Chapter 5

Generalized Limit and Cluster Point in $PN-$Spaces

For any lacunary sequence $\theta = (k_r)$, our aim, in this chapter, is to introduce the notions of $\theta-$statistical limit and $\theta-$statistical cluster points in a probabilistic normed space (briefly known as $PN-$ space) $(X, F, *)$ which arose naturally after the appearance of the interesting work (149) due to Rehmet. Subsequently, we continue with the notions of $S_\theta-$ limit superior and $S_\theta-$ limit inferior of sequences in $PN-$ spaces.

5.1 $S_\theta-$ Limit and Cluster Point in $PN-$ Space

Let $\theta = (k_r)$ be a lacunary sequence. For a $PN-$ space $(X, F, *)$, let $x = (x_k)$ be a sequence in X. Let $(x_{k(j)})$ be a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we denote $(x_{k(j)})$ by $(x)_K$. If

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0;$$

Results presented in this chapter are published in General Mathematical Notes, India & Atti della Fondazione Giorgio Ronchi, Italy.

then $(x)_K$ is called $\theta-$ thin subsequence. On the other hand, $(x)_K$ is a $\theta-$nonthin subsequence of x provided that

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k\,(j) \in I_r : j \in \mathbb{N}\}\right| > 0;$$

DEFINITION 5.1.1 *Let $\theta = (k_r)$ be a lacunary sequence and $(X, F, *)$ be a $PN-$ space. An element $\mu \in X$ is called a lacunary statistical limit point (briefly $S_\theta-$ limit point) of a sequence $x = (x_k)$ in X provided that there is a $\theta-$ nonthin subsequence of x that is convergent to μ with respect the probabilistic norm F.*

Let $\Lambda^F(S_\theta, x)$ denotes the set of all S_θ-limit points of the sequence $x = (x_k)$.

DEFINITION 5.1.2 *Let $\theta = (k_r)$ be a lacunary sequence and $(X, F, *)$ be a $PN-$ space. A point $\gamma \in X$ is said to be a lacunary statistical cluster point (briefly $S_\theta-$ cluster point) of a sequence $x = (x_k)$ in X provided that for every $\epsilon > 0$, $t \in (0, 1)$*

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : F_{x_k-\gamma}(\epsilon) > 1 - t\}\right| > 0.$$

Let $\Gamma^F(S_\theta, x)$ denotes the set of all S_θ- cluster points of the sequence $x = (x_k)$.

THEOREM 5.1.1 *Let $\theta = (k_r)$ be a lacunary sequence and $((X, F, *)$ be a $PN-$ space. For any sequence $x = (x_k)$ in X, $\Lambda^F(S_\theta, x) \subseteq \Gamma^F(S_\theta, x)$.*

Proof For $\mu \in \Lambda^F((S_\theta, x)$, there is a $\theta-$ nonthin subsequence $(x_{k(j)})$ of x that converges to μ with respect the probabilistic norm F. Since $(x_{k(j)})$ is a $\theta-$ nonthin subsequence so we have

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : F_{x_{k(j)}-\mu}(\epsilon) > 1 - t\right\}\right| > 0. \tag{5.1.1}$$

Now for every $t > 0$, the containment

$$\{k \in I_r : F_{x_k-\mu}(\epsilon) > 1 - t\} \supseteq \{k(j) \in I_r : F_{x_{k(j)}-\mu}(\epsilon) > 1 - t\}$$

gives

$$\left\{k \in I_r : F_{x_{k(j)}-\mu}(\epsilon) > 1-t\right\} \supseteq \{k(j) \in I_r : j \in \mathbb{N}\} - \left\{k(j) \in I_r : F_{x_{k(j)}-\mu}(\epsilon) \leq 1-t\right\};$$

which immediately implies

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : F_{x_{k(j)}-\mu}(\epsilon) > 1-t\right\}\right| \geq \limsup_{r\to\infty} \frac{1}{h_r} \left|\{k(j) \in I_r : j \in \mathbb{N}\}\right|$$
$$-\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k(j) \in I_r : F_{x_{k(j)}-\mu}(\epsilon) \leq 1-t\right\}\right|. \quad (5.1.2)$$

Further, the convergence of $p_{k(j)})$ to μ with respect to the norm F gives for $t > 0$, the set $\left\{k(j) \in I_r : F_{x_{k(j)}-\mu}(\epsilon) \leq 1-t\right\}$ is finite for which we have

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k(j) \in I_r : F_{x_{k(j)}-\mu}(\epsilon) \leq 1-t\right\}\right| = 0\ . \quad (5.1.3)$$

Using (5.1.1) and (5.1.3) in (5.1.2), we get

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : F_{x_{k(j)}-\mu}(\epsilon) > 1-t\right\}\right| \geq d > 0.$$

This shows that $\mu \in \Gamma^F(S_\theta, x)$ and therefore we have the containment $\Lambda^F(S_\theta, x) \subseteq \Gamma^F(S_\theta, x)$. ■

THEOREM 5.1.2 *Let $\theta = (k_r)$ be a lacunary sequence and $(X, F, *)$ be a PN-space. For any sequence $x = (x_k)$ in X, $\Gamma^F(S_\theta, x) \subseteq L^F(x)$, where $L^F(x)$ denotes the set of all limit points of $x = (x_k)$ in $(X, F, *)$.*

Proof Assume that $\gamma \in \Gamma^F(S_\theta, x)$, then for every $\epsilon > 0$, $t \in (0,1)$ we have

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : F_{x_{k(j)}-\gamma}(\epsilon) > 1-t\right\}\right| > 0. \quad (5.1.4)$$

For $\epsilon > 0$, if we denote $K = \left\{k \in I_r : F_{x_{k(j)}-\gamma}(\epsilon) > 1-t\right\}$, then the set $K = \{k_1 < k_2 < \cdots\}$ is an infinite set as otherwise *i.e.* if K is finite set then left side of (5.1.4) becomes zero and we obtain a contradiction. This shows that we have a subsequence $(x)_K$ of the sequence $x = (x_k)$ that is convergent to γ with respect to probabilistic norm F. Hence γ is a limit point of (x_k) and therefore we have the containment $\Gamma^F(S_\theta, x) \subseteq L^F(x)$. ■

THEOREM 5.1.3 *For any lacunary sequence $\theta = (k_r)$ and any sequence $x = (x_k)$ in a $PN-$ space $(X, F, *)$, $\Gamma^F(S_\theta, x)$ is a closed set.*

Proof To prove the THEOREM, it is sufficient to prove that $cl\left(\Gamma^F(S_\theta, x)\right) \subseteq \Gamma^F(S_\theta, x)$ where $cl(A)$ denotes the strong closure of any set A. Let $\mu \in cl\left(\Gamma^F(S_\theta, x)\right)$, then for any $\epsilon > 0$ and $t \in (0,1)$, $\Gamma^F(S_\theta, x)$ contains some point $\gamma \in K(\mu, \epsilon, t)$, where

$$K(\mu, \epsilon, t) = \{x \in X : F_{\mu - x}(\epsilon) > 1 - t\}$$

Choose t' such that $K(\gamma, t', \epsilon) \subset K(\mu, \epsilon, t)$. Since $\gamma \in \Gamma^F(S_\theta, x)$, therefore

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : F_{\gamma - x}(\epsilon) > 1 - t'\}\right| > 0;$$

which immediately gives

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : F_{\mu - x}(\epsilon) > 1 - t\}\right| > 0.$$

This shows that $\mu \in \Gamma^F(S_\theta, x)$ and therefore we have $cl\left(\Gamma^F(S_\theta, x)\right) \subseteq \Gamma^F(S_\theta, x)$. ■

THEOREM 5.1.4 *Let $\theta = (k_r)$ be a lacunary sequence. For any sequence $x = (x_k)$ in a $PN-$ space $(X, F, *)$, if $S_\theta^F - \lim_k x_k = x_0$, then $\Lambda^F(S_\theta, x) = \Gamma^F(S_\theta, x) = \{x_0\}$.*

Proof We first show that $\Lambda^F(S_\theta, x) = \{x_0\}$. Assume that $\Lambda^F(S_\theta, x) = \{x_0, y_0\}$ such that $x_0 \neq y_0$. By definition there exist two $\theta-$ nonthin subsequences $\left(x_{k(i)}\right)$ and $\left(x_{l(j)}\right)$ of the sequence $x = (x_k)$ which are respectively convergent to x_0 and y_0 with respect to the probabilistic norm F. Since $\left(x_{l(j)}\right)$ converges to y_0 with respect to the probabilistic norm F, therefore for any $\epsilon > 0$ and $t \in (0,1)$ there is a positive integer m such that $F_{x_{l(j)} - y_0}(\epsilon) > 1 - t$ whenever $l(j) \geq m$. This shows that for any $t \in (0,1)$

and $\epsilon > 0$, we have

$$\lim_r \frac{1}{h_r} \left| \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) \leq 1-t \right\} \right| = 0 \ . \tag{5.1.5}$$

Moreover,

$$\{l\left(j\right) \in I_r : j \in \mathbb{N}\} = \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \cup \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) \leq 1-t \right\};$$

which implies

$$\limsup_r \frac{1}{h_r} \left|\{l\left(j\right) \in I_r : j \in \mathbb{N}\}\right| = \limsup_r \frac{1}{h_r} \left| \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \right|$$

$$+\limsup_r \frac{1}{h_r} \left| \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) \leq 1-t \right\} \right|. \tag{5.1.6}$$

Since $(l(j))$ is $\theta-$ nonthin subsequence so we have together with (5.1.5),

$$\limsup_r \frac{1}{h_r} \left| \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \right| > 0. \tag{5.1.7}$$

Also using the fact $S_\theta - \lim_k x_k = x_0$, we have

$$\lim_r \frac{1}{h_r} \left|\{k \in I_r : F_{x_k-x_0}\left(\epsilon\right) \leq 1-t\}\right| = 0, \tag{5.1.8}$$

which gives for any $\epsilon > 0$

$$\limsup_r \frac{1}{h_r} \left|\{k \in I_r : F_{x_k-x_0}\left(\epsilon\right) > 1-t\}\right| > 0. \tag{5.1.9}$$

Also for $x_0 \neq y_0$,

$$\left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \cap \{k \in I_r : F_{x_k-x_0}\left(\epsilon\right) > 1-t\} = \emptyset.$$

So we have,

$$\left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \subseteq \{k \in I_r : F_{x_k-x_0}\left(\epsilon\right) \leq 1-t\},$$

which immediately with use of (5.1.8)

$$\limsup_r \frac{1}{h_r} \left| \left\{ l\left(j\right) \in I_r : F_{x_{l(j)}-y_0}\left(\epsilon\right) > 1-t \right\} \right|$$

$$\leq \limsup_r \frac{1}{h_r} \left|\{k \in I_r : F_{x_k-x_0}\left(\epsilon\right) \leq 1-t\}\right| = 0;$$

which contradict (5.1.7). Hence $\Lambda^F(S_\theta, x) = \{x_0\}$.

Now, suppose that $\Gamma^F(S_\theta, x) = \{x_0, z_0\}$ such that $x_0 \neq z_0$. Then

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : F_{x_k - z_0}(\epsilon) > 1 - t\}| > 0$$

For $x_0 \neq z_0$,

$$\{k \in I_r : F_{x_k - x_0}(\epsilon) > 1 - t\} \cap \{k \in I_r : F_{x_k - z_0}(\epsilon) > 1 - t\} = \emptyset.$$

So we have

$$\{k \in I_r : F_{x_k - x_0}(\epsilon) > 1 - t\} \subseteq \{k \in I_r : F_{x_k - z_0}(\epsilon) \leq 1 - t\},$$

Therefore,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) > 1 - t\}| \leq \limsup_r \frac{1}{h_r} |\{k \in I_r : F_{x_k - z_0}(\epsilon) \leq 1 - t\}| = 0.$$

This gives,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) > 1 - t\}| = 0,$$

which leads to a contradiction. Hence, $\Gamma^F(S_\theta, x) = \{x_0\}$. ■

THEOREM 5.1.5 *Let $\theta = (k_r)$ be a lacunary sequence. If $x = (x_k)$ and $y = (y_k)$ are two sequences in $(X, F, *)$ such that $\lim_r \frac{1}{h_r} |\{k \in I_r : x_k \neq y_k\}| = 0$, then $\Lambda^F(S_\theta, x) = \Lambda^F(S_\theta, y)$ and $\Gamma^F(S_\theta, x) = \Gamma^F(S_\theta, y)$.*

Proof Assume $\gamma \in \Lambda^F(S_\theta, x)$, then there exists a $\theta-$ nonthin subsequence $(x)_K$ of the sequence $x = (x_k)$ that converges to γ.

Since, $\lim_r \frac{1}{h_r} |\{k \in I_r : k \in K, x_k \neq y_k\}| = 0$, it follows that

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : k \in K, x_k = y_k\}| > 0 \tag{5.1.10}$$

Therefore, there exists a $\theta-$ nonthin subsequence $(y)_K$ of the sequence $y = (y_k)$ that converges to γ. This shows that $\gamma \in \Lambda^F(S_\theta, y)$ and therefore $\Lambda^F(S_\theta, x) \subseteq \Lambda^F(S_\theta, y)$. By symmetry we have $\Lambda^F(S_\theta, y) \subseteq \Lambda^F(S_\theta, x)$. Hence, we have $\Lambda^F(S_\theta, x) = \Lambda^F(S_\theta, y)$. Similarly we can prove $\Gamma^F(S_\theta, x) = \Gamma^F(S_\theta, y)$. ■

5.2 $S_\theta-$ Limit Superior and Limit Inferior

In this section, we use lacunary sequence $\theta = (k_r)$ to define lacunary statistical limit superior and lacunary statistical limit inferior of sequences on $PN-$ spaces $(X, F, *)$.

DEFINITION 5.2.1 *The real number sequence $x = (x_k)$ is said to be lacunary statistically bounded with respect to probabilistic norm F if there exists some $t \in \mathbb{R}$ and for every $b \in (0,1)$ such that*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : F_{x_k}(t) \leq 1 - b\}| = 0 \ .$$

For a real number sequence x, let us define the sets

$$M_x^F = \{m \in (0,1) : \delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m\}) \neq 0\},$$

$$B_x^F = \{b \in (0,1) : \delta_\theta(\{k : F_{x_k}(\epsilon) > 1 - b\}) \neq 0\}.$$

DEFINITION 5.2.2 *If x is a real number sequence then the lacunary statistical limit superior of x with respect to probabilistic norm F is defined by*

$$S_\theta^F - \limsup x = \begin{cases} \sup M_x^F, & \text{if } M_x^F \neq \emptyset, \\ 0, & \text{if } M_x^F = \emptyset \end{cases}$$

Also the lacunary statistical limit inferior of x with respect to probabilistic norm F is defined by

$$S_\theta^F - \liminf x = \begin{cases} \inf B_x^F, & \text{if } B_x^F \neq \emptyset, \\ 1, & \text{if } B_x^F = \emptyset \end{cases}$$

THEOREM 5.2.1 *If* $m = S_\theta^F - \limsup x$ *is finite, then for every positive numbers* ϵ *and* γ

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m + \gamma\}) \neq 0 \text{ and } \delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m - \gamma\}) = 0 \qquad (5.2.1)$$

Conversely, if the condition (5.2.1) *holds for every positive* ϵ *and* γ*, then* $m = S_\theta^F - \limsup x$.

Proof Let $m = S_\theta^F - \limsup x$, where m is finite. Then

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m\}) \neq 0. \qquad (5.2.2)$$

Since $F_{x_k}(\epsilon) < 1 - m + \gamma$ for every k and for any $\epsilon, \gamma > 0$,

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m + \gamma\}) \neq 0.$$

Using the DEFINITION of $S_\theta^F - \limsup x$ we have $1 - m$ as the least value satisfying (5.2.2). Let if possible,

$$F_{x_k}(\epsilon) < 1 - m - \gamma \text{ for some } \gamma > 0.$$

Then $1 - m - \gamma$ is another value with $1 - m - \gamma < 1 - m$ which satisfies (5.2.2). This leads to a contradiction to the fact that $1 - m$ as the least value satisfying (5.2.2). Hence,

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m - \gamma\}) = 0 \text{ for every } \gamma > 0.$$

Conversely, if the condition (5.2.1) holds for every positive ϵ and γ, then

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m + \gamma\}) \neq 0 \text{ and } \delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m - \gamma\}) = 0$$

Therefore

$$\delta_\theta(\{k : F_{x_k}(\epsilon) \leq 1 - m\}) \neq 0 \text{ and } \delta_\theta(\{k : F_{x_k}(\epsilon) = 1 - m\}) = 0.$$

This gives

$$\delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m\}) \neq 0$$

for every $\epsilon > 0$. Hence, $m = S_\theta^F - \limsup x$.

The result for $S_\theta^F - \liminf x$, we can prove similarly. ■

THEOREM 5.2.2 *If $b = S_\theta^F - \liminf x$ is finite, then for every positive numbers ϵ and γ*

$$\delta_\theta(\{k : F_{x_k}(\epsilon) > 1 - b - \gamma\}) \neq 0 \text{ and } \delta_\theta(\{k : F_{x_k}(\epsilon) > 1 - b + \gamma\}) = 0 \qquad (5.2.3)$$

Conversely, if the condition (5.2.3) *holds for every positive ϵ and γ, then $b = S_\theta^F - \liminf x$.* ■

REMARK 5.2.1 *From the definition of lacunary statistical cluster point we see that THEOREM* 5.2.1 *and* 5.2.2 *can be stated as that $S_\theta^F - \limsup x$ and $S_\theta^F - \liminf x$ are the greatest and least lacunary statistical cluster points respectively.* ■

THEOREM 5.2.3 *For any sequence x, $S_\theta^F - \liminf x \leq S_\theta^F - \limsup x$.*

Proof Firstly consider the case in which $S_\theta^F - \limsup x = 0$, which implies that

$$M_x^F = \phi.$$

Then for every $m \in (0, 1)$,

$$M_x^F = \delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m\}) = 0,$$

i.e.

$$\delta_\theta(\{k : F_{x_k}(\epsilon) \geq 1 - m\}) = 1.$$

Also for every $b \in (0,1)$, we have

$$\delta_\theta(\{k : F_{x_k}(\epsilon) \geq 1 - b\}) \neq 0.$$

Hence, $S_\theta^F - \liminf x = 0$.

The case in which $S_\theta^F - \limsup x = 1$, is trivial.

Suppose that $m = S_\theta^F - \limsup x$, and $b = S_\theta^F - \liminf x$, where b and m are finite.

Now for any γ, we show that $1 - m - \gamma \in B_x^F$. By THEOREM 5.2.1, we have

$M_x^F = \delta_\theta(\{k : F_{x_k}(\epsilon) < 1 - m - \frac{\gamma}{2}\}) = 0$, where $1 - m =$ least upper bound of M_x^F.

Therefore

$$\delta_\theta(\{k : F_{x_k}(\epsilon) \geq 1 - m - \frac{\gamma}{2}\}) = 1,$$

this implies

$$\delta_\theta(\{k : F_{x_k}(\epsilon) \geq 1 - m - \gamma\}) = 1.$$

Hence, $1 - m - \gamma \in B_x^F$.

Since $b = \inf B_x^F$, therefore $1-m-\gamma \leq 1-b$. As γ is arbitrary, we have $1-m \leq 1-b$, that is $b \leq m$. ■

THEOREM 5.2.4 *For a PN-space $(X, F, *)$ the lacunary statistically bounded sequence x is lacunary statistically convergent if and only if $S_\theta^F - \liminf x = S_\theta^F - \limsup x$.*

Proof Let $\alpha = S_\theta^F - \liminf x$ and $\beta = S_\theta^F - \limsup x$. Let $S_\theta^F - \lim x = L$. Then for every $\epsilon > 0$ and $m \in (0,1)$,

$$\delta_\theta(\{k : F_{x_k - L}(\epsilon) \leq 1 - m\}) = 0,$$

so that

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) * F_L(\frac{\epsilon}{2}) \leq 1 - m\}) = 0.$$

Let for every $\epsilon > 0$,

$$\sup_{\epsilon} F_{x_k}(\frac{\epsilon}{2}) = 1 - m_1 \text{ and } \sup_{\epsilon} F_L(\frac{\epsilon}{2}) = 1 - m_2$$

such that

$$(1 - m_1) * (1 - m_2) \leq 1 - m \tag{5.2.4}$$

Then

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) \leq 1 - m_1\}) = 0, \tag{5.2.5}$$

and therefore

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) \leq 1 - m_1 - \gamma\}) = 0, \text{ for every } \gamma > 0. \tag{5.2.6}$$

Now applying THEOREM 5.2.1 and the DEFINITION of $S_\theta^F - \limsup x$, we get

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) \leq 1 - \beta - \gamma\}) = 0, \text{ for every } \gamma > 0. \tag{5.2.7}$$

From (5.2.6) and (5.2.7) and by the DEFINITION of $S_\theta^F - \limsup x$, we get $1 - m_1 - \gamma \leq 1 - \beta - \gamma$, this gives

$$\beta \leq m_1 \tag{5.2.8}$$

Now we search those k for which $F_{x_k}(\frac{\epsilon}{2}) > 1 - m_1 + \gamma$. One can easily see that no such k exists which satisfy (5.2.4) together with the above condition. Therefore this implies that

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) > 1 - m_1 + \gamma\}) = 0.$$

Since $\alpha = S_\theta^F - \liminf x$, by THEOREM 5.2.2, we have

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) > 1 - \alpha + \gamma\}) = 0.$$

Now by the DEFINITION of $S_\theta^F - \liminf x$, we have $1 - \alpha + \gamma \leq 1 - m_1 + \gamma$, this gives

$$m_1 \leq \alpha \tag{5.2.9}$$

From (5.2.7) and (5.2.8), we have $\beta \leq \alpha$. Using (5.2.6) and THEOREM 5.2.3, we get $\alpha = \beta$.

Conversely, suppose that $\alpha = \beta$ and $\sup_\epsilon F_L(\epsilon) = 1 - \alpha$. Then for any $\gamma > 0$, THEOREM 5.2.1 and 5.2.2 give

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) < 1 - \alpha - \frac{\gamma}{2}\}) = 0, \tag{5.2.10}$$

and

$$\delta_\theta(\{k : F_{x_k}(\frac{\epsilon}{2}) < 1 - \alpha + \frac{\gamma}{2}\}) = 0. \tag{5.2.11}$$

Now

$$1 - \alpha \geq F_L(\epsilon) = F_{x_k-(x_k-L)}(\epsilon) \geq F_{x_k}(\frac{\epsilon}{2}) * F_{x_k-L}(\frac{\epsilon}{2}).$$

Therefore

$$F_{x_k}(\frac{\epsilon}{2}) * F_{x_k-L}(\frac{\epsilon}{2}) \leq 1 - \alpha \tag{5.2.12}$$

Let $\sup_\epsilon F_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_1$ where $a_1 \in (0, 1)$ and (5.2.10) and (5.2.12) hold. Then

$$\delta_\theta(\{k : F_{x_k-L}(\frac{\epsilon}{2}) < 1 - a_1 - \frac{\gamma}{2}\}) = 0,$$

which is true for every $\gamma > 0$. Hence

$$\delta_\theta(\{k : F_{x_k-L}(\frac{\epsilon}{2}) < 1 - a_1\}) = 0,$$

which is true for all $a \leq a_1 \in (0, 1)$, because $1 - a_1$ is the least upper bound. Now repeat the process by taking (5.2.11) and (5.2.12) instead of (5.2.10) and (5.2.12). If (5.2.11) and (5.2.12) are satisfied, then $\inf_\epsilon F_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_1$. Let if possible $1 - a_1 \neq \inf_\epsilon F_{x_k-L}(\frac{\epsilon}{2})$ while the conditions (5.2.11) and (5.2.12) hold. This implies that there exist some t in $\{F_{x_k-L}(\frac{\epsilon}{2}) : \epsilon > 0 \text{ is arbitrary}\}$ such that $t > 1 - a_1$. Let us suppose $\inf_\epsilon F_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_2$. Then, we have

$$1 - a_2 > 1 - a_1 \tag{5.2.13}$$

and by (12), we get

$$F_{x_k}(\frac{\epsilon}{2}) * (1 - a_2) \leq 1 - \alpha.$$

Using (11) we get,

$$(1 - \alpha + \frac{\gamma}{2}) * (1 - a_2) \leq 1 - \alpha, \text{ for all } \gamma > 0.$$

Clearly,

$$(1 - \alpha - \frac{\gamma}{2}) * (1 - a_2) \leq 1 - \alpha \tag{5.2.14}$$

Now

$1 - a_1 = \sup_\epsilon F_{x_k - L}(\frac{\epsilon}{2})$ where $a_1 \in (0, 1)$ and which satisfy (5.2.10) and (5.2.12). Hence $1 - a_2 < 1 - a_1$, which contradicts (5.2.13). Hence $1 - a_1 = \inf_\epsilon F_{x_k - L}(\frac{\epsilon}{2})$ satisfying conditions (11) and (12). Therefore the result becomes true for all $a \geq a_1 \in (0, 1)$, because $1 - a_1$ is the greatest lower bound, and hence

$$\delta_\theta(\{k : F_{x_k - L}(\frac{\epsilon}{2}) \leq 1 - a\}) = 0,$$

for each $\epsilon > 0$ and $a \in (0, 1)$. This gives $S_\theta^F - \lim x = L$. ■

The Definitions and Results presented in this chapter remain valid in a more general context, that of general probabilistic normed spaces $(V, \vartheta, \tau, \tau^*)$ defined by Alsina *et al*. (143).

5.3 Conclusion

The concept of statistical limit points, statistical cluster points have been studied in different spaces by many authors. In this chapter, we introduced the concepts of lacunary statistical limit points, lacunary statistical cluster points, lacunary statistical limit inferior and lacunary statistical limit superior in probabilistic normed spaces.

Since every ordinary norm defines a probabilistic norm, these ideas are more general than the concepts of lacunary statistical limit points, lacunary statistical cluster points, lacunary statistical limit inferior and lacunary statistical limit superior. ■