistic normed spaces. In these spaces, the norms of the vectors are represented by probability distribution functions rather than numerical values. Let $\mathbb{R}$ denotes the set of reals and $\mathbb{R}_0^+ = [0, \infty)$.

DEFINITION 1.6.1 (139) *A function $f : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ is called a distribution function if it is both non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. Also, a distance distribution function is a non-decreasing function F defined on $\mathbb{R}^+ = [0, \infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left-continuous on $(0, \infty)$.*

Let $\mathfrak{D}$ and $\mathfrak{D}^+$ denotes the set of all distribution functions and distance distribution functions respectively. The elements of $\mathfrak{D}^+$ are partially ordered via $F \leq G$ if and only if $F(x) \leq G(x)\ \forall x \in \mathbb{R}$.

DEFINITION 1.6.2 (133) *A triangular norm, briefly known as t-norm, is a binary operation $*$ on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ such that for all $a, b, c \in [0, 1]$:*

(i) $a * 1 = a$,

(ii) $a * b = b * a$,

(iii) $c * d \geq a * b$ *if* $c \geq a$ *and* $d \geq b$,

(iv) $(a * b) * c = a * (b * c)$.

The $*$ operations $a * b = max\{a + b - 1, 0\}$, $a * b = ab$, and $a * b = min\{a, b\}$ on $[0, 1]$ are t-norms.

Using *DEFINITION 1.6.1* and *DEFINITION 1.6.2*, the notion of probabilistic normed space is defined as follows.

DEFINITION 1.6.3 (139) *A triplet $(X, F, *)$ is called a probabilistic normed space (briefly, a PN− Space) if X is a real vector space, F is a mapping from X into $\mathfrak{D}$ (for $x \in X$, the distribution function $F(x)$ is denoted by F_x, and $F_x(t)$ is the value F_x at $t \in \mathbb{R}$) and $*$ is a t-norm satisfying the following conditions:*

(PF-1) $F_x(0) = 0$,

(PF-2) $F_x(t) = 1$ *for all* $t > 0$ *iff* $x = 0$,

(PF-3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$*for all* $\alpha \in \mathbb{R} - \{0\}$,

(PF-4) $F_{x+y}(s + t) \geq F_x(s) * F_x(t)$ *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}_0^+$.

Suppose that $(X, \|.\|)$ be a normed space. Let $\mu \in \mathfrak{D}$ such that $\mu(0) = 0$ and $\mu \neq h$, where

$$h(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0 \end{cases}$$

Define

$$F_x(t) = \begin{cases} h(t), & \text{if } x = 0, \\ \mu(\frac{t}{\|x\|}, & \text{if } x \neq 0 \end{cases}$$

where $x \in X, t \in \mathbb{R}$. Then $(X, F, *)$ is a PN-space. For example, if we define functions μ and $\mu^{'}$ on $\mathbb{R}$ by

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{x}{1+x}, & \text{if } x > 0 \end{cases}$$

and

$$\mu^{'}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp(\frac{-1}{x}), & \text{if } x > 0 \end{cases}$$

then, we obtain the following well known $*-$norms

$$F_x(t) = \begin{cases} h(t), & \text{if } x = 0, \\ \frac{t}{t+\|x\|}, & \text{if } x \neq 0 \end{cases}$$

and

$$F_x^{'}(t) = \begin{cases} h(t), & \text{if } x = 0, \\ \exp(\frac{-\|x\|}{t}), & \text{if } x \neq 0 \end{cases}$$

The notions of ordinary convergence and Cauchy sequences in Probabilistic Normed Spaces are given below. Let $(X, F, *)$ be a Probabilistic Normed space.

DEFINITION 1.6.4 (140) *A sequence $x = (x_k) \in X$ is said to be convergent to $L \in X$ with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $F_{x_k - L}(\epsilon) > 1 - \lambda$ whenever $k > k_0$. It is denoted by $F - \lim_{k\to\infty} x_k = L$ or $(x_k) \to L$ with respect to the probabilistic norm F.*

DEFINITION 1.6.5 (140) *A sequence $x = (x_k) \in X$ is said to be Cauchy sequence with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $F_{x_k - x_l}(\epsilon) > 1 - \lambda$ whenever $k, l \geq k_0$.*

An important result regarding convergence in a probabilistic normed space can be stated as follows.

REMARK 1.6.1 (141) *Let $(X, \|.\|)$ be a real normed space, and $F_x(t) = \frac{t}{t+\|x\|}$ where $x \in X$ and $t \geq 0$ (i.e. standard $*-$norm induced by $\|.\|$). Then, any sequence $x = (x_k) \in X$ is convergent with respect to the norm $\|.\|$, if and only if, it is convergent with respect to the probabilistic norm F. Moreover both the limits are same.*

REMARK 1.6.2 (141) *Let $(X, \|.\|)$ be a real normed space, and $F_x(t) = \frac{t}{t+\|x\|}$ where $x \in X$ and $t \geq 0$ (i.e. standard $*-$norm induced by $\|.\|$). Then, any sequence $x = (x_k) \in X$ is Cauchy with respect to the norm $\|.\|$, if and only if, it is Cauchy with respect to the probabilistic norm F.*

Karakus (141) generalized the notions of ordinary convergence, Cauchy sequence and introduced the notions of statistical convergence and statistically Cauchy sequence in a probabilistic normed space. He also introduced the notions of statistical limit point and statistical cluster point for these spaces.

DEFINITION 1.6.6 (141) *A sequence $x = (x_k) \in X$ is said to be statistically convergent to $L \in X$ with respect to probabilistic norm F provided that for every $\epsilon > 0$ and $\lambda \in (0,1)$*

$$\delta\left(\{k \in \mathbb{N} : F_{x_k - L}(\epsilon) \leq 1 - \lambda\}\right) = 0.$$

In this case, we write $S^F - \lim x = L$, where L is said to be S^F-limit.

DEFINITION 1.6.7 (141) *A sequence $x = (x_k) \in X$ is said to be statistically Cauchy with respect to the probabilistic norm F provided that for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m such that*

$$\delta\left(\{k \in \mathbb{N} : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}\right) = 0.$$

DEFINITION 1.6.8 (141) *A sequence $x = (x_k) \in X$ is said to be statistically bounded with respect to the probabilistic norm F if there exists some $t \in \mathbb{R}$ and $\lambda \in (0,1)$ such that*

$$\delta\left(\{k \in \mathbb{N} : F_{x_k}(t) \leq 1 - \lambda\}\right) = 0.$$

DEFINITION 1.6.9 (141) *An element $l \in X$ is called a limit point of the sequence $x = (x_k) \in X$ with respect to the probabilistic norm F provided that there is a subsequence of x that converges to l with respect to the probabilistic norm F.*

Let $L^F(x)$ denotes the set of all limit points of the sequence x with respect to the probabilistic norm F.

DEFINITION 1.6.10 (141) *Let $(X, F, *)$ be a $PN-$ space. An element $\mu \in X$ is called a statistical limit point of a sequence $x = (x_k)$ in X with respect the probabilistic norm F provided that there is a nonthin subsequence of x that is convergent to μ with respect the probabilistic norm F.*

Let $\Lambda^F(S, x)$ denotes the set of all statistical limit points of the sequence $x = (x_k)$.

DEFINITION 1.6.11 (141) *An element $\gamma \in X$ is said to be a statistical cluster point of a sequence $x = (x_k)$ in X with respect the probabilistic norm F provided that for every $\epsilon > 0$, $t \in (0, 1)$*

$$\limsup_{r\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k-\gamma}(\epsilon) > 1 - t\}| > 0.$$

Let $\Gamma^F(S, x)$ denotes the set of all statistical cluster points of the sequence $x = (x_k)$.

Karakus and Demirci (142) extended *DEFINITION 1.6.6* and *DEFINITION 1.6.7* from single to double sequences on these spaces and studied some of their basic properties.

Subsequently, Alsina *et al*. (143) presented more general DEFINITION of a $PN-$ space that includes the DEFINITION of Šerstnev as a special case. Now, we list some DEFINITIONS which will be needful in the sequel.

For any $a \in \mathbb{R}$, ε_a, the unit step at a, is the function in Δ given by

$$\epsilon_a(x) = \begin{cases} 0, & \text{if } -\infty \leq x \leq a, \\ 1, & \text{if } a \leq x \leq \infty \end{cases}$$

and

$$\epsilon_\infty(x) = \begin{cases} 0, & \text{if } -\infty \leq x \leq \infty, \\ 1, & \text{if } x = \infty \end{cases}$$

The distance $d_L(F,G)$ between two functions $F,G \in \mathfrak{D}^+$ is defined as the infimum of all numbers $h \in (0,1]$ such that the inequalities

$$F(x-h)-h \leq G(x) \leq F(x+h)+h, G(x-h)-h \leq F(x) \leq G(x+h)+h$$

hold for every $x \in (-\frac{1}{h}, \frac{1}{h})$. It is known that d_L is a metric on $\mathfrak{D}^+$ (143).

Convergence with respect to this metric is equivalent to weak convergence of distribution functions, *i.e.*, $\{F_n\}$ in $\mathfrak{D}^+$ converges weakly to F in $\mathfrak{D}^+$ (written $F_n \to^W F$) if and only if $F_n(x)$ converges to $F(x)$ at every point of continuity of the limit function F. Consequently, we have,

$$F_n \to^W F \text{ if and only if } d_L(F_n, F) \to 0$$

$$F(t) > 1-t \text{ if and only if } d_L(F, \epsilon_0) < t \text{ for every } t > 0.$$

DEFINITION 1.6.12 (144, 145) *A triangle function is a binary operation on* $\mathfrak{D}^+$ *namely a function* $\tau : \mathfrak{D}^+ \times \mathfrak{D}^+ \to \mathfrak{D}^+$ *such that for all* F, G *and* H *in* $\mathfrak{D}^+$*, we have,*

(i) $\tau(\tau(F,G), H) = \tau(F, \tau(G,H))$;

(ii) $\tau(F,G) = \tau(G,F)$;

(iii) $F \leq G \Rightarrow \tau(F,H) \leq \tau(G,H)$ *and*

(iv) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Typical continuous triangle functions are τ_T and τ_{T^*}, which are respectively given by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t)), \ \ \tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t))$$

for all $F,G \in \mathfrak{D}^+$ and for all $x \in \mathbb{R}$. Here T is a continuous $t-$norm and T^* is the corresponding continuous $t-$conorm. If T is a $t-$norm and T^* is defined on $[0,1] \times [0,1]$ via $T^*(x,y) = 1 - T(1-x, 1-y)$, then T^* is a $t-$conorm, specifically the $t-$conorm of T (133).

DEFINITION 1.6.13 *(PN− spaces redefined) (143) A PN− space is a quadruple* $(V, \vartheta, \tau, \tau^*)$*, where* V *is a real linear space,* τ *and* τ^* *are continuous triangle functions with* $\tau \leq \tau^*$ *and* ϑ *is a mapping (the probabilistic norm) from* V *into* $\mathfrak{D}^+$ *such that, for all* p, q *in* V*, the following conditions hold:*

PN(i) $\vartheta_p = \epsilon_0$*, if and only if,* $p = \theta$ *(*θ *is the null vector in* V*);*

PN(ii) $\vartheta_{-p} = \vartheta_p$*, for all* $p \in V$*;*

PN(iii) $\vartheta_{p+q} \geq \tau(\vartheta_p, \vartheta_q)$ *and*

PN(iv) $\vartheta_p \leq \tau^*(\vartheta_{\lambda p}, \vartheta_{(1-\lambda)p})$ *for every* $\lambda \in [0, 1]$.

If ϑ satisfies the condition, weaker than $(PN1)$, $\vartheta_\theta = \epsilon_0$, then $(V, \vartheta, \tau, \tau^*)$ is called a Probabilistic Pseudo-Normed Space (briefly $PPN-$space). If ϑ satisfies the condition, $(PN1)$ and $(PN2)$, then $(V, \vartheta, \tau, \tau^*)$ is called a Probabilistic seminormed space (briefly $PSN-$ space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for a some continuous t-norm T and its t-conorm T^*, then $(V, \vartheta, \tau, \tau^*)$ is denoted by (V, ϑ, T) and is called a Menger PN-space. A PN-space is called a Šerstnev space if it satisfies $(PN1)$, $(PN3)$ and the following condition: For all $p \in V$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $x > 0$,

$$\vartheta_{\alpha p}(x) = \vartheta_p\left(\frac{x}{|\alpha|}\right),$$

which clearly implies $(PN2)$ and also $(PN4)$ in the strengthened form.

DEFINITION 1.6.14 (133) *Let* $(V, \vartheta, \tau, \tau^*)$ *be a PN−space. For* $p \in V$ *and* $t > 0$*, the strong t−neighborhood of* p *is the set*

$$N_p(t) = \{q \in V : \vartheta_{q-p}(t) > 1 - t\} = \{q \in V : d_L(\vartheta_{q-p}, \epsilon_0) < t\} \text{ and,}$$

the strong neighborhood system for V *is the union* $\cup_{p \in V} N_p$ *where,* $N_p = \{N_p(t) : t > 0\}$.

There is a natural topology defined on a $PN-$ space $(V,\vartheta,\tau,\tau^{*})$ called the strong topology in terms of strong neighborhood system. Using these concepts, we extend the statistical convergence of sequences in $PN-$ spaces endowed with the strong topology.

DEFINITION 1.6.15 (133) *A sequence $p=(p_k)$ in a PN-space $(V,\vartheta,\tau,\tau^{*})$ is said to be strongly convergent to a point p_0 in V, symbolically, $\lim_k p_k = p_0$, if for any $t>0$ there exists a positive integer m such that p_k is in $N_{p_0}(t)$ whenever $k \geq m$. Also, the sequence $p=(p_k)$ in V is called strong Cauchy sequence if for every $t>0$, there is a positive integer N such that, $\vartheta_{p_k-p_m}(t) > 1-t$, whenever $m,n>N$.*
A PN-space $(V,\vartheta,\tau,\tau^{*})$ is said to be strongly complete in strong topology if and only if every strongly Cauchy sequence in V is strongly convergent to a point in V.

Şençimen and Pehlivan (146) introduced the concepts of strongly statistically convergent sequence, statistically Cauchy sequence, strong statistical limit points and the strong statistical cluster points in probabilistic normed space endowed with the strong topology and established some facts. Widening the work of Alsina *et al.* (147), Şençimen and Pehlivan (148) defined a more general type of continuity, namely the statistical continuity of probabilistic norms and vector space operations through the concepts of strong statistical convergence. The notion of strong statistical convergence is given as follows:

DEFINITION 1.6.16 (146) *A sequence $p=(p_k)$ in a PN-space $(V,\vartheta,\tau,\tau^{*})$ is said to be strongly statistically convergent to a point p_0 in V provided that*

$$\lim_n \frac{1}{n}\left|\{k \leq n \ : p_k \notin N_{p_0}(t)\}\right| = 0;$$

i.e., $\delta(\{k \in \mathbb{N} : p_k \notin N_{p_0}(t)\}) = 0$. In this case, p_0 is called the strong statistical limit *of the sequence* $p = (p_k)$ *and we write* $S^{\vartheta} - \lim_k p_k = p_0$.

Let S^{ϑ} be the collection of all strongly statistically convergent sequences in PN-space $(V, \vartheta, \tau, \tau^*)$.

DEFINITION 1.6.17 (146) *A sequence* $p = (p_k)$ *in a* PN*-space* $(V, \vartheta, \tau, \tau^*)$ *is said to be strongly statistically Cauchy sequence if there exists a positive integer* m *such that*

$$\lim_n \frac{1}{n} \left|\{k \leq n \ : (p_k - p_m) \notin N_\theta(t)\}\right| = 0.$$

Let $(V, \vartheta, \tau, \tau^*)$ be a PN-space and $p = (p_k)$ be any sequence in V. If $\left(p_{k(j)}\right)$ be a subsequence of (p_k) and $K = \{k\,(j) : j \in \mathbb{N}\}$, then we denote $\left(p_{k(j)}\right)$ by $(p)_K$. If $\lim_n \frac{1}{n} \left|\{k\,(j) : j \in \mathbb{N}\}\right| = 0$, then we say that $\left(p_{k(j)}\right)$ is a thin subsequence of (p_k). On the other hand, K is non-thin provided that $\limsup_n \frac{1}{n} \left|\{k\,(j) : j \in \mathbb{N}\}\right| > 0$.

DEFINITION 1.6.18 *An element* $\xi \in V$ *is a strong statistical limit point of* (p_k) *provided that there exists a non-thin subsequence of* (p_k) *that strongly converges to* ξ.

We denote the set of all strong statistical limit points of (p_k) by $\Lambda^{\vartheta}(S, p)$.

DEFINITION 1.6.19 *An element* $\eta \in V$ *is a strong statistical cluster point of* (p_k) *provided that for every* $t > 0$*, we have* $\limsup_n \frac{1}{n} \left|\{k \in \mathbb{N} : p_k \in N_\eta(t)\}\right| > 0$.

We denote the set of all strong statistical cluster points of (p_k) by $\Gamma^{\vartheta}(S, p)$.

In continuation to these works, Rehmet (149) extended the notion of statistical convergence with the help of lacunary sequences and defined the notions of lacunary statistical convergence and lacunary statistically Cauchy sequences in probabilistic normed space.

DEFINITION 1.6.20 (149) *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. A sequence $p = (p_k)$ in V is said to be strongly lacunary statistically convergent to a point p_0 in V if*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : p_k \notin N_{p_0}(t)\}| = 0 \ .$$

In this case, p_0 is called the strong lacunary statistical limit of the sequence $p = (p_k)$ and we write $S_\theta^\vartheta - \lim_k p_k \ = p_0$.

Let S_θ^ϑ be the collection of all strongly lacunary statistically convergent sequences in PN-space $(V, \vartheta, \tau, \tau^*)$.

Alotaibi (150) defined the notions of $\lambda-$statistical convergence and $\lambda-$statistically Cauchy in probabilistic normed spaces. Mursaleen and Mohiuddine (151) has defined the notion of $\mathcal{I}-$ convergence and $\mathcal{I}^*-$ convergence in probabilistic normed spaces. They also defined $\mathcal{I}-$ limit points and $\mathcal{I}-$ cluster points in probabilistic normed space and proved some interesting results. Kumar and Kumar (152) studied $\mathcal{I}-$ Cauchy and $\mathcal{I}^*-$ Cauchy sequences in probabilistic normed spaces. Kumar and Guillen (153) has extended these notions for double sequences and defined $\mathcal{I}-$ convergence and $\mathcal{I}^*-$ convergence for double sequences in probabilistic normed spaces. Hazarika (154) has defined $\lambda-$ convergence, $\mathcal{I}_\lambda-$ convergence, $\mathcal{I}_\lambda-$ limit points and $\mathcal{I}_\lambda-$ cluster points in probabilistic normed spaces as a variant of the notion of ideal convergence. For more details on these spaces, we have referred (155, 156, 157, 158, 159, 160).

1.7 Random $2-$Normed Spaces

In 2005, Goleţ (161) used the concept of 2-norm of Gähler (162) and presented generalized probabilistic normed space which is called Random 2-Normed Space. We quote some requisite definitions below.

DEFINITION 1.7.1 (162) *Let X be a real vector space of dimension $d > 1$ (d may be infinite). A real valued function $||\cdot,\cdot|| : X^2 \to \mathbb{R}$ satisfying the following conditions:*

(i) $||x_1, x_2|| = 0$, *if and only if x_1, x_2 are linearly dependent.*

(ii) $||x_1, x_2|| = ||x_2, x_1||$ *for all $x_1, x_2 \in X$,*

(iii) $||\alpha x_1, x_2|| = |\alpha|||x_1, x_2||$, *for any $\alpha \in \mathbb{R}$ and*

(iv) $||x_1 + x_2, x_3|| \leq ||x_1, x_3|| + ||x_2, x_3||$

is called a 2-norm and the pair $(X, ||\cdot,\cdot||)$ is called a 2-normed space.

For example, if we take $X = \mathbb{R}^2$ with 2-norm $||x_1, x_2|| =$ area of parallelogram spanned by the vectors x_1, x_2 which may be given explicitly by the formula

$$||x_1, x_2|| = |det(x_{ij})| = abs.\,(det\,(x_i, x_j))$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$ for each $i = 1, 2$. Then $(X, ||\cdot,\cdot||)$ is a 2-normed space.

DEFINITION 1.7.2 (161) *Let X be a real linear space of dimension $d > 1$ (d may be infinite), τ be a triangle function(a binary operation on $\mathfrak{D}^+$ which is associative, commutative, nondecreasing and ϵ_0 as a unit) and $\mathcal{F} : X \times X \to \mathfrak{D}^+$ (for $x, y \in X$, $\mathcal{F}(x, y; t)$ is the value of $\mathcal{F}(x, y)$ at $t \in \mathbb{R}$). Then $\mathcal{F}$ is called a probabilistic norm and $(X, \mathcal{F}, \tau)$ is called a probabilistic 2-normed space if the following conditions are satisfied:*

(P2N-i) $\mathcal{F}(x, y; t) = H_0(t)$ *if x, y are linearly dependent, where $H_0(t) = 0$ if $t \leq 0$ and $H_0(t) = 1$ if $t > 0$,*

(P2N-ii) $\mathcal{F}(x, y; t) \neq H_0(t)$ *if x, y are linearly independent,*

(P2N-iii) $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$ *for all* $x, y \in X$,

(P2N-iv) $\mathcal{F}(\alpha x, y; t) = \mathcal{F}(x, y; \frac{t}{|\alpha|})$ *for every* $t > 0$, $\alpha \neq 0$ *and* $x, y \in X$,

(P2N-v) $\mathcal{F}(x + y, z; t) \geq \tau\left(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t)\right)$, *where* $x, y, z \in X$.

If $(P2N\text{-}v)$ is replaced by $\mathcal{F}(x+y, z; t_1+t_2) \geq \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$ then $(X, \mathcal{F}, *)$ is called a random 2-normed space.

EXAMPLE 1.7.1 (161) *Every* 2*-normed space* $(X, ||\cdot, \cdot||)$ *can be made a random* 2*-normed space by setting* $\mathcal{F}(x, y; t) = H_0(t - ||x, y||)$ *where*

$$H_a(t) = \begin{cases} 0, & \textit{if } t \leq a, \\ 1, & \textit{if } t > a \end{cases}$$

for all $x, y \in X$, $t > 0$ *and* $a * b = ab$; $a, b \in [0, 1]$.

EXAMPLE 1.7.2 (161) *Let* $(X, ||\cdot, \cdot||)$ *be a* 2*-normed space with* $||x, z|| = |x_1 z_2 - x_2 z_1|$; $x = (x_1, x_2)$, $z = (z_1, z_2)$ *and* $a * b = ab$ *for all* $a, b \in [0, 1]$. *For every* $x, y \in X$ *and* $t > 0$, *we define* $\mathcal{F}(x, y; t) = \frac{t}{t + ||x, y||}$, *then* $(X, \mathcal{F}, *)$ *is a random* 2*-normed space.*

We next give few definitions due to M. Mursaleen (163) who extended the notion of statistical convergence in random 2-normed spaces.

DEFINITION 1.7.3(163) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. Then a sequence* $x = (x_k)$ *is said to be convergent to* $x_0 \in X$ *with respect to the norm* $\mathcal{F}$ *if for every* $\epsilon > 0$, $t \in (0, 1)$ *and* $\theta \neq z \in X$, *there exists a positive integer* k_0 *such that* $\mathcal{F}(x_k - x_0, z; \epsilon) > 1 - t$ *whenever* $k \geq k_0$. *It is denoted by* $\mathcal{F} - \lim x_k = x_0$.

DEFINITION 1.7.4(163) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. Then a* *sequence* $x = (x_k)$ *is said to be statistically convergent or* S^{R2N} *convergent to* $x_0 \in X$ *with respect to the norm* $\mathcal{F}$ *if for every* $\epsilon > 0$, $t \in (0, 1)$ *and* $\theta \neq z \in X$,

$$\delta\left(\{k \in \mathbb{N} : \mathcal{F}(x_k - x_0, z; \epsilon) \leq 1 - t\}\right) = 0.$$

In this case, we write $S^{R2N} - \lim x_k = x_0$.

Let S^{R2N} be the collection of all statistically convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

Mohiuddine and Aiyub (164) extended the idea of statistical convergence with the help of lacunary sequences and introduced the concepts of lacunary statistical convergence and lacunary statistically Cauchy in random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 1.7.5 (164) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space and* θ *be a lacunary sequence. Then a sequence* $x = (x_k)$ *is said to be lacunary statistically convergent or* S_θ^{R2N} *convergent to* $x_0 \in X$ *with respect to the norm* $\mathcal{F}$ *if for every* $\epsilon > 0$, $t \in (0, 1)$ *and non zero* $z \in X$,

$$\delta_\theta\left(\{k \in \mathbb{N} : \mathcal{F}(x_k - x_0, z; \epsilon) \leq 1 - t\}\right) = 0.$$

In this case, we write $S_\theta^{R2N} - \lim x_k = x_0$.

Let S_θ^{R2N} be the collection of all lacunary statistically convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 1.7.6 (164) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space and* θ *be a lacunary sequence. Then a sequence* $x = (x_k)$ *is said to be lacunary statistically Cauchy or* $S_\theta^{R2N}-$*Cauchy with respect to the norm* $\mathcal{F}$ *if for every* $\epsilon > 0$, $t \in (0, 1)$ *and non zero* $z \in X$, *there exists a number* $N = N(\epsilon, z)$ *such that for all* $k, l \geq N$

$$\delta_\theta\left(\{k \in \mathbb{N} : \mathcal{F}(x_k - x_l, z; \epsilon) \leq 1 - t\}\right) = 0.$$

Further Güncan *et al.* (165) introduced the concepts of lacunary statistical limit points and lacunary statistical cluster points in random 2-normed spaces. They have proved that the relations between the sets of $\theta-$statistical limit points and $\theta-$statistical cluster points of sequences of scalars are also true for the sets of $\theta-$statistical limit points and $\theta-$statistical cluster points of sequences in random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 1.7.7(165) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space and* θ *be a lacunary sequence. An element* $\mu \in X$ *is called a* $\theta-$*statistical limit point of a sequence* $x = (x_k) \in X$ *provided that there is a non-thin subsequence of* x *that converges to* μ.

Let $\Lambda^{R2N}(S_\theta, x)$ denotes the set of all $\theta-$statistical limit points of the sequence $x = (x_k)$ in random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 1.7.8(165) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space and* θ *be a lacunary sequence. An element* $\eta \in X$ *is called a* $\theta-$*statistical cluster point of a sequence* $x = (x_k) \in X$ *provided that for every* $\epsilon > 0$, $t \in (0, 1)$ *and non zero* $z \in X$,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \mathcal{F}(x_k - \eta, z; \epsilon) > 1 - t\}| > 0.$$

Let $\Gamma^{R2N}(S_\theta, x)$ denotes the set of all $\theta-$statistical cluster points of the sequence $x = (x_k)$ in random 2-normed space $(X, \mathcal{F}, *)$.

We next quote the following DEFINITION due to Mursaleen and Noman (166) on μ-convergent series.

DEFINITION 1.7.9 (166) *Let* $\mu = (\mu_k)$ *be a sequence of positive real numbers tending to infinity such that*

$$0 < \mu_0 < \mu_1 < \mu_2 \ldots \; and \, \mu_k \to \infty \; as \, k \to \infty.$$

Then a sequence $x = (x_k)$ *of numbers is said to be* μ*-convergent to a number* l *if* $\Lambda x_k \to l \, as \, k \to \infty$*, where*

$$\Lambda x_k = \frac{1}{\mu_k} \sum_{i=0}^{k} (\mu_i - \mu_{i-1}) x_i$$

Esi and Braha (3) used above DEFINITION to introduce a new notion called $\Lambda-$ statistical convergence in random 2-normed spaces and studied some of its properties.

DEFINITION 1.7.10 (3) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. Then a sequence* $x = (x_k)$ *is said to be* Λ*-convergent with respect to the norm* $\mathcal{F}$ *provided that, for every* $\epsilon > 0$*,* $t \in (0,1)$ *and* $\theta \neq z \in X$*, there exists a positive integer* n *such that* $\mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}$ *whenever* $k \geq n$*, for each non zero* $z \in X$*. In this case, we write* $\mathcal{F} - \lim \Lambda x_k = x_0$ *and* x_0 *is called* $\mathcal{F}_\Lambda$*-limit of* (x_k)*.*

DEFINITION 1.7.11 (3) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. Then a sequence* $x = (x_k)$ *is said to be* Λ*-Cauchy with respect to the norm* $\mathcal{F}$ *provided that, for every* $\epsilon > 0$*,* $t \in (0,1)$ *and* $\theta \neq z \in X$*, there exists a positive integer* n *such that* $\mathcal{F}(\Lambda x_k - \Lambda x_s, z; \epsilon) > 1 - t\}$ *whenever* $k, s \geq n$*, for each non zero* $z \in X$*.*

DEFINITION 1.7.12 (3) *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. Then a sequence* $x = (x_k)$ *is said to be* Λ*-statistically convergent with respect to the norm* $\mathcal{F}$ *provided that, for every* $\epsilon > 0$*,* $t \in (0,1)$ *and* $\theta \neq z \in X$*,*

$$\delta_\Lambda(\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}) = 0,$$

i.e.

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0.$$

In this case, we write $S_\Lambda^{R2N} - \lim x_k = x_0$*.*

Let S_{Λ}^{R2N} be the space of Λ-statistically convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 1.7.13 (3) *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be Λ-statistically cauchy with respect to the norm $\mathcal{F}$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, there exists a positive integer $k_0(\epsilon)$ such that for all $k, l \geq k_0$*

$$\delta_{\Lambda}(\{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) \leq 1 - t\}) = 0.$$

Random 2-normed spaces are further explored by various authors like Mursaleen and Alotaibi (167) defined the notions of ideal convergence in random 2-normed spaces. Mohiuddine *et al.* generalized the notion of ideal convergence for double sequences in random 2-normed spaces. Hazarika (168) defined generalized difference ideal convergence in random 2-normed spaces, whereas, the author (169) defined lacunary generalized difference statistical convergence in random 2-normed spaces. For an extensive view on these spaces, we referred (170, 171, 172, 173, 174, 175, 176, 177).