Chapter 4

Generalized Summability of Order $\tilde{\alpha}$

In the present chapter, we introduced the notions of $(\lambda, \mu)-$statistical summability and $(V, \lambda, \mu)-$summability of order $\tilde{\alpha}$ for double sequences and obtained some relations between these summability methods. Moreover, these summability methods are generalized with the help of ideals to define $\mathcal{I} - (\lambda, \mu)-$ statistical convergence, $\mathcal{I} - (V, \lambda, \mu)-$ summability of order $\tilde{\alpha}$, where $\mathcal{I} \subseteq \mathbb{N} \times \mathbb{N}$ be an admissible ideal. Throughout, we have used $\tilde{\alpha}$ as an alternative of (a, b) and $\tilde{\beta}$ as an alternative of (c, d), where $a, b, c, d \in (0, 1]$ as otherwise indicated. Also, we define: $\tilde{\alpha} \preceq \tilde{\beta} \iff a \leq c \,\text{and}\, b \leq d; \tilde{\alpha} \prec \tilde{\beta} \iff a < c \,\text{and}\, b < d; \tilde{\alpha} \cong \tilde{\beta} \iff a = c \,\text{and}\, b = d; \tilde{\alpha} \in (0, 1] \iff a, b \in (0, 1]; \tilde{\beta} \in (0, 1] \iff c, d \in (0, 1]; \tilde{\alpha} \cong 1 \,\text{in case}\, a = b = 1; \tilde{\beta} \cong 1 \,\text{in case}\, c = d = 1$ and $\tilde{\alpha} \succ 1 \,\text{in case}\, a > 1, b > 1$.

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4.1 $S^{\tilde{\alpha}}_{(\lambda,\mu)}-$ Summability

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$; $\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$ and $\tilde{\alpha} \in (0, 1]$ be given.

DEFINITION 4.1.1 *A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)-$ statistically convergent of order $\tilde{\alpha}$ if there exists a number L such that for every $\epsilon > 0$*

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0,$$

where, $\lambda^a = (\lambda_n^a) = (\lambda_1^a, \lambda_2^a, \lambda_3^a, \cdots\cdots)$; $\mu^b = (\mu_m^b) = (\mu_1^b, \mu_2^b, \mu_3^b, \cdots\cdots)$ and $\lambda_n^a\mu_m^b$ denotes the usual multiplication of the corresponding entries of the sequences λ^a and μ^b. In this case, the number L is called $(\lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$ and we write $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = L$.

Let $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$ denotes the set of all $(\lambda, \mu)-$statistically convergent double sequences of order $\tilde{\alpha}$.

For $\tilde{\alpha} = (a, b) = (1, 1)$, *DEFINITION 4.1.1* coincides with $(\lambda, \mu)-$statistical convergence of double sequences of (100).
For the choice $\lambda = (n)$ and $\mu = (m)$, *DEFINITION 4.1.1* coincides with statistical convergence of double sequences of order $\tilde{\alpha}$ of (101). Moreover, if we take $\lambda = (n)$; $\mu = (m)$ and $\tilde{\alpha} = (a, b) = (1, 1)$, *DEFINITION 4.1.1* coincides with statistical convergence of double sequences of (96).

THEOREM 4.1.1 For $\tilde{\alpha} \in (0, 1]$, if $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = x_0$, then x_0 is unique.

We next provide an EXAMPLE to show that the *DEFINITION 4.1.1* is well defined for $\tilde{\alpha} \in (0, 1]$ but not for $\tilde{\alpha} \succ 1$ in general.

EXAMPLE 4.1.1 Let $x = (x_{ij})$ be defined as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i+j \text{ even} \\ 0 & \text{if } i+j \text{ odd} \end{cases}$$

Then, for $\tilde{\alpha} \succ 1$,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 1| \geq \epsilon\}\right| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0$$

and

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}\right| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0.$$

This shows that $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 0$ and $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 1$ which leads to a contradiction to *THEOREM 4.1.1.* ■

THEOREM 4.1.2 *Let $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences of complex numbers and $\tilde{\alpha} \in (0,1]$.*

(i) If $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim x_{ij} = L$ and $c \in \mathbb{C}$, then $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim(cx_{ij}) = cL$.

(ii) If $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim x_{ij} = L$ and $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim y_{ij} = M$, then $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim(x_{ij} + y_{ij}) = L + M$.

Proof (i) It is clear for the case $c = 0$. For $c \neq 0$ and $\tilde{\alpha} \in (0,1]$ we have,

$$\frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |cx_{ij} - cL| \geq \epsilon\}\right| = \frac{1}{\lambda_n^a \mu_m^b} \left|\left\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \frac{\epsilon}{|c|}\right\}\right|.$$

Since $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim x_{ij} = L$, therefore, the above expression tends to zero as $n \to \infty$. Hence $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim(cx_{ij}) = cL$.

(ii) For $\tilde{\alpha} \in (0,1]$,

$$\frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |(x_{ij} + y_{ij}) - (L+M)| \geq \epsilon\}\right|$$
$$\leq \frac{1}{\lambda_n^a \mu_m^b} \left|\left\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \frac{\epsilon}{2}\right\}\right| + \frac{1}{\lambda_n^a \mu_m^b} \left|\left\{(i,j) \in I_n \times I_m : |y_{ij} - M| \geq \frac{\epsilon}{2}\right\}\right|.$$

Since $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim x_{ij} = L$ and $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim y_{ij} = M$, therefore, the right expression of the above inequality tends to zero as $n \to \infty$. Hence, $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim(x_{ij} + y_{ij}) = L + M$.
■

DEFINITION 4.1.2 *Let $\tilde{\alpha}$ be any real number such that $\tilde{\alpha} \in (0,1]$ and p be a positive real number. A double sequence $x = (x_{ij})$ is said to be strongly $(V, \lambda, \mu)-$ summable of order $\tilde{\alpha}$ to a number L provided that*

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$. In this case, the number L is called strong $(V, \lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$.

Let $[w_p^2]_{\tilde{\alpha}}(x)$ denote the set of all strongly $(V, \lambda, \mu)-$ summable double sequences of order $\tilde{\alpha}$.

For $\tilde{\alpha} = (a, b) = (1, 1)$, *DEFINITION 4.1.2* coincides with strong $(V, \lambda, \mu)-$ summability of double sequences of (100). For $\lambda = (n)$ and $\mu = (m)$, *DEFINITION 4.1.2* coincides with strong p-Cesàro summability of double sequences of order $\tilde{\alpha}$ of (101). However, if we take $\lambda = (n)$; $\mu = (m)$ and $\tilde{\alpha} = (a, b) = (1, 1)$, *DEFINITION 4.1.2* coincides with strong p-Cesàro summability of double sequences of (96).

THEOREM 4.1.3 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. Then, $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S^{\tilde{\beta}}_{(\lambda,\mu)}(x)$ and the inclusion is strict for some $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$.*

Proof Let $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$. Since, $\tilde{\alpha} \preceq \tilde{\beta}$ so $a \leq c$ and $b \leq d$; which, for any $\epsilon > 0$ gives the inequality

$$\frac{1}{\lambda_n^c \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| \leq \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}|;$$

and therefore, the result follows immediately from the fact that $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$. For rest part of the THEOREM, we consider the following EXAMPLE.

Define $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} ij, & \text{if } n - [\sqrt{\lambda_n}] + 1 \le i \le n \quad \text{and} \quad m - [\sqrt{\mu_m}] + 1 \le j \le m] \\ 0, & \text{otherwise} \end{cases} ; \text{then,}$$

$$\frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \ge \epsilon\}\right| = \frac{1}{\lambda_n^c \mu_m^d} \left|\left\{(i,j) \in I_n \times I_m : {}^{n-[\sqrt{\lambda_n}]+1 \le i \le n \text{ and}}_{m-[\sqrt{\mu_m}]+1 \le j \le m}\right\}\right|$$
$$\le \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n^c \mu_m^d}.$$

It follows, for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (*i.e.* for $\frac{1}{2} < c \le 1$ and $\frac{1}{2} < d \le 1$), we have,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \ge \epsilon\}\right| \le \lim_{n,m\to\infty} \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n^c \mu_m^d} = 0.$$

This shows that $x = (x_{ij}) \in S^{\tilde{\beta}}_{(\lambda,\mu)}(x)$, but one can easily verify that $x \notin S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (*i.e.* for $0 < a \le \frac{1}{2}$ and $0 < b \le \frac{1}{2}$). ■

COROLLARY 4.1.1 *Let* $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$,

(i) If $\tilde{\beta} \cong 1$, *then* $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S^{1}_{(\lambda,\mu)} = S_(\lambda, \mu)$ *and the inclusion is strict.*

(ii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) = S^{\tilde{\beta}}_{(\lambda,\mu)}(x) \iff \tilde{\alpha} \cong \tilde{\beta}$.

(iii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) = S_{(\lambda,\mu)}(x) \iff \tilde{\alpha} \cong 1$. ■

THEOREM 4.1.4 *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ *be two sequences as defined above and* $\tilde{\alpha} \in (0, 1]$, *then*

(i) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S_2(x)$ *for all* λ, μ *and* $\tilde{\alpha} \in (0, 1]$.

(ii) $S_2(x) \subseteq S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$, *if and only if,* $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$ *and* $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$.

Proof (i) By the nature of the sequences (λ_n), (μ_m) and from the expression $\frac{\lambda_n \mu_m}{nm} \leq 1$, the result follows.

(ii) Let $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$; $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$ and $x = (x_{ij}) \in S_2(x)$. For given $\epsilon > 0$, we have,

$$\{(i,j), i \leq n \, and \, j \leq m : |x_{ij} - L| \geq \epsilon\} \supset \{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\},$$

it follows that,

$$\frac{1}{nm} |\{(i,j), i \leq n \, and \, j \leq m : |x_{ij} - L| \geq \epsilon\}| \geq \frac{1}{nm} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}|$$
$$= \left(\frac{\lambda_n^a}{n}\right)\left(\frac{\mu_m^b}{m}\right) \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}|.$$

Taking limit as $n, m \to \infty$ we have, $S_2(x) \subseteq S_{(\lambda,\mu)}^{\tilde{\alpha}}(x)$.

Conversely, suppose that either $\liminf_{n\to\infty} \frac{\lambda_n^a}{n}$ or $\liminf_{m\to\infty} \frac{\mu_m^b}{m}$ or both are zero. Then, we can choose two subsequences (n_p) and (m_q) such that $\frac{\lambda_{n_p}^a}{n_p} < \frac{1}{p}$ and $\frac{\mu_{m_q}^b}{m_q} < \frac{1}{q}$. Define double sequence $x = (x_{ij})$ as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i \in I_{n_p} \quad \text{and} \quad j \in I_{m_q} \qquad (p, q = 1, 2, 3, \ldots) \\ 0 & \text{otherwise,} \end{cases}$$

Then, clearly $x \in S_2(x)$, but $x \notin S_{(\lambda,\mu)}(x)$. From *COROLLARY 4.1.1*, since $S_{(\lambda,\mu)}^{\tilde{\alpha}}(x) \subseteq S_{(\lambda,\mu)}(x)$, we have, $x \notin S_{(\lambda,\mu)}^{\tilde{\alpha}}(x)$. Hence, $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$ and $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$. ■

THEOREM 4.1.5 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. Then $[w_p^2]_{\tilde{\alpha}}(x) \subseteq [w_p^2]_{\tilde{\beta}}(x)$ and the inclusion is strict for some $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$.*

Proof Let $x = (x_{ij}) \in [w_p^2]_{\tilde{\alpha}}(x)$, then for $\tilde{\alpha} \in (0,1]$ and a positive real number p,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n \times I_m} |x_{ij} - L|^p = 0.$$

Also, for given $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \preceq \tilde{\beta}$, one can write

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p \leq \lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = 0$$

which implies $x = (x_{ij}) \in [w_p^2]_{\tilde{\beta}}(x)$. Hence, $[w_p^2]_{\tilde{\alpha}}(x) \subseteq [w_p^2]_{\tilde{\beta}}(x)$.

The following EXAMPLE will show that the inclusion is strict.

Define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} 1, & \text{if } n - \sqrt{\lambda_n} + 1 \leq i \leq n \quad \text{and} \quad m - \sqrt{\mu_m} + 1 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

Then, for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (*i.e.* for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$),

$$\frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - 0|^p \leq \frac{\sqrt{\lambda_n}\sqrt{\mu_m}}{\lambda_n^c \mu_m^d} = \frac{1}{\lambda_n^{c-\frac{1}{2}} \mu_m^{d-\frac{1}{2}}}.$$

Since $\frac{1}{\lambda_n^{c-\frac{1}{2}} \mu_m^{d-\frac{1}{2}}} \to 0$ as $n, m \to \infty$, therefore, $x = (x_{ij}) \in [w_p^2]_{\tilde{\beta}}(x)$, but for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (*i.e.* for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$)

$$\frac{(\sqrt{\lambda_n} - 1)(\sqrt{\mu_m} - 1)}{\lambda_n^a \mu_m^b} \leq \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - 0|^p$$

and $\frac{(\sqrt{\lambda_n}-1)(\sqrt{\mu_m}-1)}{\lambda_n^a \mu_m^b} \to \infty$ as $n, m \to \infty$, which implies $x = (x_{ij}) \notin [w_p^2]_{\tilde{\alpha}}(x)$. Hence, the inclusion is strict. ■

COROLLARY 4.1.2 *Let $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. Then,*

(i) $[w_p^2]_{\tilde{\alpha}}(x) = [w_p^2]_{\tilde{\beta}}(x) \Leftrightarrow \tilde{\alpha} \cong \tilde{\beta}$.

(ii) $[w_p^2]_{\tilde{\alpha}}(x) \subseteq w_p^2$ for each $\tilde{\alpha} \in (0, 1]$ and $0 < p < \infty$. ■

THEOREM 4.1.6 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. If a sequence $x = (x_{ij})$ is strongly $(V, \lambda, \mu)-$summable to L of order $\tilde{\alpha}$, then it is $(\lambda, \mu)-$statistically convergent to L of order $\tilde{\beta}$, i.e., $[w_p^2]_{\tilde{\alpha}}(x) \subset S^{\tilde{\beta}}_{(\lambda,\mu)}(x)$.*

Proof For any sequence $x = (x_{ij})$ and $\epsilon > 0$,

$$\sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = \sum_{\substack{(i,j)\in I_n\times I_m \\ |x_{ij}-L|\geq\epsilon}} |x_{ij} - L|^p + \sum_{\substack{(i,j)\in I_n\times I_m \\ |x_{ij}-L|<\epsilon}} |x_{ij} - L|^p$$

$$\geq \sum_{(i,j)\in I_n\times I_m |x_{ij}-L|\geq\epsilon} |x_{ij} - L|^p \geq |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p,$$

which implies,

$$\frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p \geq \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p$$

$$\geq \frac{1}{\lambda_n^c \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p.$$

It follows that if $x = (x_{ij})$ is strong $(V, \lambda, \mu)-$summable to L of order $\tilde{\alpha}$, then it is $(\lambda, \mu)-$statistically convergent to L of order $\tilde{\beta}$. ■

For particular choice of $\tilde{\alpha} \cong \tilde{\beta}$ in above Theorem, we have the following result.

COROLLARY 4.1.3 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$,*

(i) If $\tilde{\alpha} \cong \tilde{\beta}$ then $[w_p^2]_{\tilde{\alpha}}(x) \subset S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$.

(ii) For $\tilde{\beta} \cong 1$, $[w_p^2]_{\tilde{\alpha}}(x) \subset S_{(\lambda,\mu)}(x)$. ■

4.2 $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})-$Convergence

In this section, we have generalized the notions defined in former section with the help of ideals and defined new notions $\mathcal{I}-(\lambda, \mu)-$ statistical convergence and $\mathcal{I}-(V, \lambda, \mu)-$

summability of order $\tilde{\alpha}$, where, $\mathcal{I} \subseteq \mathbb{N}\times\mathbb{N}$ is a non trivial admissible ideal. Firstly, we recall the very interesting notion of $\mathcal{I}-$ statistical convergence of order $\alpha(\alpha \in (0,1])$ of single sequences which is due to Das and Savas (88).

DEFINITION 4.2.1 (88) *A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$ statistically convergent of order $\alpha, (\alpha \in (0,1])$ to a number L provided that for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \; |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, the number L is called $\mathcal{I}-$ statistical limit of the sequence $x = (x_k)$ of order α and we write $\mathcal{I} - S^\alpha - \lim_k x_k = L$.
Let $S^\alpha(\mathcal{I})$ denotes the set of all $\mathcal{I}-$ statistically convergent sequences of order α.

For $\mathcal{I} = \mathcal{I}_f$, the ideal of all finite subsets of $\mathbb{N}$, *DEFINITION 4.2.1* coincides with statistical convergence of order α of (**?**), for $\alpha = 1$, it coincides with $\mathcal{I}-$ statistical convergence of (86), For $\mathcal{I} = \mathcal{I}_f$ and $\alpha = 1$, it coincides with statistical convergence.

DEFINITION 4.2.2 *A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}-$ statistically convergent of order $\tilde{\alpha}$ if for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{n^a m^b} \; |\{(i,j) : i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, the number L is called $\mathcal{I}-$statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$ and we write $\mathcal{I} - S_2^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = L$. Let $S_2^{\tilde{\alpha}}(\mathcal{I})$ denotes the set of all $\mathcal{I}-$ statistically convergent double sequences of order $\tilde{\alpha}$.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to ∞ with

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1; \mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$$

and $\tilde{\alpha} \in (0,1]$ be given.

DEFINITION 4.2.3 *A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I} - (\lambda,\mu)-$statistically convergent of order $\tilde{\alpha}$ if for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}\right| \geq \delta\right\} \in \mathcal{I},$$

where $\lambda^a = (\lambda_n^a) = (\lambda_1^a, \lambda_2^a, \lambda_3^a, \cdots\cdots)$; $\mu^b = (\mu_m^b) = (\mu_1^b, \mu_2^b, \mu_3^b, \cdots\cdots)$ and $\lambda_n^a \mu_m^b$ denotes the usual multiplication of the corresponding entries of the sequences λ^a and μ^b. In this case, the number L is called $\mathcal{I} - (\lambda,\mu)-$statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$ and we write $\mathcal{I} - S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = L$.

Let $S_{(\lambda,\mu)}^{\tilde{\alpha}}(\mathcal{I})$ denotes the set of all $\mathcal{I}-(\lambda,\mu)-$statistically convergent double sequences of order $\tilde{\alpha}$.

For $\mathcal{I} = \mathcal{I}_0, (\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\text{there exist } n(A) \in \mathbb{N})(i,j \geq n(A) \Rightarrow (i,j) \notin A)\})$, *DEFINITION 4.2.3* coincides with $(\lambda,\mu)-$statistical convergence of double sequences of order $\tilde{\alpha}$ and for $\tilde{\alpha} = (a,b) = (1,1)$, *DEFINITION 4.2.3* coincides with $\mathcal{I} - (\lambda,\mu)-$statistical convergence of double sequences of (178). For the choice $\lambda = (n)$ and $\mu = (m)$, *DEFINITION 4.2.3* coincides with the notion of $\mathcal{I}-$statistical convergence of double sequences of order $\tilde{\alpha}$ of *DEFINITION 4.2.1*. Moreover, if we take, $\lambda = (n)$; $\mu = (m)$ and $\tilde{\alpha} = (a,b) = (1,1)$, *DEFINITION 4.2.1* coincides with $\mathcal{I}-$statistical convergence of double sequences of (178).

Also, if $x = (x_{ij})$ is $\mathcal{I} - (\lambda,\mu)-$ statistically convergent of order $\tilde{\alpha}$ to a number L, then L is determined uniquely. The notion of $\mathcal{I} - (\lambda,\mu)-$ statistical convergence of order $\tilde{\alpha}$ is well defined only for $\tilde{\alpha} \in (0,1]$. We next provide an example to show that the *DEFINITION 4.2.2* is well defined for $\tilde{\alpha} \in (0,1]$ but not for $\tilde{\alpha} \succ 1$ in general.

EXAMPLE 4.2.1 Let $\mathcal{I}$ be a strongly admissible ideal and $x = (x_{ij})$ be defined as

$$x_{ij} = \begin{cases} 1 & \text{if } i+j \text{ even} \\ 0 & \text{if } i+j \text{ odd} \end{cases}$$

Then for $\tilde{\alpha} \succ 1$,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 1| \geq \epsilon\}\right| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0,$$

i.e., $S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 1$. So, for each $\delta > 0$, there exists a positive integer n_1, such that

$$\frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 1| \geq \epsilon\}\right| < \delta \text{ for all } n, m \geq n_1.$$

Let $A = \{1, 2, \cdots, n_1 - 1\}$ and

$$B = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}\right| \geq \delta\right\}.$$

Then clearly, $B \subseteq (A \times \mathbb{N}) \cup (\mathbb{N} \times A)$. Since $\mathcal{I}$ is a strongly admissible ideal, the set $B \in \mathcal{I}$. This shows $\mathcal{I} - S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 1$. Also,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}\right| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0.$$

Similarly, one can easily see that $\mathcal{I} - S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 0$. This leads to a contradiction to the notion that limit of $\mathcal{I} - (\lambda, \mu)-$ statistically convergent sequence of order $\tilde{\alpha}$ is unique.

DEFINITION 4.2.4 *Let $\tilde{\alpha}$ be any real number such that $\tilde{\alpha} \in (0, 1]$. A double sequence $x = (x_{ij})$ is said to be $\mathcal{I} - (V, \lambda, \mu)-$summable of order $\tilde{\alpha}$ to a number L if for each $\delta > 0$,*

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L| \geq \delta\right\} \in \mathcal{I},$$

In this case, the number L is called $\mathcal{I}-(V,\lambda,\mu)-$ *statistical limit of the sequence* $x=(x_{ij})$ *of order* $\tilde{\alpha}$.

Let $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I})$ denotes the set of all strongly $(V,\lambda,\mu)-$ summable double sequences of order $\tilde{\alpha}$. For $\mathcal{I}=\mathcal{I}_0$, *DEFINITION 4.2.4* coincides with $(V,\lambda,\mu)-$ summability of order $\tilde{\alpha}$ of (179), for $\tilde{\alpha}=(a,b)=(1,1)$, *DEFINITION 4.2.4* coincides with $\mathcal{I}-(V,\lambda,\mu)-$ summability of double sequences of (178). For $\lambda=(n)$ and $\mu=(m)$, *DEFINITION 4.2.4* coincides with $\mathcal{I}-(C,1,1)$ summability of double sequences of order $\tilde{\alpha}$. However, if we take $\lambda=(n)$; $\mu=(m)$ and $\tilde{\alpha}=(a,b)=(1,1)$, *DEFINITION 4.2.4* coincides with $\mathcal{I}-(C,1,1)$ summability of double sequences of (178).

We next present the algebraic properties of $\mathcal{I}-(\lambda,\mu)-$ statistical convergence of order $\tilde{\alpha}$.

THEOREM 4.2.1 *Let* $x=(x_{ij})$ *and* $y=(y_{ij})$ *be two double sequences of complex numbers and* $\tilde{\alpha}\in(0,1]$.

(i) *If* $\mathcal{I}-S^{\tilde{\alpha}}_{(\lambda,\mu)}-\lim x_{ij}=L$ *and* $c\in\mathbb{C}$, *then* $\mathcal{I}-S^{\tilde{\alpha}}_{(\lambda,\mu)}-\lim(cx_{ij})=cL$.

(ii) *If* $\mathcal{I}-S^{\tilde{\alpha}}_{(\lambda,\mu)}-\lim x_{ij}=L$ *and* $\mathcal{I}-S^{\tilde{\alpha}}_{(\lambda,\mu)}-\lim y_{ij}=M$, *then* $\mathcal{I}-S^{\tilde{\alpha}}_{(\lambda,\mu)}-\lim(x_{ij}+y_{ij})=L+M$. ■

THEOREM 4.2.2 *Let* $\tilde{\alpha},\tilde{\beta}\in(0,1]$ *such that* $\tilde{\alpha}\preceq\tilde{\beta}$. *Then* $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})\subseteq S^{\tilde{\beta}}_{(\lambda,\mu)}(\mathcal{I})$ *and the inclusion is strict for some* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *such that* $\tilde{\alpha}\prec\tilde{\beta}$ *and when* $\mathcal{I}=\mathcal{I}_0$.

Proof Let $x=(x_{ij})\in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$. Since, $\tilde{\alpha}\preceq\tilde{\beta}$ so $a\leq c$ and $b\leq d$; which for any $\epsilon>0$ gives the inequality

$$\frac{1}{\lambda_n^c\mu_m^d}\left|\{(i,j)\in I_n\times I_m:|x_{ij}-L|\geq\epsilon\}\right|\leq\frac{1}{\lambda_n^a\mu_m^b}\left|\{(i,j)\in I_n\times I_m:|x_{ij}-L|\geq\epsilon\}\right|;$$

and so for any $\delta > 0$,

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^c \mu_m^d} \, |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \subseteq \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \, |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\},$$

therefore, the result follows immediately from the fact that $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$.

For rest part of the THEOREM, we consider the following EXAMPLE.

Let $\mathcal{I} = \mathcal{I}_0$ and define $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} ij, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq i \leq n \quad \text{and} \quad m - [\sqrt{\mu_m}] + 1 \leq j \leq m] \\ 0, & \text{otherwise} \end{cases};$$

Then, for $\tilde{\beta} \in (\frac{1}{2}, 1)$($i.e.$ for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$), we have $x = (x_{ij}) \in S^{\tilde{\beta}}_{(\lambda,\mu)}(\mathcal{I})$, but one can easily verify that $x = (x_{ij}) \notin S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$($i.e.$ for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$). ■

COROLLOARY 4.2.1 *Let $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$,*

(i) If $\tilde{\beta} \cong 1$, then $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \subseteq S^{1}_{(\lambda,\mu)}(\mathcal{I}) = S_{(\lambda,\mu)}(\mathcal{I})$ and the inclusion is strict.

(ii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) = S^{\tilde{\beta}}_{(\lambda,\mu)}(\mathcal{I}) \Longleftrightarrow \tilde{\alpha} \cong \tilde{\beta}$.

(iii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) = S_{(\lambda,\mu)}(\mathcal{I}) \Longleftrightarrow \tilde{\alpha} \cong 1$.■

THEOREM 4.2.3 *$S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \cap l^2_\infty$ is a closed linear subspace of the normed linear space l^2_∞.*

Proof Let $(x^{nm}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \cap l^2_\infty$ (for $\tilde{\alpha} \in (0, 1]$) and $\mathcal{I} - S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim x^{nm} = x$, where $x \in l^2_\infty$. We have to prove that $x \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$. Since $x^{nm} \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$ and $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \subseteq S_{(\lambda,\mu)}(\mathcal{I})$, there exists some number L_{nm} such that (x^{nm}) is $\mathcal{I} - (\lambda, \mu)-$statistically convergent to L_{nm} for $m, n = 1, 2, 3, \cdots$. Take a strictly decreasing sequence of positive numbers (ϵ_{mn}) converging to zero, then for every $m, n = 1, 2, 3 \cdots$, there is a positive N'_{nm} such that for $n, m \geq N'_{mn}$, $\|x - x^{nm}\|_{(\infty,2)} \leq \frac{\epsilon_{mn}}{4}$. Let $0 < \delta < 1$. Then,

$$A = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_r \mu_s} \left|\left\{(i,j) \in I_n \times I_m : |x_{ij}^{nm} - L_{mn}| \geq \frac{\epsilon_{mn}}{4}\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}),$$

$$B = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_r \mu_s} \left|\left\{(i,j) \in I_n \times I_m : |x_{ij}^{n+1,m+1} - L_{m+1,n+1}| \geq \frac{\epsilon_{m+1,n+1}}{4}\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$, we can choose $(r,s) \in A \cap B$. Then,

$$\left\{\frac{1}{\lambda_r \mu_s} \left|\left\{(i,j) \in I_n \times I_m : |x_{ij}^{nm} - L_{mn}| \geq \frac{\epsilon_{mn}}{4} \vee |x_{ij}^{n+1,m+1} - L_{m+1,n+1}| \geq \frac{\epsilon_{m+1,n+1}}{4}\right\}\right|\right\} < \delta < 1.$$

Since $\lambda_r, \mu_s \to \infty$ and $A \cap B \in \mathcal{F}(\mathcal{I})$ is infinite, we can choose r, s such that $\lambda_r > 5$ and $\mu_s > 5$. Hence, there must exist a $(i,j) \in I_n \times I_m$ for which $|x_{ij}^{nm} - L_{mn}| < \frac{\epsilon_{mn}}{4}$ and $|x_{ij}^{n+1,m+1} - L_{m+1,n+1}| < \frac{\epsilon_{m+1,n+1}}{4}$. So,

$$\begin{aligned}
|L_{mn} - L_{m+1,n+1}| &\leq |L_{mn} - x_{ij}^{nm}| + |x_{ij}^{nm} - x_{ij}^{n+1,m+1}| + |x_{ij}^{n+1,m+1} - L_{m+1,n+1}| \\
&\leq |x_{ij}^{nm} - L_{mn}| + |x_{ij}^{n+1,m+1} - L_{m+1,n+1}| + \|x - x^{nm}\|_{(\infty,2)} + \|x - x^{n+1,m+1}\|_{(\infty,2)} \\
&\leq \frac{\epsilon_{mn}}{4} + \frac{\epsilon_{m+1,n+1}}{4} + \frac{\epsilon_{m,n}}{4} + \frac{\epsilon_{m,n}}{4} \leq \epsilon_{mn}.
\end{aligned}$$

This gives (L_{mn}) is a convergent double sequence and let L be the limit of (L_{mn}). Now, we have to show that $\mathcal{I}-S_{(\lambda,\mu)}^{\tilde{\alpha}}-\lim x = L$. For any $\epsilon > 0$, choose $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that $\epsilon_{mn} < \frac{\epsilon}{4}$, $\|x - x^{nm}\|_{(\infty,2)} < \frac{\epsilon}{4}$ and $|L_{mn} - L| < \frac{\epsilon}{4}$. Then, for $\tilde{\alpha} \in (0,1]$,

$$\begin{aligned}
&\frac{1}{{\lambda_r}^a {\mu_s}^b} \left|\{(i,j) \in I_r \times I_s : |x_{ij} - L| \geq \epsilon\}\right| \\
&\leq \frac{1}{{\lambda_r}^a {\mu_s}^b} \left|\left\{(i,j) \in I_r \times I_s : |x_{ij}^{nm} - L_{mn}| + \|x - x^{nm}\|_{(\infty,2)} + |L_{mn} - L| \geq \epsilon\right\}\right| \\
&\leq \frac{1}{{\lambda_r}^a {\mu_s}^b} \left|\left\{(i,j) \in I_r \times I_s : |x_{ij}^{nm} - L_{mn}| \geq \epsilon\right\}\right|.
\end{aligned}$$

For any $\delta > 0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{{\lambda_r}^a {\mu_s}^b} \left|\{(i,j) \in I_r \times I_s : |x_{ij} - L| \geq \epsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{{\lambda_r}^a {\mu_s}^b} \left|\left\{(i,j) \in I_r \times I_s : |x_{ij}^{nm} - L_{nm}| \geq \epsilon\right\}\right| \geq \delta\right\}.$$

Since the set on right hand side belongs to $\mathcal{I}$, it follows that $\mathcal{I} - S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim x = L$. This completes the result. ∎

THEOREM 4.2.4 *Let $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ be two sequences as defined above and $\tilde{\alpha} \in (0,1]$, then,*

(i) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \subseteq S_2(\mathcal{I})$ for all λ, μ and $\tilde{\alpha} \in (0,1]$.

(ii) $S_2(\mathcal{I}) \subseteq S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$, if and only if, $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$ *and* $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$.

Proof *(i)* By the nature of the sequences (λ_n), (μ_m) and from the expression $\frac{\lambda_n \mu_m}{nm} \leq 1$, the result follows.
(ii) Let, $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$; $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$ and $x = (x_{ij}) \in S_2(\mathcal{I})$. For given $\epsilon > 0$, we have,

$$\{(i,j), i \leq n \, and \, j \leq m : |x_{ij} - L| \geq \epsilon\} \supset \{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\},$$

it follows that,

$$\frac{1}{nm} \left|\{(i,j), i \leq n \, and \, j \leq m : |x_{ij} - L| \geq \epsilon\}\right| \geq \frac{1}{nm} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}\right|$$
$$= \left(\frac{\lambda_n^a}{n}\right)\left(\frac{\mu_m^b}{m}\right) \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}\right|.$$

If $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} = n_1 > 0$ and $\liminf_{m\to\infty} \frac{\mu_m^b}{m} = m_1 > 0$ then the set $\{(n,m) \in$

$\mathbb{N}\times\mathbb{N} : \frac{\lambda_n^a}{n}.\frac{\mu_m^b}{m} < \frac{n_1 m_1}{2}\}$ is finite. So for each $\delta > 0$,

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a\mu_m^b}\ |\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq\epsilon\}|\geq\delta\right\}$$
$$\subset\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{nm}\ |\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq\epsilon\}|\geq\frac{n_1m_1}{2}\delta\right\}$$
$$\cup\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{\lambda_n^a}{n}.\frac{\mu_m^b}{m} < \frac{n_1m_1}{2}\right\}.$$

Since $\mathcal{I}$ is admissible, $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$.

Conversely, suppose that either $\liminf_{n\to\infty}\frac{\lambda_n^a}{n}$ or $\liminf_{m\to\infty}\frac{\mu_m^b}{m}$ or both are zero. Then, we can choose two subsequences (n_p) and (m_q) such that $\frac{\lambda^a_{n_p}}{n_p} < \frac{1}{p}$ and $\frac{\mu^b_{m_q}}{m_q} < \frac{1}{q}$. Define double sequence $x = (x_{ij})$ as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i\in I_{n_p} \quad \text{and} \quad j\in I_{m_q} \qquad (p,q = 1,2,3,\ldots) \\ 0 & \text{otherwise,} \end{cases}$$

Then, clearly $x \in S_2(\mathcal{I})$, but $x \notin S_{(\lambda,\mu)}(\mathcal{I})$.

From *COROLLARY 4.2.1*, since $S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I}) \subseteq S_{(\lambda,\mu)}(\mathcal{I})$, we have $x \notin S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$. Hence, $\liminf_{n\to\infty}\frac{\lambda_n^a}{n} > 0$ and $\liminf_{m\to\infty}\frac{\mu_m^b}{m} > 0$. ■

THEOREM 4.2.5 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. Then $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subseteq (V,\lambda,\mu)^{\tilde{\beta}}(\mathcal{I})$ and the inclusion is strict for some $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$ and $\mathcal{I} = \mathcal{I}_0$.*

Proof Let $x = (x_{ij}) \in (V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I})$, then for $\tilde{\alpha} \in (0,1]$ and for each $\delta > 0$,

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a\mu_m^b}\sum_{(i,j)\in I_n\times I_m}|x_{ij}-L|\geq\delta\right\}\in\mathcal{I}.$$

Also for given $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \preceq \tilde{\beta}$, one can write,

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^c\mu_m^d}\sum_{(i,j)\in I_n\times I_m}|x_{ij}-L|\geq\delta\right\}$$
$$\subseteq\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a\mu_m^b}\sum_{(i,j)\in I_n\times I_m}|x_{ij}-L|\geq\delta\right\}.$$

Since the right set belongs to $\mathcal{I}$, this gives $x = (x_{ij}) \in (V,\lambda,\mu)^{\tilde{\beta}}(\mathcal{I})$. Hence, $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subseteq (V,\lambda,\mu)^{\tilde{\beta}}(\mathcal{I})$. The following example will show that the inclusion is strict. Let $\mathcal{I} = \mathcal{I}_0$ and define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} 1, & \text{if } n - \sqrt{\lambda_n} + 1 \leq i \leq n \quad \text{and} \quad m - \sqrt{\mu_m} + 1 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

Then, for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (*i.e.*for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$),

$$\frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n \times I_m} |x_{ij} - 0|^p \leq \frac{\sqrt{\lambda_n}\sqrt{\mu_m}}{\lambda_n^c \mu_m^d} = \frac{1}{\lambda_n^{c-\frac{1}{2}} \mu_m^{d-\frac{1}{2}}} \to 0 \text{ as } n, m \to \infty.$$

So, for each $\delta > 0$,

$$\left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n \times I_m} |x_{ij} - L| \geq \delta \right\} \in \mathcal{I},$$

which implies, $x = (x_{ij}) \in (V,\lambda,\mu)^{\tilde{\beta}}(\mathcal{I})$. But for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (*i.e.* for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$), $x = (x_{ij}) \notin (V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I})$. Hence, the inclusion is strict. ■

COROLLARY 4.2.2 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. Then*

(i) $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) = (V,\lambda,\mu)^{\tilde{\beta}}(\mathcal{I}) \Leftrightarrow \tilde{\alpha} \cong \tilde{\beta}$.

(ii) $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subseteq (V,\lambda,\mu)(\mathcal{I})$ *for each* $\tilde{\alpha} \in (0,1]$. ■

THEOREM 4.2.6 *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. If a sequence $x = (x_{ij})$ is $\mathcal{I} - (V,\lambda,\mu)-$summable to L of order $\tilde{\alpha}$, then it is $\mathcal{I} - (\lambda,\mu)-$statistically convergent to L of order $\tilde{\beta}$, i.e., $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subset S^{\tilde{\beta}}_{(\lambda,\mu)}(\mathcal{I})$.*

Proof For any sequence $x = (x_{ij}) \in (V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I})$ and $\epsilon > 0$

$$\sum_{(i,j)\in I_n \times I_m} |x_{ij} - L| = \sum_{\substack{(i,j)\in I_n \times I_m \\ |x_{ij}-L|\geq\epsilon}} |x_{ij} - L| + \sum_{\substack{(i,j)\in I_n \times I_m \\ |x_{ij}-L|<\epsilon}} |x_{ij} - L|$$

$$\geq \sum_{(i,j)\in I_n \times I_m |x_{ij}-L|\geq\epsilon} |x_{ij} - L| \geq |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon,$$

which implies,

$$\frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij}-L| \geq \frac{1}{\lambda_n^a \mu_m^b} \left|\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq \epsilon\}\right|.\epsilon$$
$$\geq \frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq \epsilon\}\right|.\epsilon.$$

So for each $\delta > 0$,

$$\frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq \epsilon\}\right| \geq \delta \Rightarrow \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij}-L| \geq \epsilon\delta,$$

which gives,

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j)\in I_n\times I_m : |x_{ij}-L|\geq \epsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij}-L| \geq \epsilon\delta\right\}$$

It follows that if $x = (x_{ij})$ is $\mathcal{I}-(V,\lambda,\mu)-$summable to L of order $\tilde{\alpha}$, then it is $\mathcal{I}-(\lambda,\mu)-$statistically convergent to L of order $\tilde{\beta}$. ■

For particular choice of $\tilde{\alpha} \cong \tilde{\beta}$ in above Theorem, we have the following result.

Corollary 4.2.3 *Let $\tilde{\alpha} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$,*

(i) For $\tilde{\alpha} \cong \tilde{\beta}$, we have $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subset S^{\tilde{\alpha}}_{(\lambda,\mu)}(\mathcal{I})$, i.e., If a sequence $x = (x_{ij})$ is $\mathcal{I}-(V,\lambda,\mu)-$summable to L of order $\tilde{\alpha}$, then it is $\mathcal{I}-(\lambda,\mu)-$statistically convergent to L of order $\tilde{\alpha}$.

(ii) For $\tilde{\beta} \cong 1$, $(V,\lambda,\mu)^{\tilde{\alpha}}(\mathcal{I}) \subset S_{(\lambda,\mu)}(\mathcal{I})$. ■

REMARK 4.2.1 It is remarkable that the converse of above result does not hold good in general. The following EXAMPLE will show that an $\mathcal{I}-(\lambda,\mu)-$ statistically convergent sequence of order $\tilde{\alpha}$ need not be $\mathcal{I}-(V,\lambda,\mu)-$ summable of order $\tilde{\alpha}$, for

$\tilde{\alpha} \in (0,1]$.

Let $\mathcal{I}$ be a strongly admissible ideal and $A \in \mathcal{I}$ is fixed.

Define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} ij, & \text{if } n - \sqrt{\lambda_n^a} + 1 \leq i \leq n, m - \sqrt{\mu_m^b} + 1 \leq j \leq m, (m,n) \notin A \\ ij, & \text{if } n - \lambda_n + 1 \leq i \leq n, m - \mu_m + 1 \leq j \leq m, (m,n) \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $\epsilon > 0$,

$$\frac{1}{\lambda_n^a \mu_m^b} \, |\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}| = \frac{[\sqrt{\lambda_n^a}][\sqrt{\mu_m^b}]}{\lambda_n^a \mu_m^b} \to 0$$

as $n, m \to \infty$ and $(m,n) \notin A$. Hence, for each $\delta > 0$,

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_n^a \mu_m^b} \, |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\}$$
$$\subset A \cup \{(\mathbb{N} \times \mathbb{N} \backslash A) \cap ((\{1,2,\cdots,m_1 - 1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,m_1 - 1\}))\}$$

for some $m_1 \in \mathbb{N}$. Since $\mathcal{I}$ is strongly admissible ideal, the set on right hand side belongs to $\mathcal{I}$. This shows $\mathcal{I} - S_{(\lambda,\mu)}^{\tilde{\alpha}} - \lim_{i,j} x_{ij} = 0$, but

$$\frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - 0| \to \infty \text{ as } n, m \to \infty,$$

So, $x = (x_{ij})$ is not $\mathcal{I} - (V, \lambda, \mu)-$summable to zero of order $\tilde{\alpha}$. ■

4.3 Conclusion

We have generalized the concept of $(\lambda, \mu)-$ statistical convergence, $(V, \lambda, \mu)-$ summability of double sequences to introduce the new ideas of $(\lambda, \mu)-$ statistical convergence of order $\tilde{\alpha}$ and $(V, \lambda, \mu)-$ summability of order $\tilde{\alpha}$ for double sequences. These ideas are more general for particular choice of $\tilde{\alpha}$. For $\tilde{\alpha} = (a, b) = (1, 1)$, $(\lambda, \mu)-$ statistical convergence of order $\tilde{\alpha}$ of double sequences coincides with $(\lambda, \mu)-$ statistical convergence of double sequences of (100). Also we have generalized these concepts with the help of ideals and defined the concepts of $\mathcal{I}-(\lambda, \mu)-$ statistical convergence of order $\tilde{\alpha}$ and $\mathcal{I}-(V, \lambda, \mu)-$ summability of order $\tilde{\alpha}$. For giving particular choice to the ideal $\mathcal{I}$ and $\tilde{\alpha}$ we obtain different convergence methods. *COROLLARY* 4.2.3 is important to note that $\mathcal{I}-(V, \lambda, \mu)-$ summability of order $\tilde{\alpha}$ implies $\mathcal{I}-(\lambda, \mu)-$ statistical convergence of order $\tilde{\alpha}$ but converse is not true. ■