# Chapter 2

# $WS_\lambda(\mathcal{I})-$Convergence

In this chapter, we have introduced and studied some generalized weak convergence: $WS_\lambda-$ convergence, $WS_\lambda-$ Cauchy and $W(V,\lambda)-$ summability with the help of de la Vallée-Poussin mean. We establish some of their properties and obtained relevant connections between these notions. We provide some concrete examples which show that our methods of weak convergence are more general on these spaces. Apart from this, for an admissible ideal $\mathcal{I}$, some stronger notions namely $WS(\mathcal{I})-$ convergence, $WS(\mathcal{I})-$ Cauchy, $WS(I^*)-$ convergence, $WS(I^*)-$ Cauchy, $WS_\lambda(\mathcal{I})-$ convergence and $W[V,\lambda](\mathcal{I})-$ summability are also introduced and developed.

## 2.1 Weak $\lambda-$Statistical Convergence

For a normed linear space $X$, $\lambda = (\lambda_n)$ be a non-decreasing sequence tending to $\infty$ with $\lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 1$ and let $I_n = [n - \lambda_n + 1, n]$, we present the following definitions.

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***DEFINITION 2.1.1*** *A sequence $(x_k)$ in $X$ is said to be strongly $\lambda-$statistically convergent ($S_\lambda$-convergent) to $x \in X$ if for every $\epsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \|x_k - x\| \geq \epsilon\}| = 0.$$

*In this case, we write $S_\lambda - \lim_k x_k = x$ or $x_k \xrightarrow{S_\lambda} x$ as $k \to \infty$.*

***DEFINITION 2.1.2*** *A sequence $(x_k)$ in $X$ is said to be weakly $\lambda$-statistically convergent ($WS_\lambda-$convergent) to $x \in X$ if for every $\epsilon > 0$*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| = 0,$$

*for every $\varphi \in X^*$. In this case, we write $WS_\lambda - \lim_k x_k = x$ or $x_k \xrightarrow{WS_\lambda} x$ as $k \to \infty$.*

***DEFINITION 2.1.3*** *A sequence $(x_k)$ in $X$ is said to be weakly $\lambda-$statistically Cauchy sequence if for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x_N)| \geq \epsilon\}| = 0,$$

*for every $\varphi \in X^*$.*

***DEFINITION 2.1.4*** *A sequence $(x_k)$ in $X$ is said to be weakly $(V, \lambda)-$summable to $x \in X$ provided that*

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| = 0,$$

*for each $\varphi \in X^*$. In this case, we write $W[V, \lambda] - \lim_k x_k = x$ or $x_k \xrightarrow{W[V,\lambda]} x$ as $k \to \infty$.*

Let $S_\lambda(X)$, $WS_\lambda(X)$ and $W[V, \lambda](X)$ respectively denote the sets of all strongly

$\lambda$-statistically convergent, weakly $\lambda-$ statistically convergent and weakly $(V, \lambda)-$ summable sequences in $X$.

For the particular choice $(\lambda_n) = (n)$, *DEFINITION 2.1.1*, *DEFINITION 2.1.2* and *DEFINITION 2.1.4* coincide with norm statistical convergence of (117), weak statistical convergence of (118) and weak $C_1-$ summability of (119) respectively.

Next, we establish some properties of weak $\lambda$-statistical convergence. We start with its uniqueness.

***THEOREM 2.1.1*** *For any sequence $(x_k)$ in $X$, if $x_k \xrightarrow{WS_\lambda} x$, then $x$ must be unique.*
***Proof*** Suppose there exists $x, y \in X$ such that $x_k \xrightarrow{WS_\lambda} x$ and $x_k \xrightarrow{WS_\lambda} y$; which follows, for any $\varphi \in X^*$,

$$\varphi(x_k) \xrightarrow{S_\lambda} \varphi(x) \text{ and } \varphi(x_k) \xrightarrow{S_\lambda} \varphi(y).$$

But, then, uniqueness of $S_\lambda-$limit of a sequence of scalars immediately implies $\varphi(x) = \varphi(y)$; and so by linearity of $\varphi$ one have $\varphi(x - y) = 0$. Let, if possible, $x \neq y$ then $x - y \neq 0$ and therefore, by one of the consequences of Hahn Banach theorem there exists $\varphi \in X^*$ such that $\varphi(x - y) = \|x - y\|$ and $\|\varphi\| = 1$. Since $\|x - y\| \neq 0$, it follows that $\varphi(x - y) \neq 0$ and therefore, we obtain a contradiction as $\varphi(x - y) = 0$. Hence $x = y$.■

***THEOREM 2.1.2*** *Let $(x_k)$ and $(y_k)$ be the sequences in $X$ and $c$ be a scalar.*

*(i) If $WS_\lambda - \lim_k x_k = x$, then $WS_\lambda - \lim_k cx_k = cx$.*

*(ii) If $WS_\lambda - \lim_k x_k = x$ and $WS_\lambda - \lim_k y_k = y$ where $x, y \in X$, then $WS_\lambda - \lim_k (x_k + y_k) = (x + y)$.*

The proof of the $THEOREM$ goes on parallel lines as in *THEOREM 2.1.1*.■

In our next result, we present another important characterization of weak $\lambda$-statistical convergence.

***THEOREM 2.1.3*** *A sequence $(x_k)$ in $X$ is weakly $\lambda$-statistically convergent to $x$, if and only if, there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and $\lim_{k\in K} \varphi(x_k - x) = 0$ for each $\varphi \in X^*$.*

***Proof*** We first suppose that $WS_\lambda - \lim_k x_k = x$. Let $\epsilon > 0$ and $\varphi \in X^*$ be arbitrary. Define, $M_p := \left\{k \in I_n : |\varphi(x_k - x)| \geq \frac{1}{p}\right\}$ and $K_p := \left\{k \in I_n : |\varphi(x_k - x)| < \frac{1}{p}\right\}$ for $p = 1, 2, \cdots$. Then one can verify that $\delta_\lambda(M_p) = 0$. Also,

$$K_1 \supset K_2 \supset \cdots \supset K_i \supset K_{i+1} \supset \cdots \tag{2.1.1}$$

and

$$\delta_\lambda(K_p) = 1, \;\; p = 1, 2, \cdots . \tag{2.1.2}$$

Now, to prove the result it is sufficient to prove that $\lim_{k\in K_p} \varphi(x_k - x) = 0$. Suppose $\varphi(x_k)$ does not converge to $\varphi(x)$ over $K_p$. So, there exists $\epsilon > 0$ for which $|\varphi(x_k - x)| \geq \epsilon$ for infinitely many terms. If we take $K_\epsilon := \{k \in I_n : |\varphi(x_k - x)| < \epsilon\}$ and $\epsilon > \frac{1}{p}$, $(p = 1, 2, \cdots)$, then $\delta_\lambda(K_\epsilon) = 0$ and by (2.2.1), $K_p \subset K_\epsilon$. But then $\delta_\lambda(K_p) = 0$, which contradicts (2.2.2) and therefore we have $\lim_{k\in K_p} \varphi(x_k - x) = 0$ for each $\varphi \in X^*$, as $\varphi \in X^*$ was arbitrary.

Conversely, suppose that there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ with $\delta_\lambda(K) = 1$ and $\lim_{k\in K} \varphi(x_k - x) = 0$ for every $\varphi \in X^*$. So, we can find a positive integer $N$ such that $|\varphi(x_k - x)| < \epsilon$ for all $k \geq N$, $k \in K$ and each $\varphi \in X^*$. If we define $M_\epsilon := \{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\} \subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, k_{N+3} \cdots\}$ then $\delta_\lambda(M_\epsilon) = 0$. This shows that $(x_k)$ is weakly $\lambda$-statistically convergent to $x$. ■

***THEOREM 2.1.4*** *For any sequence $(x_k)$ in $X$, if $W - \lim_k x_k = x$, then $WS_\lambda -$*

$\lim_k x_k = x$, *however, converse need not be true in general.*

***Proof*** Let $W - \lim_k x_k = x$, then for every $\epsilon > 0$ and each $\varphi \in X^*$, there exists a positive integer $N$ such that

$$|\varphi(x_k - x)| < \epsilon$$

for all $k \geq N$. Thus the set $M(\epsilon) := \{k \in \mathbb{N} : |\varphi(x_k - x)| \geq \epsilon\}$ is finite; for which $\delta_\lambda(M(\epsilon)) = 0$. This shows that $WS_\lambda - \lim_k x_k = x$.

The converse of above result is not true in general and can be seen from the following EXAMPLE.

***EXAMPLE 2.1.1*** Let $(x_n) \in l_p$ with $1 < p < \infty$ be defined by

$$x_n(j) = \begin{cases} \sqrt[3]{n}, & \text{if } j \in I_n \text{ and } n = m^3; \\ 0, & \text{if } j \notin I_n \text{ and } n = m^3; \\ \frac{1}{n}, & \text{if } j \in I_n \text{ and } n \neq m^3; \\ 0, & \text{if } j \notin I_n \text{ and } n \neq m^3; \end{cases}$$

where $I_n = [n - \lambda_n + 1, n]$. Let $K = \{n \in \mathbb{N} : n \neq m^3\}$, then for $n \in K$ and any $\varphi \in l_p^*$ there is a unique $y \in l_q$ such that

$$|\varphi(x_n)| = \left|\sum_{j=1}^{\infty} x_n(j)\ y(j)\right| \leq \|x\|_p\ \|y\|_q \quad \text{(by Holder inequality)}$$

$$= \left(\sum_{j=1}^{\infty} |x_n(j)|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} |y(j)|^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{j\in I_n} |x_n(j)|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} |y(j)|^q\right)^{\frac{1}{q}} \text{(by structure of the sequence)}$$

$$= \left(\sum_{j\in I_n} \frac{1}{n^p}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} |y(j)|^q\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{\lambda_n}{n^p}\right)^{\frac{1}{p}} K^{\frac{1}{q}} \quad \text{for some positive constant K}$$

$$< \frac{K^{\frac{1}{q}}}{n^{1-\frac{1}{p}}} \to 0 \text{ as } n \to \infty.$$

Using *THEOREM 2.1.3*, we have $WS_\lambda - \lim_{n\to\infty} x_n = 0$.

For $n \in K^c$, Consider the functional defined by $\varphi_j(x) = x(j)$, where $x = (x_n) \in l_p$. Clearly, $\varphi_j(x_n) = x_n(j) = \sqrt[3]{n} \to \infty$ as $n \to \infty$. Hence $(x_k)$ is not weakly convergent to 0. ∎

***THEOREM 2.1.5*** *For a normed linear space $X$, strong $\lambda$-statistical convergence implies weak $\lambda$-statistical convergence with same limit but the converse is not true in general.*

***Proof*** Let $(x_k)$ in $X$, be a sequence such that $x_k \xrightarrow{S_\lambda} x$. Then for every $\epsilon > 0$,

$$\frac{1}{\lambda_n}|\{k \in I_n : \|x_k - x\| \geq \epsilon\}| = 0. \tag{2.1.3}$$

Now, for every $\epsilon > 0$ and each $\varphi \in X^*$,

$$\frac{1}{\lambda_n}\,|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \leq \frac{1}{\lambda_n}\,|\{k \in I_n : \|\varphi\|\ \|x_k - x\| \geq \epsilon\}|$$

$$= \frac{1}{\lambda_n}\left|\left\{k \in I_n : \|x_k - x\| \geq \frac{\epsilon}{\|\varphi\|}\right\}\right|.$$

Using (2.1.3), we have $\frac{1}{\lambda_n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| = 0$, but the converse of the above result is not true in general as demonstrated by the following EXAMPLE.

***EXAMPLE 2.1.2*** Consider the Hilbert space $L_2(0, 2\Pi)$ (the space of square integrable functions on the interval $(0, 2\Pi)$) as $X$ and $X^*$ be its dual space. Using Riesz representation theorem, we observe that for any $\varphi \in X^*$ there exists some $h \in X$ such that for any $x \in X$,

$$\varphi(x) = \langle x, h \rangle = \int_0^{2\Pi} x(t)h(t)\ dt.$$

Let the sequence $(x_k)$ in $X$ is defined as,

$$x_k(t) = \sin kt \text{ for } k = 1, 2, \cdots .$$

Now, we claim that $(x_k) \overset{WS_\lambda}{\to} 0$, but $(x_k)$ is not strong $\lambda-$ statistically convergent to 0. For $\varphi \in X^*$,

$$\varphi(x_k) = \int_0^{2\Pi} \sin kt\ h(t) dt.$$

According to Riemann Lebesgue lemma, $\varphi(x_k) \to 0$ for each $\varphi \in X^*$, *i.e.*, $x_k \overset{W}{\to} 0$. Using *THEOREM 2.1.4*, we have

$$x_k \overset{WS_\lambda}{\to} x.$$

Next to prove $(x_k)$ is not strongly $\lambda-$ statistically convergent to 0. Let if possible, $(x_k)$ is strongly $\lambda-$ statistically convergent to 0. Then for each $\epsilon > 0$, $\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|x_k - L\| \geq \epsilon\}| = 0$. But

$$\|x_k - 0\|^2 = \langle x_k, x_k \rangle = \int_0^{2\Pi} \sin^2 kt\ dt = \pi$$

for all $k$. This shows that $\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \neq 0$. Hence $(x_k)$ is not strongly $\lambda-$ statistically convergent to 0. ■

Our next THEOREM demonstrates the indistinguishability of strong $\lambda-$ statistical convergence and weak $\lambda-$ statistical convergence in finite dimensional spaces.

***THEOREM 2.1.6*** *For a finite dimensional normed space $X$, strong $\lambda-$ statistical convergence is equivalent to weak $\lambda-$ statistical convergence.*

***Proof*** Since we have shown that strong $\lambda-$ statistical convergence implies weak $\lambda-$ statistical convergence in a normed space $X$, therefore it is sufficient to prove that weak $\lambda-$ statistical convergence implies strong $\lambda-$ statistical convergence. Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be a basis for $X$ and $x_k \overset{WS_\lambda}{\rightarrow} x$, where

$$x_k = a_1^k e_1 + a_2^k e_2 + a_3^k e_3 + \cdots + a_n^k e_n \text{ for } k = 1, 2, 3, \ldots,$$

and

$$x = a_1 e_1 + a_2 e_2 + a_3 e_3 + \cdots + a_n e_n.$$

Consider the linear functionals $\varphi_i \in X^*(i = 1, 2, \ldots, n)$ defined as follows:

$$\varphi_i(e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Since $x_k \overset{WS_\lambda}{\rightarrow} x$, it follows that $\varphi_i(x_k) \overset{S_\lambda}{\rightarrow} \varphi_i(x)$. By definition of $\varphi_i$, $\varphi_i(x_k) = a_i^{(k)}$ and $\varphi_i(x) = a_i$. So, we have $a_i^{(k)} \overset{S_\lambda}{\rightarrow} a_i$. This implies for every $\epsilon > 0$,

$$\frac{1}{\lambda_n} |\{k \in I_n : |a_i^{(k)} - a_i| \geq \epsilon\}| = 0 \tag{2.1.4}$$

for $i = 1, 2, 3, \cdots, n$. Also,

$$\|x_k - x\| = \left\| \sum_{i=1}^{n} (a_i^{(k)} - a_i) e_i \right\| \leq \sum_{i=1}^{n} \left| a_i^{(k)} - a_i \right| \ \|e_i\| \leq M \sum_{i=1}^{n} \left| a_i^{(k)} - a_i \right|$$

where $M = \max_i \|e_i\|$, Thus for every $\epsilon > 0$,

$$\{k \in I_n : \|x_k - x\| \geq \epsilon\} \subset \left\{ k \in I_n : |a_1^{(k)} - a_1| \geq \frac{\epsilon}{M} \right\}$$
$$\cup \ldots \cup \left\{ k \in I_n : |a_n^{(k)} - a_n| \geq \frac{\epsilon}{M} \right\}.$$

Using (2.1.4), $(x_k)$ is strongly $\lambda-$ statistically convergent to $x \in X$. This completes the result. ■

***THEOREM 2.1.7*** *For a complete normed linear space, $(x_k)$ is weakly $\lambda-$statistically convergent if and only if $(x_k)$ is weakly $\lambda-$statistically Cauchy.*

***Proof*** Let $(x_k)$ be weakly $\lambda-$ statistically convergent to $x \in X$. Then for each $\epsilon > 0$, each $\varphi \in X^*$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \varphi(x_k - x) \geq \epsilon\}| = 0.$$

Choose a number $N = N(\epsilon)$, such that $\varphi(x_N - x) \geq \epsilon$. Now, let

$$M(\epsilon) := \{k \in I_n : \varphi(x_k - x_N) \geq \epsilon\};$$
$$K_1(\epsilon) := \{k \in I_n : \varphi(x_k - x) \geq \epsilon\};$$
$$K_2(\epsilon) := \{k = N \in I_n : \varphi(x_N - x) \geq \epsilon\}.$$

Then, $M(\epsilon) \subseteq K_1(\epsilon) \cup K_2(\epsilon)$. This gives $\delta_\lambda(M(\epsilon)) \leq \delta_\lambda(K_1(\epsilon)) + \delta_\lambda(K_2(\epsilon))$. Hence $(x_k)$ is weakly $\lambda-$statistically Cauchy.

Conversely, suppose that $(x_k)$ is weakly $\lambda-$statistically Cauchy but not weakly $\lambda-$statistically convergent. Then there exists a positive integer $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \varphi(x_k - x_N) \geq \epsilon\}| = 0,$$

*i.e.*,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \varphi(x_k - x_N) < \epsilon\}| = 1.$$

In particular, we can write $|\varphi(x_k - x_N)| \leq 2\,|\varphi(x_k - x)| < \epsilon$ if $|\varphi(x_k - x)| < \frac{\epsilon}{2}$. Since $(x_k)$ is not weak $\lambda-$statistically convergent, therefore

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \varphi(x_k - x) < \epsilon\}| = 0.$$

This immediately gives,

$$\lim_n \frac{1}{\lambda_n} \left|\{k \in I_n : \varphi(x_k - x_N) < \epsilon\}\right| = 0,$$

which leads to a contradiction. Hence $(x_k)$ is weakly $\lambda-$ statistically convergent. ■

Our next result shows the relation between weak $(V, \lambda)-$ summability and weak $\lambda-$statistical convergence.

***THEOREM 2.1.8*** *For a sequence $(x_k)$ in $X$, $(x_k)$ is weakly $(V, \lambda)-$summable to $x$, if and only if, $(x_k)$ is weakly $\lambda-$statistically convergent to $x$.*

***Proof*** Let $(x_k)$ is weakly $(V, \lambda)-$ summable to $x$. Then for each $\varphi \in X^*$, $\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| = 0$. Now,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| \geq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\varphi(x_k - x)| \geq \epsilon}} |\varphi(x_k - x)| \geq \frac{\epsilon}{\lambda_n} \left|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}\right|.$$

This implies $(x_k)$ is weak $\lambda-$statistically convergent to $x$.

Conversely, suppose that $(x_k)$ is weakly $\lambda-$statistically convergent to $x$. Since $\varphi \in X^*$, $\varphi$ is bounded, $|\varphi(x_k - x)| \leq M$(say) for all $k$. For $\epsilon > 0$,

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\varphi(x_k - x)| \geq \epsilon}} |\varphi(x_k - x)| + \sum_{\substack{k \in I_n \\ |\varphi(x_k - x)| < \epsilon}} |\varphi(x_k - x)| \\ &\leq \frac{M}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| + \epsilon, \end{aligned}$$

which implies that $(x_k)$ is weakly $(V, \lambda)-$ summable to $x$. ■

The next result reveals the relation between weak statistical convergence and weak $\lambda$- statistical convergence.

One can easily see that $WS_\lambda(X) \subset WS(X)$, since $\lambda_n \leq n$. To show the reverse relation, we give a necessary and sufficient condition as follows.

***THEOREM 2.1.9*** *For a normed linear space $X$, $WS(X) \subset WS_\lambda(X)$, if and only if,* $\liminf \frac{\lambda_n}{n} > 0$.

***Proof*** Firstly, suppose that $\liminf \frac{\lambda_n}{n} > 0$ and $x_k \overset{WS}{\to} x$ as $k \to \infty$. Then for every $\epsilon > 0$ and each $\varphi \in X^*$,

$$\begin{aligned}\frac{1}{n}\left|\{k \le n : |\varphi(x_k - x)| \ge \epsilon\}\right| &\ge \frac{1}{n}\left|\{k \in I_n : |\varphi(x_k - x)| \ge \epsilon\}\right| \\ &= \left(\frac{\lambda_n}{n}\right)\frac{1}{\lambda_n}\left|\{k \in I_n : |\varphi(x_k - x)| \ge \epsilon\}\right|.\end{aligned}$$

Taking the limit as $n \to \infty$ and using given condition, we have $x_k \overset{WS_\lambda}{\longrightarrow} x$.

Conversely, suppose that $WS(X) \subset WS_\lambda(X)$ and $\liminf \frac{\lambda_n}{n} = 0$. Then as in [], we can choose a subsequence $(n(j))_{j=1}^{\infty}$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$,
Define a sequence $(x_k)$ by

$$x_k = \begin{cases} p, & \text{if } i \in I_{n(j)},\ j = 1, 2, 3, \cdots, \\ \theta, & \text{otherwise;} \end{cases}$$

where $p \in X$ with $\|p\| = 1$ and $\theta$ is the zero element of $X$. Then $(x_k)$ is norm statistically convergent which implies $(x_k)$ is weak statistically convergent.

Also, for any $x \in X$, we have

$$\begin{aligned}\frac{1}{\lambda_{n(j)}}\sum_{k \in I_{n(j)}}|\varphi(x_k - x)| &= \frac{1}{\lambda_{n(j)}}\sum_{k \in I_{n(j)}}|\varphi(p - x)| \le M \,\text{for}\, j = 1, 2, \cdots \\ \text{and}\frac{1}{\lambda_n}\sum_{k \in I_n}|\varphi(x_k - x)| &= \frac{1}{\lambda_n}\sum_{k \in I_n}|\varphi(-x)| \le M \,\text{for}\, n \ne n_j\end{aligned}$$

which gives $(x_k) \notin W[V, \lambda](X)$. By *THEOREM 2.2.8*, we have $(x_k) \notin WS_\lambda(X)$. This leads to a contradiction. Hence $\liminf \frac{\lambda_n}{n} > 0$. ■

## 2.2 Weak $\mathcal{I}-$Statistical Convergence

For a non-trivial admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$, the aim of the present section is to introduce and develop $WS(\mathcal{I})-$ convergence, $WS(\mathcal{I})-$ Cauchy, $WS(I^*)-$ convergence and $WS(I^*)-$ Cauchy in a normed linear space $X$. We begin with the following definitions.

***DEFINITION 2.2.1*** *A sequence $(x_k)$ in $X$ is said to be norm statistically convergent to $x \in X$ with respect to the ideal $\mathcal{I}$ (or $S(\mathcal{I})-$ convergent) if for every $\epsilon > 0$ and every $\gamma > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - x)\| \geq \epsilon\}| \geq \gamma\right\} \in \mathcal{I}.$$

*In this case, we write $S(\mathcal{I}) - \lim_k x_k = x$ or $x_k \stackrel{S(\mathcal{I})}{\longrightarrow} x$.*

Let $S(\mathcal{I}, X)$ denotes the set of all strongly $S(\mathcal{I})-$convergent sequences in normed linear space $X$.

***DEFINITION 2.2.2*** *A sequence $(x_k)$ in $X$ is said to be weakly statistically convergent to $x \in X$ with respect to the ideal $\mathcal{I}$ (or $WS(\mathcal{I})-$convergent) if $\varphi(x_k) \stackrel{S(\mathcal{I})}{\longrightarrow} \varphi(x)$ for every $\varphi \in X^*$. This means that for every $\epsilon > 0$ and every $\gamma > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \in \mathcal{I},$$

*for every $\varphi \in X^*$. In this case, we write $WS(\mathcal{I}) - \lim_k x_k = x$ or $x_k \stackrel{WS(\mathcal{I})}{\longrightarrow} x$.*

Let $WS(\mathcal{I}, X)$ denotes the set of all $WS(\mathcal{I})-$convergent sequences in $X$.

For $\mathcal{I} = \mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is finite}\}$, *DEFINITION 2.2.1* and *DEFINITION 2.2.2* respectively coincide with norm statistical convergence and weak statistical convergence.

Further, we would like to establish some properties of $WS(\mathcal{I})-$ convergence. We begin with its uniqueness.

***THEOREM 2.2.1*** *For any sequence* $(x_k)$ *in* $X$*, if* $x_k \stackrel{WS(\mathcal{I})}{\longrightarrow} x$*, then* $x$ *is unique.*
***Proof*** Suppose there exists $x, y \in X$ such that $x_k \stackrel{WS(\mathcal{I})}{\longrightarrow} x$ and $x_k \stackrel{WS(\mathcal{I})}{\longrightarrow} y$; which follows, for any $\varphi \in X^*$,

$$\varphi(x_k) \stackrel{S(\mathcal{I})}{\longrightarrow} \varphi(x) \text{ and } \varphi(x_k) \stackrel{S(\mathcal{I})}{\longrightarrow} \varphi(y).$$

Since $S(\mathcal{I})-$convergence of scalars always leads to a unique limit, therefore, by above assumption, one has $\varphi(x) = \varphi(y)$; which implies immediately by linearity of $\varphi$ that $\varphi(x - y) = 0$. Let, if possible, $x \neq y$ then $x - y \neq 0$ and therefore, by one of consequences of Hahn Banach Theorem, there exists $\varphi \in X^*$ such that $\varphi(x - y) = \|x - y\|$ and $\|\varphi\| = 1$. Since $\|x - y\| \neq 0$, it follows that $\varphi(x - y) \neq 0$ and therefore we obtain a contradiction to $\varphi(x - y) = 0$. Hence, $x = y$. ■

***THEOREM 2.2.2*** *Let* $(x_k)$ *and* $(y_k)$ *be the sequences in* $X$ *and* $c$ *being any scalar.*

*(i) If* $WS(\mathcal{I}) - \lim_k x_k = x$*, then* $WS(\mathcal{I}) - \lim_k cx_k = cx$*.*

*(ii) If* $WS(\mathcal{I}) - \lim_k x_k = x$ *and* $WS(\mathcal{I}) - \lim_k y_k = y$ *where* $x, y \in X$*, then* $WS(\mathcal{I}) - \lim_k (x_k + y_k) = (x + y)$*.*

The proof of the *THEOREM* follows parallel lines as in *THEOREM 2.2.1*. ■

***THEOREM 2.2.3*** *For any sequence* $(x_k)$ *in* $X$*, if* $WS - \lim_k x_k = x$*, then* $WS(\mathcal{I}) - \lim_k x_k = x$*, however, converse need not be true in general.*
***Proof*** Let $WS - \lim_k x_k = x$, then for every $\epsilon > 0$ and each $\varphi \in X^*$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| = 0.$$

So, for each $\gamma > 0$, there exists a positive integer $N$ such that

$$\frac{1}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| < \gamma \text{ for all } k \geq N$$

and therefore, we have,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| \geq \gamma\right\} \subseteq \{1, 2, \ldots, N-1\}.$$

Now, the result follows immediately by the admissibility of the ideal $\mathcal{I}$.

The converse of above result is not true in general as it can be seen from the *REMARK 2.2.1.* ■

***THEOREM 2.2.4*** *$S(\mathcal{I})-$convergence implies $WS(\mathcal{I})-$convergence to the same limit in $X$, but the converse need not be true in general.*

***Proof*** For $x_k \xrightarrow{S(\mathcal{I})} x$, we have for every $\epsilon > 0$ and each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|x_k - x)\| \geq \epsilon\}\right| \geq \gamma\right\} \in \mathcal{I}. \tag{2.2.1}$$

For each $\varphi \in X^*$,

$$\begin{aligned} \frac{1}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| &\leq \frac{1}{n}\left|\{k \leq n : \|\varphi\| \, \|x_k - x\| \geq \epsilon\}\right| \\ &= \frac{1}{n}\left|\left\{k \leq n : \|x_k - x\| \geq \frac{\epsilon}{\|\varphi\|}\right\}\right|; \end{aligned}$$

which gives, for each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \|x_k - x\| \geq \frac{\epsilon}{\|\varphi\|}\right\}\right| \geq \gamma\right\},$$

and therefore, the result follows using (2.2.1). We next give an EXAMPLE to show that the converse of the above result is not true in general.

***EXAMPLE 2.2.1*** Consider the Hilbert space $L_2(0, 2\pi)$ (the space of square integrable functions on the interval $(0, 2\pi)$) by $X$ and $X^*$ be its dual space. By Riesz representation theorem, for any $\varphi \in X^*$, there exists some $h \in X$ such that,

$$\varphi(x) = \langle x, h \rangle = \int_0^{2\pi} x(t)\ h(t)\ dt,$$

for any $x \in X$. Define a sequence $(x_k)$ in $X$ by

$$x_k(t) = \sin kt \text{ for } k = 1, 2, \cdots .$$

By the use of Riemann Lebesgue lemma, we have $x_k \xrightarrow{W} 0$ and therefore, $x_k \xrightarrow{WS} 0$, as weak convergence implies weak statistical convergence. Hence, by *THEOREM 2.2.3*, we have, $x_k \xrightarrow{WS(\mathcal{I})} 0$.

Next, we show that $x_k \xrightarrow{S(\mathcal{I})} 0$ does not hold. Suppose $x_k \xrightarrow{S(\mathcal{I})} 0$ holds, then for each $\epsilon > 0$ and every $\gamma > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \|x_k - 0\| \geq \epsilon \} \right| \geq \gamma \right\} \in \mathcal{I},$$

or

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \|x_k - 0\| \geq \epsilon \} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

Choose $0 < \epsilon < 1$ and $0 < \gamma < 1$. Since $\emptyset \notin \mathcal{F}(\mathcal{I})$, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} \left| \{ k \leq n : \|x_k - 0\| \geq \epsilon \} \right| < \gamma < 1$, which gives immediately some $k \leq n$ such that

$$\|x_k - 0)\| < \epsilon \tag{2.2.2}$$

Now, the observation

$$\|x_k - 0\|^2 = \langle x_k, x_k \rangle = \int_0^{2\pi} \sin^2 kt\ dt = \pi$$

for all $k$, follows that, for $0 < \epsilon < 1$, there does not exist any $k \leq n$ for which $\|x_k - 0\| < \epsilon$. Thus, we obtain a contradiction to (2.2.2) and hence, $x_k \xrightarrow{S(\mathcal{I})} 0$ does not hold. ■

The next Theorem demonstrates the indistinguishability of $S(\mathcal{I})-$convergence and $WS(\mathcal{I})-$convergence on finite dimensional normed spaces.

***THEOREM 2.2.5*** *For a finite dimensional normed space $X$, $S(\mathcal{I})-$convergence is equivalent to $WS(\mathcal{I})-$convergence.*

***Proof*** In view of *THEOREM 2.2.4*, it is sufficient to prove that $WS(\mathcal{I})-$convergence implies $S(\mathcal{I})-$convergence. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for $X$ and $x_k \xrightarrow{WS(\mathcal{I})} x$, where

$$x_k = a_1^k e_1 + a_2^k e_2 + \ldots + a_n^k e_n \text{ for } k = 1, 2, \ldots, \quad \text{and}$$

$$x = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n.$$

Consider the linear functionals $\varphi_i \in X^*(i = 1, 2, \ldots, n)$ defined as follows:

$$\varphi_i(e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Since $x_k \xrightarrow{WS(\mathcal{I})} x$, it follows that $\varphi_i(x_k) \xrightarrow{S(\mathcal{I})} \varphi_i(x)$ for each $\varphi_i \in X^*(1 \leq i \leq n)$. This implies $a_i^{(k)} \xrightarrow{S(\mathcal{I})} a_i$ as $\varphi_i(x_k) = a_i^{(k)}$ and $\varphi_i(x) = a_i$.

Thus, for each $\epsilon > 0$ and each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |a_i^{(k)} - a_i| \geq \epsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I} \tag{2.2.3}$$

for $i = 1, 2, \ldots, n$. Further, we observe

$$\|x_k - x\| = \left\|\sum_{i=1}^{n}(a_i^{(k)} - a_i)e_i\right\| \leq \sum_{i=1}^{n}\left|a_i^{(k)} - a_i\right| \ \|e_i\| \leq M\sum_{i=1}^{n}\left|a_i^{(k)} - a_i\right|,$$

where $M = \max_i \|e_i\|$. So for each $\epsilon > 0$ and each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - x\| \geq \epsilon\}| \geq \gamma\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \sum_{i=1}^{n} |a_i^{(k)} - a_i| \geq \frac{\epsilon}{M}\right\}\right| \geq \gamma\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |a_1^{(k)} - a_1| \geq \frac{\epsilon}{M}\right\}\right| \geq \gamma\right\} \cup \ldots$$
$$\cup \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |a_n^{(k)} - a_n| \geq \frac{\epsilon}{M}\right\}\right| \geq \gamma\right\};$$

and therefore, the result follows by using (2.3.3). ■

We next define $WS(\mathcal{I})-$ Cauchy sequences and present Cauchy-convergence criterion for $WS(\mathcal{I})-$ convergence as follows.

***DEFINITION 2.2.4*** *A sequence $(x_k)$ in $X$ is said to be weakly statistically Cauchy with respect to the ideal $\mathcal{I}$ (or $WS(\mathcal{I})-$Cauchy) if there is a subsequence $(x_{k'(n)})$ of $(x_k)$ such that for each $n$, $W - \lim_n x_{k'(n)} = x$ and for every $\epsilon > 0$ and every $\gamma > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x_{k'(n)})| \geq \epsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I},$$

*for every $\varphi \in X^*$.*

***THEOREM 2.2.6*** *For a Banach space $X$, $(x_k)$ is $WS(\mathcal{I})-$convergent if and only if $(x_k)$ is $WS(\mathcal{I})-$Cauchy.*

***Proof*** Let $x_k \xrightarrow{WS(\mathcal{I})} x$, then for every $\epsilon > 0$, each $\gamma > 0$ and every $\varphi \in X^*$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2}\right\}\right| \geq \gamma\right\} \in \mathcal{I}, \text{ or}$$
$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2}\right\}\right| < \gamma\right\} \in \mathcal{F}(\mathcal{I}).$$

We first construct a subsequence $(x_{k'(n)})$ of $(x_k)$ such that for each $n$, $W - \lim_n x_{k'(n)} =$

$x$. For this, let $0 < \gamma < 1$ and for each $i \in \mathbb{N}$, we define,

$$M_i := \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2} \right\} \right| < \frac{1}{i} \right\}.$$

Clearly, $M_i \supseteq M_{i+1}$ and $M_i \in \mathcal{F}(\mathcal{I})$. As $\emptyset \notin \mathcal{F}(\mathcal{I})$, we have,

$$\frac{1}{n} \left| \left\{ k \leq n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2} \right\} \right| < \frac{1}{i}.$$

Choose $k_1$ such that $k_1 \leq n$, and

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k_1 \leq n : |\varphi(x_{k_1} - x)| \geq \frac{\epsilon}{2} \right\} \right| < 1 \right\} \in \mathcal{F}(\mathcal{I}).$$

Further, choose $k_2 > k_1$ such that $k_2 \leq n$, then

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k_2 \leq n : |\varphi(x_{k_2} - x)| \geq \frac{\epsilon}{2} \right\} \right| < \frac{1}{2} \right\} \in \mathcal{F}(\mathcal{I}).$$

Thus, for each $n$ satisfying $k_1 \leq n \leq k_2$, choose $k^{'}(n) \leq n$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k^{'}(n) \leq n : |\varphi(x_{k'(n)} - x)| \geq \frac{\epsilon}{2} \right\} \right| < 1 \right\} \in \mathcal{F}(\mathcal{I}).$$

In general, we choose $k_{p+1} > k_p$ such that $k_{p+1} \leq n$ and satisfying

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k_{p+1} \leq n : |\varphi(x_{k_{p+1}} - x)| \geq \frac{\epsilon}{2} \right\} \right| < \frac{1}{p} \right\} \in \mathcal{F}(\mathcal{I}).$$

Thus, for each $n$ satisfying $k_p \leq n \leq k_{p+1}$, we can choose $k^{'}(n) \leq n$, for which

$$\left| \varphi(x_{k'(n)} - x) \right| < \frac{\epsilon}{2}.$$

This shows that for each $n$, $W - \lim_n x_{k'(n)} = x$.

Next, to prove the another requirement, let $\epsilon > 0$. Further, we observe

$$\frac{1}{n} \left| \left\{ k \in I_n : |\varphi(x_k - x_{k'(n)})| \geq \epsilon \right\} \right| \leq \\ \frac{1}{n} \left| \left\{ k \in I_n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2} \right\} \right| + \frac{1}{n} \left| \left\{ k \in I_n : |\varphi(x_{k'(n)} - x)| \geq \frac{\epsilon}{2} \right\} \right|;$$

for each $\varphi \in X^*$, which gives for each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x_{k'(n)})| \geq \epsilon\right\}\right| \geq \gamma\right\} \subseteq$$
$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x)| \geq \frac{\epsilon}{2}\right\}\right| \geq \gamma\right\} \cup$$
$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_{k'(n)} - x_k)| \geq \frac{\epsilon}{2}\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

This shows, $(x_k)$ is $WS(\mathcal{I})-$Cauchy.

Conversely, suppose that $(x_k)$ is $WS(\mathcal{I})-$Cauchy. Let $\epsilon > 0$ be arbitrary, then, for each $\varphi \in X^*$, we can write

$$\frac{1}{n}\left|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}\right| \leq \frac{1}{n}\left|\left\{k \in I_n : |\varphi(x_k - x_{k'(n)})| \geq \frac{\epsilon}{2}\right\}\right|$$
$$+\frac{1}{n}\left|\left\{k \in I_n : |\varphi(x_{k'(n)} - x)| \geq \frac{\epsilon}{2}\right\}\right|;$$

which gives for each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}\right| \geq \gamma\right\} \subseteq$$
$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_k - x_{k'(n)})| \geq \frac{\epsilon}{2}\right\}\right| \geq \gamma\right\}$$
$$\cup\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |\varphi(x_{k'(n)} - x)| \geq \frac{\epsilon}{2}\right\}\right| \geq \gamma\right\}$$

Since, $(x_k)$ is $WS(\mathcal{I})-$Cauchy therefore, the union of last two members of above expression belongs to $\mathcal{I}$. Hence, $(x_k)$ is $WS(\mathcal{I})-$convergent. ■

In the subsequent part of this chapter, another kind of convergence is introduced which we call $WS(\mathcal{I}^*)-$ convergence. We also establish relation between $WS(\mathcal{I})-$ convergence and $WS(\mathcal{I}^*)-$ convergence.

***DEFINITION 2.2.5*** *A sequence $(x_k)$ in $X$ is said to be norm $\mathcal{I}^*-$ statistically convergent (or $S(\mathcal{I}^*)-$ convergent) to $x \in X$ if and only if there exists a set $M =$*

$\{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I})$ *such that for every* $\epsilon > 0$*,*

$$\lim_{k\to\infty} \frac{1}{m_k}|\{r \leq m_k : ||x_r - x|| \geq \epsilon\}| = 0.$$

*In this case, we write* $S(\mathcal{I}^*)-\lim_k x_k = x$ *or* $x_k \overset{S(\mathcal{I}^*)}{\longrightarrow} x$*.*

Let $S(\mathcal{I}^*,X)$ denotes the set of all $\mathcal{I}^*-$statistically convergent sequences in $X$.

***DEFINITION 2.2.6*** *A sequence* $(x_k)$ *in* $X$ *is said to be weakly* $\mathcal{I}^*-$*statistically convergent (or* $WS(\mathcal{I}^*)-$*convergent) to* $x \in X$ *if and only if there exists a set* $M = \{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I})$ *such that for every* $\epsilon > 0$ *and every* $\varphi \in X^*$*,*

$$\lim_{k\to\infty} \frac{1}{m_k}|\{r \leq m_k : |\varphi(x_r - x)| \geq \epsilon\}| = 0.$$

*In this case, we write* $WS(\mathcal{I}^*)-\lim_k x_k = x$ *or* $x_k \overset{WS(\mathcal{I}^*)}{\longrightarrow} x$*.*

Let $WS(\mathcal{I}^*,X)$ denotes the set of all weakly $\mathcal{I}^*-$ statistically convergent sequences in $X$.

***THEOREM 2.2.7*** *Let* $\mathcal{I}$ *be an admissible ideal. If* $WS(\mathcal{I}^*)-\lim_k x_k = x$*, then* $WS(\mathcal{I})-\lim_k x_k = x$*. If the ideal* $\mathcal{I}$ *satisfying the property* $(AP)$*, then* $WS(\mathcal{I})-$ *convergence implies* $WS(\mathcal{I}^*)-$ *convergence for any sequence* $(x_k)$ *in* $X$*.*

***Proof*** Let $WS(\mathcal{I}^*)-\lim_k x_k = x$, then there exists a set $M = \{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I})$ such that for every $\epsilon > 0$ and every $\varphi \in X^*$,

$$\lim_{k\to\infty} \frac{1}{m_k}|\{r \leq m_k : |\varphi(x_r - x)| \geq \epsilon\}| = 0.$$

It follows that, for every $\gamma > 0$, there exists a positive integer $N$ such that, $\frac{1}{m_k}|\{r \leq m_k : |\varphi(x_r - x)| \geq \epsilon\}| < \gamma$ for all $k \geq N$ and therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\} \geq \gamma\right\} \subseteq M^c \cup \{m_1, m_2, \ldots, m_{N-1}\} \quad (2.2.4)$$

Since $\mathcal{I}$ is admissible, the right part of the above equation belongs to $\mathcal{I}$. Therefore, $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\} \in \mathcal{I}$. Hence, $WS(\mathcal{I}) - \lim_k x_k = x$.

Next, suppose $WS(\mathcal{I}) - \lim_k x_k = x$, which means, for each $\varphi \in X^*$, $\varphi(x_k) \xrightarrow{S(\mathcal{I})} \varphi(x)$.

Obviously, the sequence $\frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| > \epsilon\}|$ is $\mathcal{I}-$convergent to 0. Since, the ideal $\mathcal{I}$ has property $(AP)$, the sequence $\frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| > \epsilon\}|$ is $\mathcal{I}^*-$convergent to 0. Therefore, $WS(\mathcal{I}^*)-\ \lim_k x_k = x$. ■

***DEFINITION 2.2.7*** *A sequence $(x_k)$ in $X$ is said to be weakly $\mathcal{I}^*-$ statistically Cauchy (or $WS(\mathcal{I}^*)-$ Cauchy) if there is a subsequence $(x_{k'(n)})$ of $(x_k)$, such that, for each n, $W - \lim_n x_{k'(n)} = x$ and there exists a set $M = \{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I})$, such that, for every $\epsilon > 0$ and every $\varphi \in X^*$,*

$$\lim_{k\to\infty} \frac{1}{m_k} \left|\left\{r \leq m_k : |\varphi(x_r - x_{k'(n)})| \geq \epsilon\right\}\right| = 0.$$

***THEOREM 2.2.8*** *If $(x_k)$ is $WS(\mathcal{I}^*)-$ Cauchy, then $(x_k)$ is $WS(\mathcal{I})-$ Cauchy. Furthermore, if $\mathcal{I}$ is an admissible ideal satisfying the property $(AP)$, then $WS(\mathcal{I})-$ Cauchy coincides with $WS(\mathcal{I}^*)-$ Cauchy for any sequence $(x_k)$ in $X$.*

The proof of the THEOREM follows parallel lines as in *THEOREM 2.2.7.*

## 2.3 Weak $\mathcal{I} - \lambda-$ Statistical Convergence

In the present section, we generalize the notion of weak $\lambda-$ statistical convergence defined in earlier section with the help of ideals and introduce $WS_\lambda(\mathcal{I})-$ convergence and $W[V,\lambda](\mathcal{I})-$ summability in a normed linear space $X$, where $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a non-trivial admissible ideal. We also provide examples which clearly show that for particular choice of the ideal $\mathcal{I}$, some of these notions coincide with the notions introduced in the earlier section.

***DEFINITION 2.3.1*** *A sequence $(x_k)$ in $X$ is said to be weakly $\lambda-$ statistically convergent to $x \in X$ with respect to the ideal $\mathcal{I}$ (or $WS_\lambda(\mathcal{I})-$ convergent) if for every $\epsilon > 0$ and every $\gamma > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma \right\} \in \mathcal{I},$$

*for every $\varphi \in X^*$. In this case, we write $WS_\lambda(\mathcal{I})-\lim_k x_k = x$ or $x_k \xrightarrow{WS_\lambda(\mathcal{I})} x$.*

Let $WS_\lambda(\mathcal{I},X)$ denotes the set of all weakly $\lambda-$ statistically convergent sequences with respect to the ideal $\mathcal{I}$ in $X$.

***DEFINITION 2.3.2*** *A sequence $(x_k)$ in $X$ is said to be weakly $[V,\lambda]-$ summable to $x \in X$ with respect to ideal $\mathcal{I}$ (or $W[V,\lambda](\mathcal{I})-$ summable) if for every $\epsilon > 0$ and every $\varphi \in X^*$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| \geq \epsilon \right\} \in \mathcal{I}.$$

*In this case, we write $W[V,\lambda](\mathcal{I})-\lim_k x_k = x$ or $x_k \xrightarrow{W[V,\lambda](\mathcal{I})} x$.*

Let $W[V,\lambda](\mathcal{I},X)$ denotes the set of all weakly $W[V,\lambda](\mathcal{I})-$summable sequences with respect to the ideal $\mathcal{I}$ in $X$. For $\mathcal{I} = \mathcal{I}_f$, *DEFINITION 2.3.1* and *DEFINITION 2.3.2* coincide with *DEFINITION 2.1.2* and *DEFINITION 2.1.4* respectively.

***THEOREM 2.3.1*** *If $x_k \xrightarrow{WS_\lambda} x$, then $x_k \xrightarrow{WS_\lambda(\mathcal{I})} x$, however, the converse does not hold generally.*

***Proof*** The proof of the first part is similar to that of *THEOREM 2.2.3*, but the next EXAMPLE shows that the converse of the above result is not true in general.

***EXAMPLE 2.3.1*** Let $X = c_{00}$ be the normed linear space with $\|.\|_p$ $(1 < p < \infty)$,

$\mathcal{I}$ be an admissible ideal, $A \in \mathcal{I}$ is fixed and $\lambda = (\lambda_n)$ be a non-decreasing sequence as defined above.

Define a sequence $(x_k)$ in $c_{00}$ by

$$x_j^{(k)} = \begin{cases} ku, & \text{if } j \leq k, k \in I_n^{'} \text{ and } n \notin A; \\ ku, & \text{if } j \leq k, k \in I_n \text{ and } n \in A; \\ 0, & \text{otherwise.} \end{cases}$$

where $u$ is a fixed element in $X$ with $||u|| = 1$, $I_n^{'} = [n - [\sqrt{\lambda_n}] + 1, n]$ and $I_n = [n - \lambda_n + 1, n]$. For arbitrary $\varphi \in X^*$, there is unique $y \in l_q$ such that

$$|\varphi(x_k)| = \left| \sum_{j=1}^{\infty} x_j^{(k)} y_j \right| \leq \|x\|_p \|y\|_q \text{ (by Holder inequality)} \tag{2.3.1}$$

So, for each $\epsilon > 0$ $(0 < \epsilon < 1)$ and each $\varphi \in X^*$,

$$\begin{aligned} \frac{1}{\lambda_n}|\{k \in I_n : |\varphi(x_k - 0)| \geq \epsilon\}| &\leq \frac{1}{\lambda_n}|\{k \in I_n : \|x\|_p \|y\|_q \geq \epsilon\}| \\ &= \frac{1}{\lambda_n}\left|\left\{k \in I_n : \|x\|_p \geq \frac{\epsilon}{\|y\|_q}\right\}\right| \\ &= \frac{1}{\lambda_n}\left|\left\{k \in I_n : x_j^{(k)} = ku\right\}\right| = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \longrightarrow 0 \end{aligned}$$

as $n \to \infty$ and $n \notin A$. So, for each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|k \in I_n : |\varphi(x_k - 0)| \geq \epsilon\}| \geq \gamma\right\} \subset A \cup \{1, 2, \ldots, m\}$$

for some $m \in \mathbb{N}$. Since, $A \in \mathcal{I}$ and $\mathcal{I}$ is admissible, it gives that $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} 0$. Also, by (2.3.1) we can see that,

$$\begin{aligned} |\varphi(x_k)| &= \left|\sum_{j=1}^{\infty} x_j^{(k)} y(j)\right| \leq \|x\|_p \|y\|_q \\ &= \left(\sum_{j=1}^{\infty} \left|x_j^{(k)}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} |y(j)|^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{j=1}^{k} \left|x_j^{(k)}\right|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y(k)|^q\right)^{\frac{1}{q}} \text{ (by structure of sequence)} \end{aligned}$$

$$= \left( \sum_{j=1}^{k} k^p \right)^{\frac{1}{p}} M^{\frac{1}{q}} (\text{for some positive constant M}) = k^{\frac{p+1}{p}} M^{\frac{1}{q}}.$$

So,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - 0)| \leq \frac{1}{\lambda_n} \sum_{k \in I_n} k^{\frac{p+1}{p}} M^{\frac{1}{q}} \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Hence, $x_k \overset{WS_\lambda}{\longrightarrow} 0$ does not hold. ■

***THEOREM 2.3.2*** *For a sequence $(x_k)$ in $X$, $x_k \overset{W[V,\lambda](\mathcal{I})}{\longrightarrow} x$ if and only if $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} x$.*

***Proof*** Let $x_k \overset{W[V,\lambda](\mathcal{I})}{\longrightarrow} x$. Then for each $\varphi \in X^*$ and each $\epsilon > 0$,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| \geq \frac{1}{\lambda_n} \sum_{k \in I_n : |\varphi(x_k - x)| \geq \epsilon} |\varphi(x_k - x)| \geq \frac{\epsilon}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}|.$$

So, for each $\gamma > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| \geq \epsilon\gamma \right\} \in \mathcal{I}.$$

Hence, $x_k \overset{W[V,\lambda](\mathcal{I})}{\longrightarrow} x$ implies $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} x$.

Conversely, suppose $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} x$. Since $\varphi \in X^*$, $\varphi$ is bounded, $|\varphi(x_k - x)| \leq M$ (say) for all $k$. For $\epsilon > 0$,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |\varphi(x_k - x)| = \frac{1}{\lambda_n} \sum_{k \in I_n : |\varphi(x_k - x)| \geq \epsilon} |\varphi(x_k - x)| + \sum_{k \in I_n : |\varphi(x_k - x)| < \epsilon} |\varphi(x_k - x)|$$
$$\leq \frac{M}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| + \epsilon,$$

which implies,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum |\varphi(x_k - x)| \geq \epsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \frac{\epsilon}{M} \right\} \in \mathcal{I}.$$

Thus, $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} x$ implies $x_k \overset{W[V,\lambda](\mathcal{I})}{\longrightarrow} x$. ■

***THEOREM 2.3.3*** *If* $\liminf_{n\to\infty}(\frac{\lambda_n}{n}) > 0$, *then* $WS(\mathcal{I}, X) \subset WS_\lambda(\mathcal{I}, X)$.

***Proof*** For each $\epsilon > 0$ and each $\varphi \in X^*$,

$$\begin{aligned}\frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| &\geq \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \\ &= \left(\frac{\lambda_n}{n}\right)\frac{1}{\lambda_n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}|.\end{aligned}$$

Let, $\liminf_{n\to\infty}(\frac{\lambda_n}{n}) = m > 0$, by definition $\{n \in \mathbb{N} : \frac{\lambda_n}{n} < \frac{m}{2}\}$ is finite. Thus, for $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \frac{m}{2}\gamma\right\} \cup \left\{n \in \mathbb{N} : \frac{\lambda_n}{2} < \frac{m}{2}\right\}.$$

Since, $\mathcal{I}$ is admissible, the set on right side belongs to $\mathcal{I}$.

Hence, $WS(\mathcal{I}, X) \subset WS_\lambda(\mathcal{I}, X)$. ∎

***THEOREM 2.3.4*** *If* $\liminf_{n\to\infty}(\frac{\lambda_n}{n}) = 1$, *then* $WS_\lambda(\mathcal{I}, X) \subset WS(\mathcal{I}, X)$.

***Proof*** Since $\liminf_{n\to\infty}(\frac{\lambda_n}{n}) = 1$, for each $\gamma > 0$, there exists a positive integer $m$ such that $|\frac{\lambda_n}{n} - 1| < \frac{\gamma}{2}$, for all $n \geq m$. Also, for each $\epsilon > 0$,

$$\begin{aligned}&\frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \\ &= \frac{1}{n}|\{k \leq n - \lambda_n : |\varphi(x_k - x)| \geq \epsilon\}| + \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \\ &\leq \frac{n - \lambda_n}{n} + \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \\ &\leq 1 - (1 - \frac{\gamma}{2}) + \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \\ &= \frac{\gamma}{2} + \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}|,\end{aligned}$$

for all $n \geq m$. Hence,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \gamma\right\} \subset \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \in I_n : |\varphi(x_k - x)| \geq \epsilon\}| \geq \frac{\gamma}{2}\right\} \cup \{1, 2, \ldots m\}.$$

Since $x_k \overset{WS_\lambda(\mathcal{I})}{\longrightarrow} x$, the set on the right side belongs to $\mathcal{I}$.
Hence, $WS_\lambda(\mathcal{I}, X) \subset WS(\mathcal{I}, X)$.

***REMARK 2.3.1*** Consider the sequence $(\lambda_n)$ where $(\lambda_n) = 1$ for $n = 1, 2, \ldots, 10$ and $\lambda_n = n - 10$ for all $n \geq 10$. Define the sequence $(x_k)$ as in *EXAMPLE 2.4.1*. Take $A = \{1^2, 2^2, \ldots\}$ and $\mathcal{I} = \mathcal{I}_d$ (the ideal of density zero sets of $\mathbb{N}$). Then, the sequence $(x_k)$ is $WS(\mathcal{I})-$convergent but not $WS-$convergent. ■

## 2.4 Conclusion

We have generalized the concepts of weak convergence with the help of a non-decreasing sequence $\lambda = (\lambda_n)$ and an admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. It is noted that giving particular choice to the sequence $(\lambda_n) = (n)$ we obtained the corresponding weak statistical convergence. THEOREM 2.1.4 shows that weak $\lambda-$ statistical convergence is more general than weak convergence. Also, by giving particular choice to admissible ideal $\mathcal{I}$, the generalized convergence coincides with weak statistical convergence and weak $\lambda-$ statistical convergence. THEOREM 2.3.1 shows that $WS_\lambda(\mathcal{I})-$convergence is more general than $WS_\lambda-$ convergence. ■