CHAPTER 3

SYNTHESIS OF INTERVAL SYSTEM BY POLE CLUSTERING METHOD

A detailed analysis of interval system using mixed order reduction method is investigated in this chapter. Further, the chapter attempts to prove the stability of the reduced order model through Routh – Hurwitz Criterion, Nyquist Criterion, Bode plot and Root locus Techniques.

3.1 INTRODUCTION

Systems of practical importance are often of high dimensionality. Quite often it is always desirable to represent such systems by models of lower order for the purpose of computer simulation and/or control system design. Hence the purpose of approximating a higher order system into a lower order model is of considerable interest (Prasad et al 2009). The problem of approximating a higher order system by lower order models is considered important in analysis, simulation of practical systems and control system designs (Shiv Kumar Tomar and Rajendra Prasad 2008). The analysis of the higher order interval system is complex task. It is often necessary to reduce the order of the interval system for simplicity of analysis. The systems whose coefficients vary in a specified manner within an interval are called interval systems. Stability of interval systems can be checked by Kharitonov theorem is an extension of Routh’s array and it deals only with the extreme points of intervals.

The determination of stability and performance of a system is the major concern of control engineers. A control system assumed to be described by a linear differential equation is stable if and only if all the roots of the characteristic equation of the system lie in the left half of the s-plane. In classical control, these are some powerful tools for a fixed nominal system. These include the Routh-Hurwitz criterion.
and the Nyquist criterion. However, in real physical systems, the parameters of the plant transfer function model will not be fixed but vary with time. Thus the fundamental stability problem in the study of control systems with parametric uncertainty is to determine whether or not all the polynomials in what now becomes family of characteristic polynomials are Hurwitz stable. This property is known as robust stability.

When the system parameters are bound to vary within the close limits the system has structured uncertainty. This also called parametric uncertainty and the other type of uncertainty is called unstructured uncertainty such as when dynamics of the system in the higher frequency ranges are not incorporated in the system model. Uncertainty in modeling the system leads to degradation of system performance and loss of stability or coefficient with the interval. Recent analysis of system with parametric uncertainty has been inspired by the development of Kharitonov theorem.

3.2 STATEMENT OF PROBLEM

The general mathematical representation of an interval system is given by

\[ G(s) = \frac{[a_{n+1}, a_{n+1}]}{[b_{n+1}, b_{n+1}]} s^{n+1} + [a_{n-2}, a_{n-2}] s^{n-2} + ... + [a_i, a_i] s + [a_0, a_0] = \frac{N(s)}{D(s)} \]

Where,

\[ [a_i, a_i] \quad \text{for } i = 0 \text{ to } n-1 \quad \text{and} \quad [b_i, b_i] \quad \text{for } i = 0 \text{ to } n \] are the interval parameters.

By applying the proposed model order reduction techniques, the corresponding reduced order model is obtained as,

\[ G_r(s) = \frac{[d_0, d_0] + [d_1, d_1] s + ... + [d_{k-1}, d_{k-1}] s^{k-1}}{[e_0, e_0] + [e_1, e_1] s + ... + [e_{k-1}, e_{k-1}] s^{k-1} + [e_k, e_k] s^k} = \frac{N_r(s)}{D_r(s)} \]
Instead of applying the Model Order Reduction (MOR) procedure to single interval system as a whole the system can be split up to 4 Kharitonov polynomials of the denominator and 4 Kharitonov polynomials of the numerator. After arriving at the reduced denominator and each denominator has 4 possible combinations of the numerator and for the 4 denominator functions totally 16 possible system models are obtained.

3.3 POLE CLUSTERING

The Pole Clustering technique is used to obtain a denominator polynomial of model. The cluster center using pole clustering technique is obtained by grouping the poles of higher order interval systems. In this process, separate cluster partitions should be made for real poles and complex conjugate poles and then cluster centers for these cluster partitions are obtained. Each real cluster center or pair of complex conjugate cluster centers are, respectively, replaced by real pole or pair of complex conjugate poles of model, respectively.

Procedure to obtain Denominator Polynomial of Model

The ‘k’ number of poles from the given higher order system denominator polynomial are calculated. The number of cluster centers to be calculated is equal to the order of the reduced system. The poles are distributed into the cluster centre for the calculation such that none of the repeated poles is present in the same cluster centre. Minimum number of poles distributed per each cluster centre is at least one. There is no limitation for the maximum number poles per cluster centre. Let \(k\) be the number of poles available in a cluster centre: \(p_1, p_2, p_3, \ldots, p_k\). The poles are arranged in a manner such that \(|p_1| < |p_2| < \cdots < |p_k|\). The cluster centre for the reduced order model can be obtained by using the following procedure. The procedure is similar as in case of method proposed by Vishwakarma and Prasad (2009) but pole cluster calculated in proposed scenario is based on dominant pole in that particular cluster centre.
**Step 1:** Arrange the poles of higher order system in r cluster partitions collecting the real and complex conjugate poles in separate cluster partitions.

**Step 2:** Obtain the cluster center: The cluster center for real poles is obtained as

\[ p_c = \left( \sum_{i=1}^{k} \left( \frac{1}{p_i} \right) \right)^{-1} \]

Where, \( p_c \) is cluster center of cluster partition containing k real poles \( P_1, P_2, \) and \( P_3 \) of \( n^{th} \) order system.

The cluster center for complex conjugate is obtained as

\[ p_R = \left( \sum_{i=1}^{k} \left( \frac{1}{p_R^i} \right) \right)^{-1} \]

\[ p_I = \left( \sum_{i=1}^{k} \left( \frac{1}{p_I^i} \right) \right)^{-1} \]

Where, \( P_R \) and \( P_I \) are real and imaginary parts of cluster pair \( P_R \pm j P_I \).

**Step 3:** Obtain the Denominator Polynomial of Reduced Model

The denominator polynomial of \( k^{th} \) order model is given as

**Case 1:** When all the denominator poles are real, the corresponding reduced order denominator polynomial can be obtained as,

\[ D_k(s) = (s - p_{c1})(s - p_{c2}) \ldots \ldots (s - p_{ck}) \]

**Case 2:** If one pair of cluster center is complex conjugate and (k-2) cluster center are real.

\[ D_k(s) = \left( s - \left( p_R^g + j p_I^g \right) \right) \left( s - \left( p_R^g - j p_I^g \right) \right) (s - p_{ci}) \]
**Case 3**: When all the denominator poles are complex conjugates, the corresponding reduced order denominator polynomial can be obtained as,

\[ D_R(s) = (s - (p_i^k + j\gamma_i)) (s - (p_i^k - j\gamma_i)) \]

**Procedure to obtain Numerator Polynomial of Model**

A popular approach, known as Pade approximation method for deriving reduced order models has been based on matching the original and reduced order systems. This technique has a number of useful properties, such as, computational simplicity, fitting of the initial time moments and the steady state values of the output of original and reduced order systems being the same for input of the form. \( \sum a_i \). This simple technique usually gives good results and is not computationally demanding.

The transfer function of original higher order (n\textsuperscript{th}) system is considered as,

\[ G(s) = \frac{a_0 + a_1s + a_2s^2 + \cdots + a_{m-1}s^{m-1}}{b_0 + b_1s + b_2s^2 + \cdots + b_{n-1}s^{n-1} + b_ns^n} \]

\( G(s) \) can be expanded into a power series about \( s = 0 \) of the form,

\[ G(s) = c_0 + c_1s + c_2s^2 + \cdots \]

Where,

\[ c_0 = \frac{a_0}{b_0} \]

The general form of reduced order model is given by,

\[ G_r(s) = \frac{d_0 + d_1s + d_2s^2 + \cdots + d_{r-1}s^{r-1}}{e_0 + e_1s + e_2s^2 + \cdots + e_{r-1}s^{r-1} + e_rs^r} \]
On equating and cross multiplying the transfer functions $G(s)$ and $G_r(s)$ by using the first order pade approximation technique.

\[
a_0 = b_0 c_0 \\
a_1 = b_0 c_1 + b_1 c_0 \\
\vdots \\
a_{k-1} = b_0 c_{k-1} + b_1 c_{k-2} + \ldots + b_{k-2} c_1 + b_{k-1} c_0
\]

Therefore

\[
c_0 = \frac{a_0}{b_0} = \frac{d_0}{e_0}
\]

According to the above equation

\[
\begin{align*}
a_0 &= d_0 \\
b_0 &= e_0
\end{align*}
\]

The first order Pade Approximation technique is used to obtain a Numerator Polynomial of Model. The given higher order system transfer function of $n^{th}$ order is equated and cross multiplied with $r^{th}$ order general transfer function. This process yields $(n+2)$ equations with $(2r-1)$ unknown reduced order transfer function coefficients. This step is similar to the model order reduction method proposed in Manigandan et al (2005), where the values of $e_0$ or $d_0$ are kept as equal to ‘1’ irrespective of the system condition to obtain the values of unknown coefficients in the reduced order model transfer function. But in this proposed method, the values of $e_0$ and $d_0$ are obtained through Pade approximation method. This leads to better system approximation as compared to the model order reduction method proposed by Manigandan et al (2005). Equating the higher order and lower order system as,
After cross multiplying the terms, the coefficients of same power of ‘s’ on both sides of above Equation is equated and they are given by,

\[
\begin{align*}
\frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots + b_{n-1} s^{n-1} + b_n s^n} &= \frac{d_0 + d_1 s + d_2 s^2 + \ldots + d_{r-1} s^{r-1}}{e_0 + e_1 s + e_2 s^2 + \ldots + e_{r-1} s^{r-1} + e_r s^r} \\
\left(a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}\right)\left(e_0 + e_1 s + e_2 s^2 + \ldots + e_{r-1} s^{r-1} + e_r s^r\right) &= \left(b_0 + b_1 s + b_2 s^2 + \ldots + b_{n-1} s^{n-1} + b_n s^n\right)\left(d_0 + d_1 s + d_2 s^2 + \ldots + d_{r-1} s^{r-1}\right)
\end{align*}
\]

The (n+2) number of equations is solved with the values of \( d_0, e_0 \). This leads to have different equations for solving the remaining unknown parameters. From the above resulting values, the reduced order interval system transfer function can be obtained as,

\[
\frac{d_0 + d_1 s}{e_0 + e_1 s + e_2 s^2} = \frac{\left(\frac{d_0}{e_2}\right) + \left(\frac{d_1}{e_2}\right)s}{\left(\frac{e_0}{e_2}\right) + \left(\frac{e_1}{e_2}\right)s + s^2} = \frac{A_0 + A_1 s}{B_0 + B_1 s + s^2}
\]

Where, the parameters \( A_0, B_0 \) and \( B_1 \) values can be changed as a function undamped natural frequency \( (\omega_n) \) and damping ratio \( (\xi) \).

The parameters values obtained through the proposed method are considered as initial parameters and are fine-tuned with the help of Genetic Algorithm (GA) by considering Mean Square Error between the higher order and lower order systems as an objective function. The performance of the reduced order model system is compared by computing the error index criterion ‘J’ in between the transient parts of the original and reduced order model system.
3.4 NUMERICAL ILLUSTRATION

The validity of the proposed methods is investigated with the help of some numerical examples.

The 3rd order interval system stated in Ismail et al (1997) is considered as

$$G(s) = \frac{[1,2]s^2 + [3,4]s + [8,10]}{[6,6]s^3 + [9,9.5]s^2 + [4.9,5]s + [0.8,0.85]}$$

(3.1)

The given interval system transfer function is in the form of

$$G(s) = \frac{[a_2,\bar{a}_2]s^2 + [a_1,\bar{a}_1]s + [a_0,\bar{a}_0]}{[b_3,\bar{b}_3]s^3 + [b_2,\bar{b}_2]s^2 + [b_1,\bar{b}_1]s + [b_0,\bar{b}_0]}$$

(3.2)

For which, the reduced order interval system is to be derived in the form of

$$G_r(s) = \frac{[d_1,\bar{d}_1]s + [d_0,\bar{d}_0]}{[e_2,\bar{e}_2]s^2 + [e_1,\bar{e}_1]s + [e_0,\bar{e}_0]}$$

(3.3)

From Equation (3.1), the following values are noted down and are used to obtain the Kharitonov polynomials.

\[
\begin{align*}
a_0 &= 8 & \bar{a}_0 &= 10 \\
a_1 &= 3 & \bar{a}_1 &= 4 \\
a_2 &= 1 & \bar{a}_2 &= 2 \\
b_0 &= 0.8 & \bar{b}_0 &= 0.85 \\
b_1 &= 4.9 & \bar{b}_1 &= 5 \\
b_2 &= 9 & \bar{b}_2 &= 9.5 \\
b_3 &= 6 & \bar{b}_3 &= 6
\end{align*}
\]

The Kharitonov polynomials are obtained separately for the higher order interval system’s numerator and denominator polynomials.
Numerator Kharitonov polynomials

Consider the numerator polynomial,

\[
N(s) = [a_{2s}, \bar{a}_3] s^2 + [a_1, \bar{a}_0] s + [a_0, \bar{a}_0] = [1, 2] s^2 + [3, 4] s + [8, 10]
\] (3.4)

The polynomial N(s) is split into two components g(s) and h(s) where, g(s) is polynomial of even degree and h(s) is polynomial of odd degree.

Defining two even polynomials,

\[
K_{1\text{even.min}}(s) = g_1(s) = a_0 + \bar{a}_2 s^2 = 8 + 2 s^2
\] (3.5)

\[
K_{1\text{even.max}}(s) = g_2(s) = \bar{a}_0 + a_2 s^2 = 10 + s^2
\] (3.6)

Defining two odd polynomials,

\[
K_{1\text{odd.min}}(s) = h_1(s) = a_1 s = 3 s
\] (3.7)

\[
K_{1\text{odd.min}}(s) = h_2(s) = \bar{a}_1 s = 4 s
\] (3.8)

The four Kharitonov polynomials would be

\[
K_{11}(s) = g_1(s) + h_1(s) = 2 s^2 + 3 s + 8
\] (3.9)

\[
K_{12}(s) = g_1(s) + h_2(s) = 2 s^2 + 4 s + 8
\] (3.10)

\[
K_{21}(s) = g_2(s) + h_1(s) = s^2 + 3 s + 10
\] (3.11)

\[
K_{22}(s) = g_2(s) + h_2(s) = s^2 + 4 s + 10
\] (3.12)

Denominator Kharitonov polynomials

Consider the denominator polynomial,

\[
D(s) = [b_{2s}, \bar{b}_3] s^3 + [b_2, \bar{b}_1] s^2 + [b_1, \bar{b}_0] s + [b_0, \bar{b}_0]
\]
The polynomial $D(s)$ is split into two components $g(s)$ and $h(s)$ where, $g(s)$ is polynomial of even degree and $h(s)$ is polynomial of odd degree.

Defining two even polynomials,

$$K_1(s) = g_1(s) + h_1(s) = 6s^3 + 9s^2 + 5s + 0.85$$
$$K_2(s) = g_2(s) + h_2(s) = 6s^3 + 4.9s^2 + 0.85$$

Defining two odd polynomials,

$$K_3(s) = g_3(s) + h_3(s) = 6s^3 + 4.9s^2 + 5s$$
$$K_4(s) = g_4(s) + h_4(s) = 6s^3 + 9s^2 + 0.85$$

The four Kharitonov polynomials would be

$$K_1(s) = \frac{6s^3 + 9s^2 + 5s + 0.85}{6s^3 + 4.9s^2 + 0.85}$$
$$K_2(s) = \frac{6s^3 + 9s^2 + 5s}{6s^3 + 4.9s^2 + 0.85}$$
$$K_3(s) = \frac{6s^3 + 4.9s^2 + 5s}{6s^3 + 4.9s^2 + 0.85}$$
$$K_4(s) = \frac{6s^3 + 4.9s^2 + 0.85}{6s^3 + 9s^2 + 0.85}$$

The Kharitonov polynomials available for numerator and denominator of higher order interval system, the following four interval system transfer functions may be obtained.

$$G(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^3 + a_1 s^2 + a_2 s + a_3}{b_0 s^3 + b_1 s^2 + b_2 s + b_3} = \frac{2s^3 + 3s + 8}{6s^3 + 9s^2 + 0.85}$$

$$N(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$$
$$D(s) = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

The polynomial $D(s)$ is split into two components $g(s)$ and $h(s)$ where, $g(s)$ is polynomial of even degree and $h(s)$ is polynomial of odd degree.
\[ G_2(s) = \frac{N_2(s)}{D_2(s)} = \frac{a_2s^2 + a_1s + a_0}{b_2s^3 + b_1s^2 + b_0s + b_0} = \frac{2s^2 + 4s + 8}{6s^3 + 9.5s^2 + 5s + 0.8} \]  
(3.23)

\[ G_3(s) = \frac{N_3(s)}{D_3(s)} = \frac{a_2s^2 + a_1s + a_0}{b_2s^3 + b_1s^2 + b_0s + b_0} = \frac{s^2 + 3s + 10}{6s^3 + 9s^2 + 4.9s + 0.85} \]  
(3.24)

\[ G_4(s) = \frac{N_4(s)}{D_4(s)} = \frac{a_2s^2 + a_1s + a_0}{b_2s^3 + b_1s^2 + b_0s + b_0} = \frac{s^2 + 4s + 10}{6s^3 + 9s^2 + 5s + 0.85} \]  
(3.25)

**Order reduction by pole clustering method**

Kharitonov polynomial transfer functions available form equations (3.22) – (3.25) are used to obtain the reduced order interval system for the given higher order interval system. The pole clustering technique is only applied to the denominator polynomial transfer functions. Using the resultant, the reduced order denominator polynomial transfer function is constructed. The same procedure stated below can be repeated to the rest of the Kharitonov polynomial transfer functions to obtain similar polynomial transfer function.

**Order reduction of** \( G_1(s) \)

**Step-1: Obtain the reduced order denominator polynomial**

Consider the system transfer function

\[ G_1(s) = \frac{N_1(s)}{D_1(s)} = \frac{\tilde{a}_2s^2 + \tilde{a}_1s + \tilde{a}_0}{\tilde{b}_2s^3 + \tilde{b}_1s^2 + \tilde{b}_0s + \tilde{b}_0} = \frac{2s^2 + 3s + 8}{6s^3 + 9.5s^2 + 4.9s + 0.8} \]  
(3.26)

The corresponding generalised second order system transfer function is to derived in the form of

\[ G_{11}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} \]
The roots of the corresponding characteristic equation of lower limit interval system transfer function stated in $G_1(s)$ are $P_1 = -0.3333$, $P_2$, $P_3 = -0.6250 \pm j 0.0968$. Where, the poles $P_2$ and $P_3$ are in complex form. These poles are divided into two cluster groups as follows

Cluster group-1: $-0.6250 \pm j 0.0968$

Cluster group-2: $-0.3333$

The cluster centre for first cluster group is determined as follows.

Let, $k = \text{Numbers of poles available} = 2$

Find the pole cluster as,

$$p_R = \left\{ \sum_{i=1}^{k} \left( \frac{1}{p^R_i} \right) \right\}^{-1}$$

$$p_I = \left\{ \sum_{i=1}^{k} \left( \frac{1}{p^I_i} \right) \right\}^{-1}$$

Check for the condition $L = k$. Since the required condition is met, the pole clustering process is terminated and resultant pole cluster is $P_c = -0.6250$. The corresponding reduced order denominator polynomial is obtained as,

$$D_{r1}(s) = (s - p_{r1})(s - p_{r2})$$

$$= (s + 0.6250)(s + 0.3333)$$

$$D_{r1}(s) = s^2 + 0.9583s + 0.2083$$

(3.27)
Step 2: Obtain the reduced order numerator polynomial

The given higher order system transfer function and the general form of reduced system transfer function are equated. Where, the reduced order denominator polynomial obtained from step 1 is utilized here to obtain the unknown values of reduced order system coefficients.

\[
\frac{2s^2 + 3s + 8}{6s^3 + 9.5s^2 + 4.9s + 0.8} = \frac{d_1s + d_0}{s^2 + 0.9583s + 0.2083}
\]  
(3.28)

On cross multiplying the above equation, the following condition is obtained.

\[
(2s^2 + 3s + 8)(s^2 + 0.9583s + 0.2083) = (d_1s + d_0)(6s^3 + 9.5s^2 + 4.9s + 0.8)
\]

\[
2s^4 + 4.9166s^3 + 11.2915s^2 + 8.2913s + 1.6664 = 6d_0s^4 + (6d_0 + 9.5d_1)s^3 + (9.5d_0 + 4.9d_1)s^2 + (4.9 + 0.8d_1)s + 0.8d_0
\]

On comparing the coefficients of same power of \(s\) term on both sides, the following equations are obtained.

Coefficient of \(s^4\): \[6d_1 = 2\]  
(3.29)

Coefficient of \(s^3\): \[6d_0 + 9.5d_1 = 4.9166\]  
(3.30)

Coefficient of \(s^2\): \[9.5d_0 + 4.9d_1 = 11.2915\]  
(3.31)

Coefficient of \(s^1\): \[4.9d_0 + 0.8d_1 = 8.2913\]  
(3.32)

Coefficient of \(s^0\): \[0.8d_0 = 1.6664\]  
(3.33)

The last Equation (3.33) gives \(d_0 = 2.083\). With the known value of \(d_0\), last but two Equation (3.29) gives the values of \(d_1\) as 0.333. The ISE value involved by last but two Equation is less.
The value $d_i$ is determined using other Equations (3.30) – (3.31) and (3.32) will give large error in the system approximation. Therefore, the corresponding reduced order model is obtained as,

$$G_{r1}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} = \frac{-0.7980s + 2.083}{s^2 + 0.9583s + 0.2083}$$

(3.34)

**Order reduction of $G_2(s)$, $G_3(s)$ and $G_4(s)$**

The pole clustering technique is applied to the systems $G_2(s)$, $G_3(s)$ and $G_4(s)$ as in the case of systems $G_1(s)$. The resultant reduced order models obtained as,

$$G_{r2}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} = \frac{-0.5739s + 1.888}{s^2 + 0.9379s + 0.1888}$$

(3.35)

$$G_{r3}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} = \frac{-1.0704s + 2.258}{s^2 + 0.9137s + 0.1919}$$

(3.36)

$$G_{r4}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} = \frac{-0.4562s + 2.1094}{s^2 + 0.8992s + 0.1793}$$

(3.37)

From the reduced order system transfer functions which are available in Equations (3.34) – (3.37), the following conditions are obtained.

$$d_0 = \min (2.083, 1.888) = 1.888$$

$$d_i = \min (-0.7980, -1.0704) = -1.0704$$

$$e_0 = \min (0.2083, 0.1888) = 0.1888$$

$$e_i = \min (0.9583, 0.9137) = 0.9137$$

$$e_2 = \min (1, 1) = 1$$

$$\bar{d}_0 = \max (2.258, 2.1094) = 2.258$$

$$\bar{d}_i = \max (-0.5739, -0.4562) = -0.4562$$
From the above resultant values, the reduced order interval system transfer function can be obtained as,

\[
G_r(s) = \frac{[d_1, \overline{d}_1]s + [d_0, \overline{d}_0]}{[\varepsilon_1, \overline{\varepsilon}_1]s^2 + [\varepsilon_0, \overline{\varepsilon}_0]s + [\varepsilon_0, \overline{\varepsilon}_0]} = \frac{[-1.0704, -0.4562]s + [1.888, 2.258]}{[1, 1]s^2 + [0.9137, 0.9379]s + [0.1888, 0.1919]}
\]  \tag{3.38}

These parameters values obtained through the proposed method are considered as initial parameters and these can be fine-tuned with the help of Genetic Algorithm (GA) by considering Mean Square Error between the higher order and lower order systems as an objective function to obtain the following reduced order interval system.

\[
G_r(s) = \frac{[-1.0704, -0.4562]s + [1.7383, 2.1008]}{[1, 1]s^2 + [0.8264, 0.9421]s + [0.1738, 0.1787]}
\]  \tag{3.39}

The variation in the error rate over the iterations for lower and upper limit interval systems is shown in Figure 3.1 and Figure 3.2 respectively. The unit step responses of higher order interval system and reduced order interval system obtained through proposed method (pole clustering method) are shown in Figure 3.3. The closeness between the higher order model and reduced order lower limit system is analyzed with the help of ISE value and is listed in Table 3.1. The results available in this table show the validity of the proposed model order reduction methods in their approximation process.
Figure 3.1  Change in error rate in lower limit interval parameters tuning process of Example 3.4

Figure 3.2  Change in error rate in upper limit interval parameters tuning process of Example 3.4
Figure 3.3  Step responses of higher order and reduced order interval systems through pole clustering method

Table 3.1  Comparison of error index values with existing methods

<table>
<thead>
<tr>
<th></th>
<th>ISE</th>
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<tbody>
<tr>
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<td>Lower limit</td>
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<tr>
<td>Balanced reduction</td>
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<td>Balanced model truncation via square root method</td>
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<td>Henkel minimum degree optimization</td>
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<tr>
<td>Balanced stochastic model truncation via schur method</td>
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<tr>
<td>Proposed System</td>
<td>1.5640</td>
</tr>
</tbody>
</table>

The closeness between the higher order model and reduced order lower limit system is analyzed with the help of ISE value and is listed in Table 3.1.
3.5 STABILITY ANALYSIS

The most important factor which is to be considered in system design is its stability. Practical control systems are subjected to uncertain parameters. These uncertain parameters are often constants rather than unknown functions of time. The Kharitonov theorem states that the stability of a polynomial can be determined by testing just the four Kharitonov polynomials which are obtained by using upper and lower bounds of the unknown parameters. In spite of applying Kharitonov theorem, there are chances for getting an unstable reduced order model. Hence, verification of stability of the reduced order model becomes necessary. To verify the stability of the reduced order interval systems Routh –Hurwitz criteria, Nyquist plot, Bode plot and Root locus model are used in this thesis to discuss the stability.

3.5.1 Routh – Hurwitz Criterion

The stability of an interval system can also be determined by applying the Routh-Hurwitz criterion to the selected Kharitonov polynomials of given interval system. If the elements in the first column of the Routh array are positive then the interval system is stable, otherwise it is not.

The Routh stability criterion states as follows.

“For a system to be stable, it is necessary and sufficient that each term of first column of Routh array of its characteristics equation to positive if \( a_0 > 0 \). If this condition is no met, the system is unstable and number of sign changes of the terms of the first column of the Routh array corresponds to the number of roots of the characteristics equation in the right of the s-plane”.

Routh-Hurwitz criterion is easily used to check the stability of a simple polynomial which becomes difficult to apply to families of polynomials because it gives nonlinear equations with the unknown parameters. The polynomial \( G_1(s) \) is Hurwitz stable, if and only if the following polynomials are Hurwitz stable according to the Kharitonov theorem.
The 3rd order Kharitonov polynomial Equation (3.22) of original interval system stated by Ismail et al (1997) is given as,

\[
G_i(s) = \frac{N_i(s)}{D_i(s)} = \frac{2s^2 + 3s + 8}{6s^3 + 9.5s^2 + 4.9s + 0.8}
\]

\[
D_i(s) = 6s^3 + 9.5s^2 + 4.9s + 0.8
\]

The stability of the above nominal system is determined by applying the Routh-Hurwitz criterion and is depicted as given below. Since there is no sign change in the first column of Routh array, there are no roots in the right half of the \(s\)-plane. Hence, the system is stable. This array can be obtained and there is no sign change in the elements of first column of the array. This indicates the interval system is stable. The same procedure may be repeated to check the stability of other models.

\[
\begin{array}{c|cc}
S^3 & 6 & 4.9 \\
S^2 & 9.5 & 0.8 \\
S^1 & 4.395 & 0 \\
S^0 & 0.8 \\
\end{array}
\]

Now consider the reduced order system interval transfer function obtained in Equation (3.34) as,

\[
G_{r1}(s) = \frac{d_1s + d_0}{\epsilon_2s^2 + \epsilon_1s + \epsilon_0} = \frac{-0.7980s + 2.083}{s^2 + 0.9583s + 0.2083}
\]

The corresponding characteristic equation is obtained as,

\[
s^2 + 0.9583s + 0.2083 = 0
\]
One can note that there is no sign change in the elements of first column of the array as shown below. This indicates the reduced order system is stable. In a similar fashion, the stability can be analyzed for remaining three possible groups of Kharitonov polynomial transfer functions. Thus the reduced order interval systems obtained through the proposed model order reduction methods are stable.

\[
\begin{array}{c|cc}
S^2 & 1 & 0.2083 \\
S^1 & 0.9583 & 0 \\
S^0 & 0.2083 \\
\end{array}
\]

3.5.2 **Nyquist Criterion**

The stability of a system depends upon the location of the roots of its characteristic equation in the s-plane. The important technique, which relates the location of the roots of the characteristic equation to the open loop frequency response of a system, is called as Nyquist stability criterion.

The Nyquist stability criterion as follows:

“If the contour \( \Gamma (GH) \) of the open loop transfer function \( G(s) H(s) \) corresponding to the Nyquist contour in the s-plane encircles that point \((-1+j0)\) in the counter clockwise direction as many as the number of right half s-plane poles of \( G(s) H(s) \), then the closed loop system is said to stable”. In the commonly occurring case of the open loop stable system, the closed loop system is stable if the contour \( \Gamma (GH) \) of \( G(s) H(s) \) does not encircle \((-1+j0)\) point, i.e., the net encircle is zero.

**Nyquist Plots for Kharitonov Polynomials**

\[
G_{r_1}(s) = \frac{d_1 s + d_0}{e_2 s^3 + e_1 s + e_0} = \frac{-0.7980s + 2.083}{s^3 + 0.9583s + 0.2083}
\]
Figure 3.4  Nyquist Plot of Reduced Order System $G_{r1}(s)$

Figure 3.5  Nyquist Plots of Reduced Order Kharitonov Plants

Figure 3.4 is the Nyquist plots for the reduced order Kharitonov polynomials $G_{r1}(s)$. Figure 3.5 shows the Nyquist plot rest of the four Kharitonov
polynomials. From the figure 3.4 and 3.5, it is clear that there is no encirclement of 
(-1+j0) point. This implies that the system is stable.

3.5.3 Bode Plot

One of the most useful representation of a transfer function is a logarithmic plot which consists of two graphs, one giving the logarithm of $|G(j\omega)|$ and the other phase angle of $G(j\omega)$ both plotted against frequency in logarithmic scale. These plots are called Bode plots. Consider the reduced order Kharitonov transfer function given in equation (3.34)

$$G_{r1}(s) = \frac{d_3 s + d_0}{e_2 s^2 + e_1 s + e_0} = \frac{-0.7980s + 2.083}{s^2 + 0.9583s + 0.2083}$$

![Bode Diagram](image)

**Figure 3.6** Bode Plot of Reduced Order System $G_{r1}(s)$
Figure 3.7  Bode plot of Reduced Order Kharitonov Plants

The Bode plot for the reduced order Kharitonov polynomials \(G_{r1}(s)\) is shown in figure 3.6. It can be noted that the gain margin and phase margin of the \(G_1(s), G_2(s), G_3(s)\) and \(G_4(s)\) are positive as indicated in figure 3.7. This implies the system is stable.

3.5.4  Root Locus

Root locus provides a graphical method of plotting the locus of the roots in the s-plane as a given system parameter is varied over the complete ranges of values (may be zero to infinity). It gives a visual impression about the stability of family of systems and it is a plot of the location of the roots of the characteristic equation in terms of a single parameter \(K \geq 0\). Consider the reduced order Kharitonov transfer function given in equation (3.34)

\[
G_{r1}(s) = \frac{d_1s + d_0}{e_2s^2 + e_1s + e_0} = \frac{-0.7980s + 2.083}{s^2 + 0.9583s + 0.2083}
\]
Figure 3.8  Root Locus of Reduced Order System $G_{r1}(s)$

Figure 3.9  Root Locus of Reduced Kharitonov plants
The Root Locus plot for the reduced order Kharitonov polynomials $G_1(s)$ is shown in figure 3.8. Figure 3.9 illustrates the placement of pole for all four Kharitonov polynomials. As there are no roots in the right hand side of the s-plane in the above figures, it implies that the system is stable. This proves that the reduced order model derived using the proposed method remains stable, which are proved using Nyquist Plot, Bode plot and Root locus.

3.6 **SUMMARY**

The unit step response of original interval system and reduced order interval system using proposed method were plotted for different illustrations. The step responses of the original and reduced order model interval systems are closer to each other. Hence, depending upon the closeness of approximation desired, the order of reduction in modeling is chosen. The proposed methods are versatile and simple. The time response pattern is excellently preserved in reduced order systems even for lower order approximations.

The proposed method can be extended for the MIMO systems by using the following rules of interval arithmetic,

(i) Addition : $[a, b] + [c, d] = [a+c, b+d]$

(ii) Subtraction : $[a, b] - [c, d] = [a-d, b-c]$

Controller parameters $K_p, K_i$ so obtained will lead to good performance on system response. But these parameter values may not be final values and these can be tuned further by means of controller tuning methods.