Chapter 6

Generalized Statistical Convergence in Random 2-Normed Space

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. In the present chapter, we introduce the notion of $\Lambda-$ statistical convergence of order α, $\Lambda-$ statistical Cauchy sequences of order α in random 2-normed spaces and obtain some results. We display examples which shows our method of convergence is more general in random 2-normed space.

6.1 $S_{\Lambda}^{R2N}(\alpha)-$ Convergence

In this section, we begin with the following definition of statistical and $\Lambda-$ statistical convergence of order α in random 2-normed spaces. Before we start, it would be convenient to recall the DEFINITION of $\mu-$convergent sequences due to Mursaleen

Results presented in this chapter are accepted in Miskolc Mathematical Notes, Hungary.

and Noman (166).

Let $\mu = (\mu_k)$ be a sequence of positive real numbers tending to infinity such that

$$0 < \mu_0 < \mu_1 < \mu_2 \ldots \text{ and } \mu_k \to \infty \text{ as } k \to \infty.$$

Then, a sequence $x = (x_k)$ of numbers is said to be μ-convergent to a number l if $\Lambda x_k \to l \text{ as } k \to \infty$, where

$$\Lambda x_k = \frac{1}{\mu_k} \sum_{i=0}^{k} (\mu_i - \mu_{i-1}) x_i$$

DEFINITION 6.1.1 *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be statistically convergent of order α $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1.$$

In this case, we write $S^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$.

Let, $S^{R2N}(\alpha)$ denotes the set of all statistically convergent sequences of order α in a random 2-normed space $(X, \mathcal{F}, *)$.

DEFINITION 6.1.2 *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be $\Lambda-$ statistically convergent of order α $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1,$$

where λ_n^α *denote the* α*th power of* λ_n*, i.e.,* $(\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha, \cdots\cdots)$*. In this case, we write* $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$*.*

Let, $S_\Lambda^{R2N}(\alpha)$ denotes the set of all Λ- statistically convergent sequences of order α in a random 2-normed space $(X, \mathcal{F}, *)$.

For the particular choice $\alpha = 1$, DEFINITION 6.1.2 coincides with the notion of $\Lambda-$ statistical convergence of (3); For $\lambda_n = n$, DEFINITION 6.1.2 coincides with the notion of statistical convergence of order α in random 2-normed space; For $\lambda_n = n$ and $\alpha = 1$, DEFINITION 6.1.2 coincides with the notion of statistical convergence in random 2-normed space (163).

We next give EXAMPLE that shows DEFINITION 6.1.2 is well defined for $(0 < \alpha \leq 1)$ but not for $\alpha > 1$. In view of this we need the following theorem with LEMMA.

LEMMA 6.1.1 *Let* $\lambda = (\lambda_n)$ *be a non-decreasing sequence as defined above and* $(X, \mathcal{F}, *)$ *be a random 2-normed space. Let* $0 < \alpha \leq 1$ *and* $x = (x_k)$ *be a sequence in* X*. Then, for* $\epsilon > 0$*,* $t \in (0, 1)$ *and* $\theta \neq z \in X$*, the following statements are equivalent:*

(i) $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$,

(ii) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0$,

(iii) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1$,

(iv) $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) = 1$.

THEOREM 6.1.1 *Let* $(X, \mathcal{F}, *)$ *be a random 2-normed space and* $0 < \alpha \leq 1$ *be* *given. If $S_{\Lambda}^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$, then x_0 must be unique.*

Proof Suppose $S_{\Lambda}^{R2N}(\alpha) - \lim_{k\to\infty} x_k = y_0$ where $y_0 \neq x_0$. Given $\epsilon > 0$ and $t > 0$, choose $\eta > 0$ such that $(1-\eta) * (1-\eta) > 1-\epsilon$. For $\theta \neq z \in X$, define

$$K_1(\eta, t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1-\eta\right\};$$
$$K_2(\eta, t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right) \leq 1-\eta\right\}.$$

Since $S_{\Lambda}^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$ and $S_{\Lambda}^{R2N}(\alpha) - \lim_{k\to\infty} x_k = y_0$, it follows for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K_1(\eta, t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K_2(\eta, t)| = 0$$

Let $K(\eta, t) = K_1(\eta, t) \cup K_2(\eta, t)$, then clearly $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K(\eta, t)| = 0$ which immediately implies $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K^c(\eta, t)| = 1$. Let $k \in K^c(\eta, t) = K_1^c(\eta, t) \cap K_2^c(\eta, t)$. Now one can write,

$$\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) \geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right) > (1-\eta) * (1-\eta) > 1-\epsilon.$$

Since ϵ is arbitrary, it follows that $\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) = 1$, for $t > 0$ and $\theta \neq z \in X$. This shows that $x_0 = y_0$. ■

EXAMPLE 6.1.1 Let $X = \mathbb{R}^2$ with the 2− norm $||x, z|| = ||x_1 z_2 - x_2 z_1||$ where $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t + ||x,z||}$ where $x \in X, t \in (0, 1)$ and $\theta \neq z \in X$. Then $(\mathbb{R}^2, \mathcal{F}, *)$ is a random 2-normed space. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1, 0), & \text{if } k \text{ is even,} \\ (0, 0), & \text{if } k \text{ is odd.} \end{cases}$$

For $\epsilon > 0$, $t \in (0,1)$, if we define

$$\begin{aligned} K(\epsilon,t) &= \{k \in I_n : \mathcal{F}(\Lambda x_k - \theta, z; t) \leq 1-\epsilon\}, \theta = (0,0) \\ &= \left\{k \in I_n : \frac{t}{t + ||\Lambda x_k - \theta, z||} \leq 1-\epsilon\right\} \\ &= \left\{k \in I_n : ||\Lambda x_k - \theta, z|| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\} \\ &= \{k \in I_n : \Lambda x_k = (1,0)\} \\ &= \{k \in I_n : k \text{ is even}\}; \end{aligned}$$

then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|\{k \in I_n : k \text{ is even}\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.

Similarly, for $\epsilon > 0$ and $t \in (0,1)$ if we define

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}, x_0 = (1,0)$$

then

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|B(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|\{k \in I_n : \text{k is odd }\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.

This shows that $S_\Lambda^{R2N}(\alpha) - \lim_k x_k$ is not unique and we obtain a contradiction to THEOREM 6.1.1. ■

The next THEOREM reveals the relation between ordinary convergence and $\Lambda-$ statistical convergence of order α in a random 2-normed space.

THEOREM 6.1.2 *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. For a sequence $x = (x_k)$ in X if $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, then $S_\Lambda^{R2N}(\alpha) - \lim_k x_k = x_0$. However, the converse need not be true in general.*

Proof Since $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, so for $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$ there exists a positive integer n_0 such that $\mathcal{F}(\Lambda x_k - x_0, z; t) > 1 - \epsilon$ for all $k \geq n_0$. Hence the set, $K(\epsilon, t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1 - \epsilon\} \subset \{1, 2, 3, \ldots, n_0 - 1\}$, for which we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left| \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1 - \epsilon\} \right| = 0.$$

This shows that $S_\Lambda^{R2N}(\alpha) - \lim_k x_k = x_0$.

We next give the following EXAMPLE which shows that the converse need not be true.

EXAMPLE 6.1.2 Consider the random 2-normed space as in EXAMPLE 6.1.1. Define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (k, 0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0, 0), & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$ and $t > 0$ if we define $K(\epsilon, t) = \{k \in I_n : \mathcal{F}(\Lambda x_k, z; t) \leq 1 - \epsilon\}$, then one can write as in EXAMPLE 6.1.1, $K(\epsilon, t) = \left\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\right\}$.

Thus,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon, t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left| \left\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\right\} \right|$$

$$\leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n^\alpha} = 0$$

for $\frac{1}{2} < \alpha \leq 1$. This shows that $S_\Lambda^{R2N}(\alpha) - \lim_k x_k = 0$. But $\mathcal{F}_\Lambda - \lim_k x_k \neq 0$, since

$$\mathcal{F}(\Lambda x_k, z; t) = \frac{t}{t + ||\Lambda x_k, z||} = \begin{cases} \frac{t}{t + kz_2}, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ 1, & \text{otherwise.} \end{cases}$$

which implies

$$\lim_{k\to\infty} \mathcal{F}\left(\Lambda x_k, z; t\right) = \begin{cases} 0, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ 1, & \text{otherwise.} \end{cases} \qquad \blacksquare$$

THEOREM 6.1.3 *Let* $(X, \mathcal{F}, *)$ *be a random 2-normed space and* $0 < \alpha \leq 1$ *be given. Let* $x = (x_k)$ *and* $y = (y_k)$ *be two sequences in* X*.*

(i) If $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$ *and* $0 \neq c \in \mathbb{R}$*, then* $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} cx_k = cx_0$*.*

(ii) If $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} x_k = x_0$ *and If* $S_\Lambda^{R2N}(\alpha) - \lim_{k\to\infty} y_k = y_0$*, then* $S_\Lambda^{R2N}(\alpha) - \lim(x_k + y_k) = x_0 + y_0$*.*

Proof The proof of the Theorem is not so hard so is omitted here. $\blacksquare$

THEOREM 6.1.4 *Let* $(X, \mathcal{F}, *)$ *be a random 2-normed space and* $0 < \alpha \leq \beta \leq 1$ *be given. Then* $S_\Lambda^{R2N}(\alpha) \subset S_\Lambda^{R2N}(\beta)$ *and the inclusion is strict for some* α *and* β *such that* $\alpha < \beta$*.*

Proof If $0 < \alpha \leq \beta \leq 1$, then for every $\epsilon > 0$, $t > 0$ and $\theta \neq z \in X$, we have

$$\frac{1}{\lambda_n^\beta} \left|\{k \in I_n : \mathcal{F}\left(\Lambda x_k - l, z; t\right) \leq 1 - \epsilon\}\right| \leq \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : \mathcal{F}\left(\Lambda x_k - l, z; t\right) \leq 1 - \epsilon\}\right|;$$

which immediately implies the inclusion $S_\Lambda^{R2N}(\alpha) \subset S_\Lambda^{R2N}(\beta)$.

We next give an EXAMPLE that shows the inclusion in $S_\Lambda^{R2N}(\alpha) \subset S_\Lambda^{R2N}(\beta)$ is strict for some α and β with $\alpha < \beta$.

EXAMPLE 6.1.3 Let $(\mathbb{R}^2, \mathcal{F}, *)$ be a random 2-normed space as defined above. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1,0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0,0), & \text{otherwise.} \end{cases}$$

Then one can easily see $S_\Lambda^{R2N}(\beta)-\lim_k x_k=0$, *i.e.*, $x\in S_\Lambda^{R2N}(\beta)$ for $\frac{1}{2}<\beta\le 1$ but $x\notin S_\Lambda^{R2N}(\alpha)$ for $0<\alpha\le\frac{1}{2}$. This shows that the inclusion in $S_\Lambda^{R2N}(\alpha)\subset S_\Lambda^{R2N}(\beta)$ is strict.

THEOREM 6.1.5 *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $0<\alpha\le 1$ be given. If $x=(x_k)$ be a sequence in X, then $S_\Lambda^{R2N}(\alpha)-\lim_k x_k=x_0$ if and only if there exists a subset $K=\{k_m:k_1<k_2<\ldots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|K|=1$ and $\mathcal{F}_\Lambda-\lim_k x_k=x_0$.*

Proof First suppose that $S_\Lambda^{R2N}(\alpha)-\lim_k x_k=x_0$. For $t>0$, $\theta\neq z\in X$ and $p\in\mathbb{N}$, if we define

$$K(p,t)=\left\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)\le 1-\frac{1}{p}\right\}$$
$$M(p,t)=\left\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)>1-\frac{1}{p}\right\};$$

then,

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|K(p,t)|=0.$$

Also, for $p=1,2,3,\ldots$

$$M(1,t)\supset M(2,t)\supset\ldots M(i,t)\supset M(i+1,t)\supset\ldots \tag{6.1.1}$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|M(p,t)|=1. \tag{6.1.2}$$

Now to prove the result in one way it is sufficient to prove that $\mathcal{F}_\Lambda-\lim_k x_k=x_0$ over $M(p,t)$. Suppose x_k is not convergent to x_0 over $M(p,t)$ with respect to the norm $\mathcal{F}_\Lambda$. Then, there exists some $\eta>0$ such that

$$\{k\in\mathbb{N}:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)\le 1-\eta\}$$

for infinitely many terms x_k. Let,

$$M(\eta,t)=\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)>1-\eta\}.$$

and $\eta>\frac{1}{p}$ for $p=1,2,3\cdots$. This implies that $\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|M(\eta,t)|=0$. Also from (6.1.1), we have $K_1(p,t)\subset M(\eta,t)$ which gives that $\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|M(p,t)|=0$, this contradicts (6.1.2). Hence $\mathcal{F}_\Lambda-\lim_k x_k=x_0$. Conversely, suppose that there exists a subset $K=\{k_m:k_1<k_2<\ldots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|K|=1$ and $\mathcal{F}_\Lambda-\lim_k x_k=x_0$. Then for every $t>0$, $\epsilon>0$ and $\theta\neq z\in X$ there exists a positive integer k_0 such that,

$$\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)>1-\epsilon\}$$

for all $k\geq k_0$. Since the set $\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)\leq 1-\epsilon\}$ is contained in $\mathbb{N}-\{k_0+1,k_0+2,k_0+3,\cdots\}$ therefore,

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\mathcal{F}\left(\Lambda x_k-x_0,z;t\right)\leq 1-\epsilon\}\right|=0.$$

Hence, $S_\Lambda^{R2N}(\alpha)-\lim_k x_k=x_0$. ■

DEFINITION 6.1.3 *Let $(X,\mathcal{F},*)$ be a random 2-normed space. A sequence $x=(x_k)$ is said to be $\Lambda-$ statistically Cauchy of order α $(0<\alpha\leq 1)$ if for every $\epsilon>0$, $t\in(0,1)$ and $\theta\neq z\in X$ there exists a positive integer k_0 such that for all $k,l\geq k_0$*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|k\in I_n:\mathcal{F}(\Lambda x_k-\Lambda x_l,z;\epsilon)\leq 1-t\}\right|=0$$

or equivalently

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|k\in I_n:\mathcal{F}(\Lambda x_k-\Lambda x_l,z;\epsilon)>1-t\}\right|=1.$$

THEOREM 6.1.6 *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $0<\alpha\leq 1$ be* *given. Then a sequence $x = (x_k)$ is said to be $\Lambda-$ statistically convergent of order α iff it is $\Lambda-$ statistically Cauchy of order α.*

Proof Let $x = (x_k)$ be a $\Lambda-$ statistically convergent sequence of order α. Suppose that $S_\Lambda^{R2N}(\alpha) - \lim_k x_k = x_0$. Let $\epsilon > 0$. Choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$. If we define

$$K(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1 - r\right\}, \text{then}$$
$$K^c(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1 - r\right\};$$

which gives by virtue of $S_\Lambda^\alpha - \lim_k x_k = x_0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K(r,t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K^c(r,t)| = 1.$$

Let $m \in K^c(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1 - r$. If we take,

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) \leq 1 - \epsilon\},$$

then to prove the first part it is sufficient to prove that $B(\epsilon,t) \subset K(r,t)$. Let $k \in B(\epsilon,t)$, which gives $\mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) \leq 1 - \epsilon$. Suppose $k \notin K(r,t)$, then $\mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1 - r$. Now, we can observe that,

$$\begin{aligned} 1 - \epsilon \geq \mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) \geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ \geq (1-r)*(1-r) \\ > 1 - \epsilon. \end{aligned}$$

This contradiction clearly shows that $B(\epsilon,t) \subset K(r,t)$ and therefore one way of the THEOREM is proved.

Conversely, Suppose that $x = (x_k)$ is $\Lambda-$ statistically Cauchy sequence of order α but not $\Lambda-$ statistically convergent of order α with respect to $\mathcal{F}$. Then for every $t > 0$,

$\epsilon > 0$ and $\theta \neq z \in X$ there exists a positive integer m such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K(\epsilon,t)| = 0 \text{ where } K(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) \leq 1-\epsilon\}.$$

This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K^c(\epsilon,t)| = 1$. Choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$. Let

$$B(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r\right\}.$$

Let $m \in B(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1-r$.

Since

$$\begin{aligned}\mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) \geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ > (1-r)*(1-r) \\ > 1-\epsilon;\end{aligned}$$

therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_m, z; t) > 1-\epsilon\}| = 0.$$

i.e. $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K^c(\epsilon,t)| = 0$ which leads to a contradiction. Hence $x = (x_k)$ is $\Lambda-$ statistically convergent of order α. ■

6.2 Conclusion

We have defined generalized statistical convergence of order α in Random $2-$normed space which is more general than $\Lambda-$ statistical convergence of order α in Random $2-$normed space. The particular choices of α give different convergence methods that shows our method of convergence is more general. For the particular choice $\alpha = 1$, this generalized concept coincides with the notion of $\Lambda-$ statistical convergence of (3); For $\lambda_n = n$, it coincides with the notion of statistical convergence of order α in random 2-normed space; For $\lambda_n = n$ and $\alpha = 1$, it coincides with the notion of statistical convergence in random 2-normed space.■