Chapter 3

$WS_\theta(\mathcal{I})-$Convergence

We continue in this chapter with the generalization of weak convergence by use of lacunary sequence $\theta = (k_r)$ and a non-trivial admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$. The whole chapter is divided into two sections. In first section, we begin by exploring a variety of definitions of $WS_\theta-$ limit and $WS_\theta-$ cluster points which arised naturally due to the introduction of the interesting idea of $WS_\theta-$ convergence by Nuray (119). However, in second section of this chapter, we will define and develop $WS_\theta(\mathcal{I})-$ convergence, $WN_\theta(\mathcal{I})-$ convergence, $WS_\theta(\mathcal{I})-$ limit points and $WS_\theta(\mathcal{I})-$ cluster points, which are new generalizations of weak convergence in a Banach space X.

3.1 $WS_\theta-$ Limit and Cluster Point

For any lacunary sequence $\theta = (k_r)$, we begin with recalling the following definition due to Nuray (119), which forms the base for the next new definitions.

A sequence $(x_k) \in X$ is said to be weak lacunary statistically convergent to $x \in X$ provided that, for each $\varphi \in X^*$, each $\epsilon > 0$, $\delta_\theta\left(\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}\right) = 0$.

Results presented in this chapter are published in Atti della Fondazione Giorgio Ronchi, Italy & Pure and Applied Mathematics Letters, India.

In this case, we write $WS_\theta - \lim_{k\to\infty} x_k = x$.

Let $x = (x_k)$ be a sequence in X, $(x_{k(j)})$ be a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we denote $(x_{k(j)})$ by $(x)_K$. If

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0;$$

then $(x)_K$ is called $\theta-$thin subsequence. On the other hand, $(x)_K$ is a $\theta-$nonthin subsequence of x provided that

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

We now give our main definitions.

DEFINITION 3.1.1 *An element* $x' \in X$ *is called weak lacunary statistical limit point (briefly weak* $S_\theta-$*limit point) of a sequence* $x = (x_k)$ *in* X *provided that there is a* $\theta-$*nonthin subsequence of* x *that is weakly convergent to* x'.

Let $\Lambda(WS_\theta, x)$ denotes the set of all weak $S_\theta-$ limit points of the sequence $x = (x_k)$.

DEFINITION 3.1.2 *For any lacunary sequence* $\theta = (k_r)$*, a point* $x'' \in X$ *is said to be weak lacunary statistical cluster point (briefly weak* $S_\theta-$ *cluster point) of a sequence* $x = (x_k)$ *in* X *provided that for each* $\epsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \left|\varphi(x_k - x'')\right| < \epsilon\}\right| > 0.$$

Let $\Gamma(WS_\theta, x)$ denotes the set of all weak $S_\theta-$ cluster points of the sequence $x = (x_k)$.

THEOREM 3.1.1 *For any sequence* $x = (x_k)$ *in* X, $\Lambda(WS_\theta, x) \subseteq \Gamma(WS_\theta, x)$.

Proof For $x' \in \Lambda((WS_\theta, x)$, there is a $\theta-$nonthin subsequence $(x_{k(j)})$ of x that weakly converges to x'. Since $(x_{k(j)})$ is a $\theta-$ nonthin subsequence so

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0. \tag{3.1.1}$$

Now the containment $\left\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\right\} \supseteq \{k(j) \in I_r : \left|\varphi(x_{k(j)} - x^{'})\right| < \epsilon\}$ gives

$$\left\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\right\} \supseteq \{k\,(j) \in I_r : j \in \mathbb{N}\} - \left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x^{'})\right| \geq \epsilon\right\};$$

which immediately implies,

$$\limsup_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\right\}\right| \geq \limsup_r \frac{1}{h_r}\left|\{k\,(j) \in I_r : j \in \mathbb{N}\}\right|$$
$$- \limsup_r \frac{1}{h_r}\left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x^{'})\right| \geq \epsilon\right\}\right|. \quad (3.1.2)$$

Further, the weak convergence of $(x_{k(j)})$ to $x^{'}$ gives for each $\epsilon > 0$ and each $f \in X^*$, the set $\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x^{'})\right| \geq \epsilon\right\}$ is finite for which we have,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x^{'})\right| \geq \epsilon\right\}\right| = 0. \quad (3.1.3)$$

i.e. $WS_\theta - \lim x_{k(j)} = x^{'}$. Using equations (3.1.1) and (3.1.3) in (3.1.2), we get

$$\limsup_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\right\}\right| > 0.$$

This shows that $x^{'} \in \Gamma\,(WS_\theta, x)$ and therefore, we have the containment $\Lambda\,(WS_\theta, x) \subseteq \Gamma\,(WS_\theta, x)$. ■

THEOREM 3.1.2 *For any sequence $x = (x_k)$ in X, $\Gamma\,(WS_\theta, x) \subseteq WL(x)$, where $WL(x)$ denotes the set of all weak limit points of $x = (x_k)$.*

Proof Assume that $x^{'} \in \Gamma\,(WS_\theta, x)$, then for each $\epsilon > 0$, we have,

$$\limsup_{r\to\infty} \frac{1}{h_r}\left|\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\}\right| > 0. \quad (3.1.4)$$

For $\epsilon > 0$, if we denote $K = \left\{k \in I_r : \left|\varphi(x_k - x^{'})\right| < \epsilon\right\}$, then the set $K = \{k_1 < k_2 < \cdots\}$ is an infinite set as otherwise *i.e.* if K is a finite set then left side of equation (3.1.4) becomes zero and we obtain a contradiction. This shows that we have a subsequence $(x)_K$ of the sequence $x = (x_k)$ that is weakly convergent to x'. Hence x' is a weak limit point of (x_k) and therefore we have the containment $\Gamma(WS_\theta, x) \subseteq WL(x)$. ■

THEOREM 3.1.3 *For any sequence $x = (x_k) \in X$, if $WS_\theta - \lim_k x_k = x_0$, then $\Lambda(WS_\theta, x) = \Gamma(WS_\theta, x) = \{x_0\}$.*

Proof We first show that $\Lambda(WS_\theta, x) = \{x_0\}$. Let $\Lambda(WS_\theta, x) = \{x_0, y_0\}$ such that $x_0 \neq y_0$. By definition there exist two $\theta-$nonthin subsequences $\left(x_{k(i)}\right)$ and $\left(x_{l(j)}\right)$ of the sequence $x = (x_k)$ which are respectively weakly convergent to x_0 and y_0. Choose $\epsilon > 0$ such that $0 < \epsilon < \frac{|\varphi(x_0 - y_0)|}{2}$. Since $\left(x_{l(j)}\right)$ weakly converges to y_0, therefore,

$$\lim_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right| = 0 \ . \tag{3.1.5}$$

Moreover, we can write,

$$\{l(j) \in I_r : j \in \mathbb{N}\} = \left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \cup \left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\};$$

which implies,

$$\begin{aligned} \limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| &= \limsup_r \frac{1}{h_r} \left|\{l(j) \in I_r : j \in \mathbb{N}\}\right| \\ &\quad - \limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right|. \end{aligned}$$

Since $(l(j))$ is $\theta-$nonthin subsequence so we have together with equation (3.1.5),

$$\limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| > 0. \tag{3.1.6}$$

Also, using the fact $WS_\theta - \lim_k x_k = x_0$, we have,

$$\lim_r \frac{1}{h_r} \left|\{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\}\right| = 0. \tag{3.1.7}$$

For $0 < 2\epsilon < |\varphi(x_0 - y_0)|$,

$$\left\{l(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \cap \{k \in I_r : |\varphi(x_k - x_0)| < \epsilon\} = \emptyset.$$

So, we have,

$$\left\{l\left(j\right) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \subseteq \{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\};$$

which implies,

$$\limsup_r \frac{1}{h_r}\left|\left\{l\left(j\right) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \leq \limsup_r \frac{1}{h_r}\left|\{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\}\right|.$$

Using equation (3.1.7), we can see that,

$$\limsup_r \frac{1}{h_r}\left|\left\{l\left(j\right) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \leq 0,$$

which contradicts equation (3.1.6). Hence, $\Lambda\left(WS_\theta, x\right) = \{x_0\}$. Similarly, we can show that $\Gamma\left(WS_\theta, x\right) = \{x_0\}$. ■

THEOREM 3.1.4 *For any lacunary sequence $\theta = (k_r)$, if $x = (x_k)$ and $y = (y_k)$ are two sequences in X such that $\lim_r \frac{1}{h_r}\ |\{k \in I_r : x_k \neq y_k\}| = 0$, then $\Lambda\left(WS_\theta, x\right) = \Lambda\left(WS_\theta, y\right)$ and $\Gamma\left(WS_\theta, x\right) = \Gamma\left(WS_\theta, y\right)$.*

Proof Let $x' \in \Lambda\left(WS_\theta, x\right)$, then there exists a θ-nonthin subsequence $(x)_K$ of the sequence $x = (x_k)$ that converges to x'.

Since, $\lim_r \frac{1}{h_r}\left|\{k \in I_r : k \in K, x_k \neq y_k\}\right| = 0$, it follows that,

$$\limsup_r \frac{1}{h_r}\left|\{k \in I_r : k \in K, x_k = y_k\}\right| > 0 \tag{3.1.8}$$

Therefore, there exists a θ-nonthin subsequence $(y)_K$ of the sequence $y = (y_k)$ that converges to x'. This shows that $x' \in \Lambda\left(WS_\theta, y\right)$ and therefore, $\Lambda\left(WS_\theta, x\right) \subseteq \Lambda\left(WS_\theta, y\right)$. By symmetry, we have $\Lambda\left(WS_\theta, y\right) \subseteq \Lambda\left(WS_\theta, x\right)$. Hence, we have, $\Lambda\left(WS_\theta, x\right) = \Lambda\left(WS_\theta, y\right)$. Similarly, we can prove $\Gamma\left(WS_\theta, x\right) = \Gamma\left(WS_\theta, y\right)$. ■

THEOREM 3.1.5 *Let $x = (x_k)$ be a sequence in X, then we have*

(i) If $\liminf_r q_r > 1$ then $\Lambda\left(WS_\theta, x\right) \subseteq \Lambda(WS, x)$;

(ii) If $\limsup_r q_r < \infty$ *then* $\Lambda(WS,x) \subseteq \Lambda(WS_\theta,x)$ *and*

(iii) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ *then* $\Lambda(WS,x) = \Lambda(WS_\theta,x)$.

Proof (i) Let $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r > 1+\delta$ for sufficiently large r which implies that $\frac{k_r}{h_r} \leq \frac{\delta+1}{\delta}$. Let $x^{'} \in \Lambda(WS_\theta,x)$, then by definition, there exists a set $K = \{k(j) : j \in \mathbb{N}\}$ such that $\lim_{j\to\infty} f(x_{k(j)} - x^{'}) = 0$ and

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k(j) \in I_r : j \in \mathbb{N}\}\right| > 0 \tag{3.1.9}$$

Since,

$$\begin{aligned}\frac{1}{k_r}\left|\{k(j) \leq k_r : j \in \mathbb{N}\}\right| &\geq \frac{1}{k_r}\left|\{k(j) \in I_r : j \in \mathbb{N}\}\right| \\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\left|\{k(j) \in I_r : j \in \mathbb{N}\}\right| \\ &\geq \left(\frac{\delta}{\delta+1}\right)\frac{1}{h_r}\left|\{k(j) \in I_r : j \in \mathbb{N}\}\right|;\end{aligned}$$

it follows by equation (3.1.9) that,

$$\limsup_{r\to\infty} \frac{1}{k_r}\left|\{k(j) \leq k_r : j \in \mathbb{N}\}\right| > 0.$$

Since $\left(x_{k(j)}\right)$ is weakly convergent to $x^{'}$, it follows that $x^{'} \in \Lambda(WS,x)$. Hence, we have, $\Lambda(WS_\theta,x) \subseteq \Lambda(WS,x)$.

(ii) If $\limsup_r q_r < \infty$, then there exists a real number H such that $q_r < H$ for all r. Without loss of generality, we can assume $H > 1$. Now for all r,

$$\frac{h_r}{k_{r-1}} = \frac{k_r - k_{r-1}}{k_{r-1}} = q_r - 1 \leq H - 1.$$

Now, Let $x^{'} \in \Lambda(WS,x)$, then by definition there is a set $K = \{k(j) : j \in \mathbb{N}\}$ with $\delta(K) > 0$ and $\lim_{j\to\infty} \varphi(x_{k(j)} - x^{'}) = 0$. Let $N_r = |\{k \in I_r : k \in K\}| = |K \cap I_r|$ and

$t_r = \frac{N_r}{h_r}$. For any integer n satisfying $k_{r-1} < n \leq k_r$, we can write,

$$\begin{aligned} \frac{1}{n}\left|\{k \leq n : k \in K\}\right| &\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : k \in K\}\right| \\ &= \frac{1}{k_{r-1}}\{N_1 + N_2 + N_3 + \cdots + N_r\} \\ &= \frac{1}{k_{r-1}}\{t_1 h_1 + t_2 h_2 + t_3 h_3 + \cdots + t_r h_r\} \\ &= \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\ &\leq \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + (H-1)t_r. \end{aligned}$$

Suppose $t_r \to 0$ as $r \to \infty$. Since θ is a lacunary sequence and the first part on the right side of above expression is a regular weighted mean transform of the sequence $t = (t_r)$, therefore it too tends to zero as $r \to \infty$. Since $n \to \infty$ as $r \to \infty$, it follows that $\delta(K) = 0$ which is a contradiction as $\delta(K) \neq 0$. Thus we have, $\lim_{r\to\infty} t_r \neq 0$ and therefore, by definition $\delta_\theta(K) \neq 0$. This shows that $x' \in \Lambda(WS_\theta, x)$. Hence, $\Lambda(WS, x) \subseteq \Lambda(WS_\theta, x)$.

(iii) This is an immediate consequence of (i) and (ii). ■

THEOREM 3.1.6 *Let $x = (x_k)$ be a sequence in X, then we have*

(i) If $\liminf_r q_r > 1$ then $\Gamma(WS_\theta, x) \subseteq \Gamma(WS, x)$;

(ii) If $\limsup_r q_r < \infty$ then $\Gamma(WS, x) \subseteq \Gamma(WS_\theta, x)$ and

(iii) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ then $\Gamma(WS, x) = \Gamma(WS_\theta, x)$.

Proof of this theorem goes on the similar lines as for *THEOREM 3.1.5*. ■

3.2 Generalized Weak Statistical Convergence

The present section is divided into two subsections in which the former is devoted to development of the basic theory of $WS_\theta(\mathcal{I})-$convergence, $WN_\theta(\mathcal{I})-$convergence and however, in second subsection $WS_\theta(\mathcal{I})-$limit points and $WS_\theta(\mathcal{I})-$ cluster points are defined and studied in a Banach space X.

3.2.1 $WS_\theta(\mathcal{I})-$Convergence

We begin by giving the following definition which is a more generalized form of weak convergence in a Banach space X.

DEFINITION 3.2.1.1 *A sequence (x_k) is said to be weak lacunary statistically convergent to x with respect to $\mathcal{I}$ in X if for every $\epsilon > 0$, every $\gamma > 0$ and each $\varphi \in X^*$, where X^* is the continuous dual of X we have*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma \right\} \in \mathcal{I}.$$

In this case, we write $WS_\theta(\mathcal{I})-\lim x_k = x$.

We will denote the set of all weak $\mathcal{I}-$lacunary statistically convergent sequences by $WS_\theta(\mathcal{I}, X)$. For $\mathcal{I} = \mathcal{I}_f$, weak $\mathcal{I}-$ lacunary statistical convergence coincides with weak lacunary statistical convergence of (119).

DEFINITION 3.2.1.2 *A sequence (x_k) is said to be weak $N_\theta-$convergent to x with respect to $\mathcal{I}$ in X if for every $\epsilon > 0$ and each $f \in X^*$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\varphi(x_k - x)| \geq \epsilon \right\} \in \mathcal{I}.$$

In this case, we write $WN_\theta(\mathcal{I})-\lim x_k = x$.

We will denote the set of all weak $N_\theta(\mathcal{I})-$ convergent sequences by $WN_\theta(\mathcal{I}, X)$. For $\mathcal{I} = \mathcal{I}_f$, weak $N_\theta(\mathcal{I})-$ convergence coincides with weak $N_\theta-$convergence of (119).

THEOREM 3.2.1.1 *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal. If $WS_\theta - \lim x_k = x$, then $WS_\theta(\mathcal{I})- \lim x_k = x$.*

Proof Suppose $WS_\theta - \lim x_k = x$, then for each $\epsilon > 0$, each $\varphi \in X^*$,

$$\lim_{r\to\infty} \frac{1}{h_r}|k \in I_r : |\varphi(x_k - x)| \geq \epsilon| = 0.$$

This immediately follows that for each $\gamma > 0$, there exists a positive integer m such that $\frac{1}{h_r}|k \in I_r : |\varphi(x_k - x)| \geq \epsilon| < \gamma$ for all $n \geq m$.
i.e. $\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \subseteq \{1, 2, 3, ..., m - 1\}$.
Since $\mathcal{I}$ is admissible, therefore, $\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \in \mathcal{I}$.
Thus, $WS_\theta(\mathcal{I})- \lim x_k = x$. ■

THEOREM 3.2.1.2 *For a lacunary sequence $\theta = (k_r)$, the sequence (x_k) is weak $N_\theta(\mathcal{I})-$convergent to x if and only if (x_k) is weak $\mathcal{I}-$lacunary statistically convergent to x.*

Proof Let (x_k) is weak $N_\theta(\mathcal{I})-$convergent to x, then for every $\epsilon > 0$ and each $\varphi \in X^*$, we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} |\varphi(x_k - x)| \geq \epsilon\right\} \in \mathcal{I}.$$

Also for given $\epsilon > 0$, we have,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} |\varphi(x_k - x)| &\geq \frac{1}{h_r} \sum_{\substack{k\in I_r \\ |\varphi(x_k - x)|\geq\epsilon}} |\varphi(x_k - x)| \\ &\geq \frac{\epsilon}{h_r}\left|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}\right|.\end{aligned}$$

Hence, for every $\epsilon > 0$, each $\gamma > 0$ and each $\varphi \in X^*$ we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}|\varphi(x_k - x)| \geq \epsilon\gamma\right\} \in \mathcal{I}.$$

This immediately gives (x_k) is weak $\mathcal{I}-$ lacunary statistically convergent to x.

Conversely, suppose that (x_k) is weak $\mathcal{I}-$ lacunary statistically convergent to x. Then for every $\epsilon > 0$, every $\gamma > 0$ and each $\varphi \in X^*$ we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \in \mathcal{I}.$$

Since $\varphi \in X^*$, therefore φ is bounded. Let $|\varphi(x_k - x)| \leq M$ for all k. For given $\epsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k \in I_r}|\varphi(x_k - x)| &= \frac{1}{h_r}\sum_{\substack{k \in I_r \\ |\varphi(x_k - x)| \geq \epsilon}}|\varphi(x_k - x)| + \sum_{\substack{k \in I_r \\ |\varphi(x_k - x)| < \epsilon}}|\varphi(x_k - x)| \\ &\leq \frac{M}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| + \epsilon.\end{aligned}$$

This implies

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}|\varphi(x_k - x)| \geq \epsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - x)| \geq \epsilon\}| \geq \frac{\epsilon}{M}\right\} \in \mathcal{I}.$$

Hence, (x_k) is weak $N_\theta(\mathcal{I})-$ convergent to x. ■

THEOREM 3.2.1.3 *For any lacunary sequence* $\theta = (k_r)$ *and* $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ *be an admissible ideal,* $WS(\mathcal{I}, X) \subseteq WS_\theta(\mathcal{I}, X)$ *if and only if* $\liminf_r q_r > 1$.

Proof Let $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r > 1+\delta$ for sufficiently large r which implies that $\frac{k_r}{h_r} \leq \frac{\delta+1}{\delta}$.

Let $WS(\mathcal{I})-\lim x_k = x$, such that $(x_k) \in WS(\mathcal{I}, X)$, then for every $\epsilon > 0$, every $\gamma > 0$ and each $f \in X^*$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \in \mathcal{I}.$$

Also, one can easily observe that

$$\begin{aligned}\frac{1}{k_r}\left|\{k\le k_r:|\varphi(x_k-x)|\ge\epsilon\}\right| &\ge \frac{1}{k_r}\left|\{k\in I_r:|\varphi(x_k-x)|\ge\epsilon\}\right|\\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\left|\{k\in I_r:|\varphi(x_k-x)|\ge\epsilon\}\right|\\ &\ge \left(\frac{\delta}{\delta+1}\right)\frac{1}{h_r}\left|\{k\in I_r:|\varphi(x_k-x)|\ge\epsilon\}\right|;\end{aligned}$$

Thus for every $\gamma>0$, we get,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\{k\in I_r:|\varphi(x_k-x)|\ge\epsilon\}\right|\ge\gamma\right\}\subseteq$$
$$\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\{k\le k_r:|\varphi(x_k-x)|\ge\epsilon\}\right|\ge\frac{\gamma\delta}{\delta+1}\right\}\in\mathcal{I},$$

which gives the result.

Conversely, suppose that $\liminf_r q_r=1$, then using as in (39), one can find a subsequence $(k_{r(j)})$ of $\theta=(k_r)$ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}}<1+\frac{1}{j}\quad\text{and}\quad\frac{k_{r(j)-1}}{k_{r(j-1)}}>j\quad\text{where}\quad r(j)\ge r(j-1)+2.$$

Define a sequence (x_k) as follows

$$x_k=\begin{cases}1, & \text{if } k\in I_{r(j)},\\ 0, & \text{otherwise.}\end{cases}$$

Since $\varphi\in X^*$, therefore φ is bounded. Let $|\varphi(x_k-x)|\le M$ for all k. Thus for any $\varphi\in X^*$, $x\in X$,

$$\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}}|\varphi(x_k-x)|=\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}}|\varphi(1-x)|\le M\,\text{for}\,j=1,2,\cdots$$
$$\text{and}\qquad\frac{1}{h_r}\sum_{k\in I_r}|\varphi(x_k-x)|=\frac{1}{h_r}\sum_{k\in I_r}|\varphi(-x)|\le M\,\text{for}\,r\ne r_j$$

which gives $(x_k)\notin WN_\theta(\mathcal{I},X)$. By *THEOREM 3.2.1.2*, we have, $(x_k)\notin WS_\theta(\mathcal{I},X)$.

Let $k_{r(j-1)} \leq n \leq k_{r(j)}$, then

$$\frac{1}{n}\sum_{k=1}^{n}|\varphi(x_k - x)| \geq \frac{1}{n}\sum_{\substack{k=1\\|\varphi(x_k-x)|\geq\epsilon}}^{n}|\varphi(x_k - x)|$$
$$\geq \frac{\epsilon}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right|.$$

We can write,

$$\frac{\epsilon}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| \leq \frac{1}{n}\sum_{k=1}^{n}|\varphi(x_k - x)| \leq M(\frac{1}{j} + \frac{1}{j}) = \frac{2M}{j}$$

Thus (x_k) is weak $\mathcal{I}-$ statistically convergent to x. ■

In (119), it has been shown that for any lacunary sequence $\theta = (k_r)$, $WS_\theta \subseteq WS$ if and only if $\limsup_r q_r < \infty$, but for any admissible ideal $\mathcal{I}$ its not sure.

THEOREM 3.2.1.4 *Let $\mathcal{I}$ be an admissible ideal having condition (AP) and $\theta \in F(\mathcal{I})$. If $x \in WS(\mathcal{I}, X) \cap WS_\theta(\mathcal{I}, X)$, then $WS(\mathcal{I})-\lim x_k = WS_\theta(\mathcal{I})-\lim x_k$.*

Proof Suppose $WS(\mathcal{I})-\lim x_k = x$ and $WS_\theta(\mathcal{I})-\lim x_k = y$ and $x \neq y$. Let $|x - y| > 2\epsilon$, $(\epsilon > 0)$. Also if $\mathcal{I}$ has the condition (AP) then there exists $M \in F(\mathcal{I})$ such that

$$\lim_{r\to\infty}\frac{1}{m_r}\left|\{k \leq m_r : |\varphi(x_k - x)| \geq \epsilon\}\right| = 0, \text{ where } M = \{m_1, m_2, m_3....\}.$$

Let $G = \{k \leq m_r : |\varphi(x_k - x)| \geq \epsilon\}$ and $H = \{k \leq m_r : |\varphi(x_k - y)| \geq \epsilon\}$. Since $\lim_{r\to\infty}\frac{|G|}{m_r} = 0$, therefore $\lim_{r\to\infty}\frac{1}{m_r}\left|\{k \leq m_r : |\varphi(x_k - y)| \geq \epsilon\}\right| = 1$.

Let $M^* = k_{l1}, k_{l2}, k_{l3}, \ldots = M \cap \theta \in F(\mathcal{I})$. Then k_{lp}^{th} term of the weak statistical limit expression $\frac{1}{m_r}\left|\{k \leq m_r : |\varphi(x_k - y)| \geq \epsilon\}\right|$:

$$\frac{1}{k_{l_p}}\left|\left\{k \in \bigcup_{r=1}^{l_p} I_r : |\varphi(x_k - y)| \geq \epsilon\right\}\right| = \frac{1}{k_{l_p}}\sum_{r=1}^{l_p}\left|\{k \in I_r : |\varphi(x_k - y)| \geq \epsilon\}\right|$$

$$= \frac{1}{\sum_{r=1}^{l_p} h_r}\sum_{r=1}^{l_p} h_r t_r$$

where $t_r = \frac{1}{h_r}|\{k \in I_r : |\varphi(x_k - y)| \geq \epsilon\}|$ and $t_r \to^{\mathcal{I}} 0$ as $WS_\theta(\mathcal{I})-\lim x_k = y$.

Since θ is a lacunary sequence and the right side of above expression is a regular weighted mean transform of the sequence $t = (t_r)$, therefore it is also $\mathcal{I}$-convergent to zero as $p \to \infty$ and so it has a subsequence which is convergent to zero since it satisfies condition (AP). But, since this is a subsequence of $\frac{1}{m_r}|\{k \leq m_r : |\varphi(x_k - y)| \geq \epsilon\}|$, we infer that $\frac{1}{m_r}|\{k \leq m_r : |\varphi(x_k - y)| \geq \epsilon\}|$ is not convergent to 1 which is a contradiction. This completes the proof. ■

3.2.2 $WS_\theta(\mathcal{I})-$Limit and Cluster Point

This subsection starts with the generalized definitions of limit and cluster points in a Banach space X. Before starting we recall the following terminology.

For a Banach space X, let $x = (x_k)$ be a sequence in X. Let $(x_{k(j)})$ be a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we denote $(x_{k(j)})$ by $(x)_K$. If for every $\gamma > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\right\} \in \mathcal{I};$$

then $(x)_K$ is called $\theta(\mathcal{I})$-thin subsequence.

On the other hand, $(x)_K$ is a $\theta(\mathcal{I})$-nonthin subsequence of x provided that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\right\} \notin \mathcal{I};$$

DEFINITION 3.2.2.1 *An element $x \in X$ is called a weak $S_\theta(\mathcal{I})-$ limit point of a sequence $x = (x_k)$ in X provided that there is a $\theta(\mathcal{I})-$ nonthin subsequence of (x_k) that is weak convergent to x.*

Let $\Lambda(WS_\theta(\mathcal{I}), x)$ denotes the set of all $WS_\theta(\mathcal{I})-$ limit points of the sequence $x = (x_k)$.

DEFINITION 3.2.2.2 *Let X be a Banach space, $\theta = (k_r)$ be a lacunary sequence and $f \in X^*$. An element $x \in X$ is called a weak $S_\theta(\mathcal{I})-$ cluster point of a sequence $x = (x_k)$ in X provided that for every $\epsilon > 0$ and each $f \in X^*$, we have*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| < \epsilon\}| \geq \gamma\right\} \notin \mathcal{I}.$$

Let $\Gamma(WS_\theta(\mathcal{I}), x)$ denotes the set of all $WS_\theta(\mathcal{I})-$cluster points of the sequence $x = (x_k)$.

THEOREM 3.2.2.1 *Let $\theta = (k_r)$ be a lacunary sequence and X be a Banach space. For any sequence (x_k) in X, $\Lambda(WS_\theta(\mathcal{I}), x) \subseteq \Gamma(WS_\theta(\mathcal{I}), x)$.*

Proof For $x \in \Lambda(WS_\theta(\mathcal{I}), x)$, there exists a $\theta(\mathcal{I})-$nonthin subsequence $(x_{k(j)})$ of (x_k) that weakly converges to x. Since $(x_{k(j)})$ is a θ-nonthin subsequence, so, for every $\gamma > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k\,(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\right\} \notin \mathcal{I}. \tag{3.2.1}$$

Now the containment, $\{k \in I_r : |\varphi(x_k - x)| < \epsilon\} \supseteq \{k(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| < \epsilon\}$ gives

$$\{k \in I_r : |\varphi(x_k - x)| < \epsilon\} \supseteq \{k\,(j) \in I_r : j \in \mathbb{N}\} - \left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\};$$

which immediately implies,

$$\frac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| < \epsilon\}| \geq \\ \frac{1}{h_r} |\{k\,(j) \in I_r : j \in \mathbb{N}\}| - \frac{1}{h_r} \left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}\right|.$$

So for each $\gamma > 0$ and $\epsilon > 0$,

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| < \epsilon\}| \geq \gamma\right\}$$

$$\supseteq \{r \in \mathbb{N} : |\{k\,(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\} - \left\{r \in \mathbb{N} : \left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}\right| \geq \gamma\right\}$$

$$= \{r \in \mathbb{N} : |\{k\,(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\} \cap \left\{r \in \mathbb{N} : \left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}\right| \geq \gamma\right\}^c$$

$$= \{r \in \mathbb{N} : |\{k\,(j) \in I_r : j \in \mathbb{N}\}| \geq \gamma\} \cap \left\{r \in \mathbb{N} : \left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}\right| < \gamma\right\} \tag{3.2.2}$$

Further, the weak convergence of $(x_{k(j)})$ to x gives that for each $\epsilon > 0$ and each $\varphi \in X^*$, the set $\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}$ is finite for which we have,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k\,(j) \in I_r : \left|\varphi(x_{k(j)} - x)\right| \geq \epsilon\right\}\right| = 0,$$

i.e., $WS_\theta - \lim x_{k(j)} = x$. Using THEOREM *3.2.1.1*, we get, $WS_\theta(\mathcal{I})-\lim x_{k(j)} = x$. So, for each $\epsilon > 0$, each $\gamma > 0$ and each $\varphi \in X^*$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |k \in I_r : |\varphi(x_k - x)| \geq \epsilon| \geq \gamma\right\} \in \mathcal{I}$$

or

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |k \in I_r : |\varphi(x_k - x)| \geq \epsilon| < \gamma\right\} \in F(\mathcal{I}). \tag{3.2.3}$$

Using (3.2.2) and (3.2.4) in (3.2.3), we get,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| < \epsilon\}| \geq \gamma\right\} \in F(\mathcal{I}).$$

i.e.

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |\varphi(x_k - x)| < \epsilon\}| \geq \gamma\right\} \notin \mathcal{I}.$$

This shows that $x \in \Gamma(WS_\theta(\mathcal{I}), x)$ and therefore, we have the containment, $\Lambda(WS_\theta(\mathcal{I}), x) \subseteq \Gamma(WS_\theta(\mathcal{I}), x)$. ■

THEOREM 3.2.2.2 *Let $\theta = (k_r)$ be a lacunary sequence. For any sequence $x = (x_k)$ in a Banach space X, if $WS_\theta(\mathcal{I}) - \lim_k x_k = x_0$, then $\Lambda\,(WS_\theta(\mathcal{I}), x) = \Gamma\,(WS_\theta(\mathcal{I}), x) = \{x_0\}$.*

Proof We first show that $\Lambda\,(WS_\theta(\mathcal{I}), x) = \{x_0\}$. Let $\Lambda\,(WS_\theta(\mathcal{I}), x) = \{x_0, y_0\}$ such that $x_0 \neq y_0$. By definition there exist two $\theta(\mathcal{I})$-nonthin subsequences $\left(x_{k(i)}\right)$

and $\left(x_{l(j)}\right)$ of the sequence $x = (x_k)$ which are respectively weakly convergent to x_0 and y_0. Choose $\epsilon > 0$ such that $0 < \epsilon < \frac{|\varphi(x_0)-\varphi(y_0)|}{2}$. Since $\left(x_{l(j)}\right)$ weakly converges to y_0, therefore, by the same lines as in THEOREM *3.2.2.1*, one can easily see that $WS_\theta(\mathcal{I})-\lim x_{l(j)} = y_0$. Moreover, we can also write,

$$\{l\,(j) \in I_r : j \in \mathbb{N}\} = \left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \cup \left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\};$$

which implies,

$$\frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| = \frac{1}{h_r}\left|\{l\,(j) \in I_r : j \in \mathbb{N}\}\right| - \frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right|.$$

Now, for $\epsilon > 0$ and $\gamma > 0$, the set,

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \geq \gamma\right\}$$

$$= \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{l\,(j) \in I_r : j \in \mathbb{N}\}\right| \geq \gamma\right\} - \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right| \geq \gamma\right\}$$

$$= \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{l\,(j) \in I_r : j \in \mathbb{N}\}\right| \geq \gamma\right\} \cap \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right| \geq \gamma\right\}^c$$

Since $(l(j))$ is $\theta(\mathcal{I})-$nonthin subsequence and $WS_\theta(\mathcal{I})-\lim x_{l(j)} = y_0$, therefore,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{l\,(j) \in I_r : j \in \mathbb{N}\}\right| \geq \gamma\right\} \cap \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|f(x_{l(j)} - y_0)\right| \geq \epsilon\right\}\right| \geq \gamma\right\}^c$$

lies in $F(\mathcal{I})$. This gives,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \geq \gamma\right\} \in F(\mathcal{I}). \tag{3.2.4}$$

Also $WS_\theta(\mathcal{I}) - \lim_k x_k = x_0$, we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\{l\,(j) \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\}\right| \geq \gamma\right\} \in \mathcal{I}. \tag{3.2.5}$$

For $0 < 2\epsilon < |\varphi(x_0 - y_0)|$,

$\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \cap \{k \in I_r : |\varphi(x_k - x_0)| < \epsilon\} = \emptyset$. So, we have,

$$\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\} \subseteq \{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\};$$

which implies,

$$\frac{1}{h_r} \left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \leq \frac{1}{h_r} \left|\{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\}\right|.$$

For $\epsilon > 0$, $\gamma > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \geq \gamma\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\{k \in I_r : |\varphi(x_k - x_0)| \geq \epsilon\}\right| \geq \gamma\right\}.$$

Using (3.2.6) we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{l\,(j) \in I_r : \left|\varphi(x_{l(j)} - y_0)\right| < \epsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I};$$

which contradicts (3.2.5). Hence, $\Lambda\left(WS_\theta(\mathcal{I}), x\right) = \{x_0\}$. Similarly, we can show that, $\Gamma\left(WS_\theta(\mathcal{I}), x\right) = \{x_0\}$. ■

THEOREM 3.2.2.3 *Let $\theta = (k_r)$ be a lacunary sequence. If $x = (x_k)$ and $y = (y_k)$ are two sequences in X such that $\lim_r \frac{1}{h_r}\ |\{k \in I_r : x_k \neq y_k\}| = 0$, then $\Lambda\left(WS_\theta(\mathcal{I}), x\right)$ $= \Lambda\left(WS_\theta(\mathcal{I}), y\right)$ and $\Gamma\left(WS_\theta(\mathcal{I}), x\right) = \Gamma\left(WS_\theta(\mathcal{I}), y\right)$.*

Proof Suppose $x' \in \Lambda\left(WS_\theta(\mathcal{I}), x\right)$, then there exists a $\theta(\mathcal{I})-$nonthin subsequence $(x)_K$ of the sequence $x = (x_k)$ that weakly converges to x'. *i.e.* If $K = \{k\,(j) : j \in \mathbb{N}\}$, then for every $\gamma > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\{k\,(j) \in I_r : j \in \mathbb{N}\}\right| \geq \gamma\right\} \notin \mathcal{I} \text{ and } \lim_k \varphi(x_{k(j)} - x') = 0.$$

Since, $\lim_r \frac{1}{h_r} |\{k \in I_r : k \in K, x_k \neq y_k\}| = 0$, it follows that,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : k \in K, x_k = y_k\}| > 0 \tag{3.2.6}$$

Therefore, there exists a $\theta(\mathcal{I})-$nonthin subsequence $(y)_K$ of the sequence $y = (y_k)$ that weakly converges to $x^{'}$. This shows that $x^{'} \in \Lambda(WS_\theta(\mathcal{I}), y)$ and therefore, $\Lambda(WS_\theta(\mathcal{I}), x) \subseteq \Lambda(WS_\theta(\mathcal{I}), y)$.
By symmetry, we have, $\Lambda(WS_\theta(\mathcal{I}), y) \subseteq \Lambda(WS_\theta(\mathcal{I}), x)$. Hence, we have,

$\Lambda(WS_\theta(\mathcal{I}), x) = \Lambda(WS_\theta(\mathcal{I}), y)$. Similarly, we can prove,

$\Gamma(WS_\theta(\mathcal{I}), x) = \Gamma(WS_\theta(\mathcal{I}), y)$. ■

3.3 Conclusion

The idea of weak lacunary statistical convergence was given by Nuray (119). We have generalized this idea with the help of lacunary sequences and a non trivial admissible ideal $\mathcal{I}$. The resulting convergence method is more general than weak lacunary statistical convergence. Also, for particular choice of the ideal $\mathcal{I}$ this generalized form coincides with the ordinary idea of (119). ■