CHAPTER 2
DOUBLE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

This chapter is devoted to the study of some double sequence spaces defined by a sequence of Orlicz functions. The inspiration to write this chapter came after study the following papers ([Har 17], [Mor 91], [Tri 03], [MuE 03], [Mur 04], [AlB 05], [BaS 09]). The chapter is divided into three sections. The section I deals with the study of some double difference sequence spaces $c^2(\Delta^m, M, u, p, q, s)$, $c_0^2(\Delta^m, M, u, p, q, s)$ and $l^2_\infty(\Delta^m, M, u, p, q, s)$. We study some topological properties and prove some inclusion relation between these sequence spaces. It is very interesting in this section if the seminorms $q_1 \equiv q_2$ (equivalent), then $Z^2(\Delta^m, M, u, p, q_1, s) = Z^2(\Delta^m, M, u, p, q_2, s)$ for $Z^2 = c^2, c_0^2$ and $l^2_\infty$. In second section an attempt has been made to study the sequence spaces $\Lambda^2_M(\Delta^m, u, p, q)$ and $\Gamma^2_M(\Delta^m, u, p, q)$. The main purpose of section second of this chapter is to study the property of linearity, paranormed and various inclusion relations. The third section of this chapter is a study of double chi and double analytic sequences $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ and $\Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$. It is shown that they are linear as well as paranormed spaces. A necessary and sufficient condition for the inclusion relations is given in the end of this section. The following inequality will be used throughout this chapter. Let $p = (p_{k,l})$ be a sequence of positive real numbers with $0 \leq p_{k,l} \leq \sup p_{k,l} = H$, $K = \max(1, 2^{H-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K \{ |a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}} \}$$  \hspace{1cm} (2.1)

for all $k, l$ and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^{H})$ for all $a \in \mathbb{C}$.

2.1 Difference double sequence spaces

A study of topological properties and inclusion relation between difference sequence spaces $Z^2(\Delta^m, M, u, p, q, s)$ for $Z^2 = c^2, c_0^2$ and $l^2_\infty$ defined by a sequence of Orlicz function has been initiated in this section.

Suppose $M = (M_{k,l})$ is a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Also let $X$ be a seminormed space over the field of complex numbers $\mathbb{C}$ with the seminorm $q$. We define the following classes of sequences in this section:

$$c^2(\Delta^m, M, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m x_{k,l}}{\rho} - L \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } L, \rho > 0 \text{ and } s \geq 0 \right\},$$
\[ c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\} \]

and

\[ l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\}. \]

If we take \( \mathcal{M}(x) = x \), we get

\[ c^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } L, \rho > 0 \text{ and } s \geq 0 \right\}, \]

\[ c_0^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\} \]

and

\[ l_\infty^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\}. \]

If we take \( p = (p_{k,l}) = 1 \), we get

\[ c^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right] = 0, \text{ for some } L, \rho > 0 \text{ and } s \geq 0 \right\}, \]

\[ c_0^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\} \]

and

\[ l_\infty^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\}. \]

If we take \( m = n = 0 \) and \( q(x) = |x| \), then we get new double sequence spaces as follows :

\[ c^2(\mathcal{M}, u, p, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (\Delta_n^m x_{k,l})^{-s} u_{k,l} \left[ q \left( \frac{|x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right\} \]
for some \( L, \rho > 0 \) and \( s \geq 0 \),

\[
\ell_0^2(\mathcal{M}, u, p, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\
\left. \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}
\]

and

\[
\ell_\infty^2(\Delta, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \right. \\
\left. \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}.
\]

If we take \( m = n = 1 \) and \( q(x) = |x| \), then we get new double sequence spaces as follows:

\[
c_0^2(\Delta, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\
\left. \quad \text{for some } L, \rho > 0 \text{ and } s \geq 0 \right\},
\]

\[
c_0^2(\Delta, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\
\left. \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\},
\]

and

\[
\ell_\infty^2(\Delta, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \right. \\
\left. \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}.
\]

**Theorem 2.1.1** Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then the classes of sequences \( c_0^2(\Delta^m_n, \mathcal{M}, u, p, q, s) \), \( c^2(\Delta^m_n, \mathcal{M}, u, p, q, s) \) and \( \ell_\infty^2(\Delta^m_n, \mathcal{M}, u, p, q, s) \) are linear spaces over the field of complex numbers \( \mathbb{C} \).

**Proof.** Let \( x = (x_{k,l}), y = (y_{k,l}) \in c_0^2(\Delta^m_n, \mathcal{M}, u, p, q, s) \) and \( \alpha, \beta \in \mathbb{C} \). Then there exist positive numbers \( \rho_1 \) and \( \rho_2 \) such that

\[
\lim_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|x_{k,l}|}{\rho_1} \right) \right]^{p_{k,l}} = 0
\]

and

\[
\lim_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( \frac{|y_{k,l}|}{\rho_2} \right) \right]^{p_{k,l}} = 0.
\]
Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing, convex function and so by using inequality (2.1), we have

$$\lim_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m(\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}\right)\right)^{p_{k,l}}\right]$$

$$= \lim_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho_1}\right)\right)\right]^{p_{k,l}}$$

$$\leq K \lim_{k,l} \frac{1}{2p_{k,l}}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m y_{k,l}}{\rho_2}\right)\right)\right]^{p_{k,l}}$$

$$\leq K \lim_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho_1}\right)\right)\right]^{p_{k,l}}$$

$$+ K \lim_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m y_{k,l}}{\rho_2}\right)\right)\right]^{p_{k,l}}$$

$$= 0.$$

So, $\alpha x + \beta y \in c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Hence $c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ is a linear space. Similarly, we can prove that $c^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ and $l^2_\infty(\Delta_n^m, \mathcal{M}, u, p, q, s)$ are linear spaces.

**Theorem 2.1.2**  Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. For $Z^2 = l^2_\infty$, $c^2$ and $c_0^2$, the spaces $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ are paranormed spaces, paranormed by

$$g(x) = \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf \left\{ \rho^{p_{k,l}} : \sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho}\right)\right) \leq 1 \right\},$$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

**Proof.**  Clearly $g(-x) = g(x)$, $g(0) = 0$. Let $(x_{k,l})$ and $(y_{k,l})$ be any two sequences belong to any one of the spaces $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$, for $Z^2 = c_0^2$, $c^2$ and $l^2_\infty$. Then, we get $\rho_1, \rho_2 > 0$ such that

$$\sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho_1}\right)\right) \leq 1$$

and

$$\sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l} \left(q\left(\frac{\Delta_n^m y_{k,l}}{\rho_2}\right)\right) \leq 1.$$
Let \( \rho = \rho_1 + \rho_2 \). Then by convexity of \( \mathcal{M} = (M_{k,l}) \), we have

\[
\sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m(x_{k,l} + y_{k,l})}{\rho} \right) \right)
\]

\[
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho_1} \right) \right) + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m y_{k,l}}{\rho_2} \right) \right)
\]

\[
\leq 1.
\]

Hence we have,

\[
g(x + y) = \sum_{k,l=1}^{mn} q(x_{k,l} + y_{k,l}) + \inf \left\{ \rho \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \leq 1 \right\}
\]

\[
\leq \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf \left\{ \rho_1 \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho_1} \right) \right) \leq 1 \right\}
\]

\[
+ \sum_{k,l=1}^{mn} q(y_{k,l}) + \inf \left\{ \rho_2 \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m y_{k,l}}{\rho_2} \right) \right) \leq 1 \right\}.
\]

This implies that

\[
g(x + y) \leq g(x) + g(y).
\]

The continuity of the scalar multiplication follows from the following inequality

\[
g(\mu x) = \sum_{k,l=1}^{mn} q(\mu x_{k,l}) + \inf \left\{ \rho \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m \mu x_{k,l}}{\rho} \right) \right) \leq 1 \right\}
\]

\[
= |\mu| \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf \left\{ (t|\mu|) \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{t} \right) \right) \leq 1 \right\},
\]

where \( t = \frac{\rho}{|\mu|} \). Hence the space \( Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s) \), for \( Z^2 = c_0^2, c_2 \) and \( l_\infty^2 \) is a paranormed space, paranormed by \( g \).

**Theorem 2.1.3** Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. For \( Z^2 = l_\infty^2, c_2 \) and \( c_0^2 \), the spaces \( Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s) \) are complete paranormed spaces, paranormed by

\[
g(x) = \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf \left\{ \rho \frac{p_{k,l}}{p_{k,l}} : \sup_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \leq 1 \right\},
\]
where $H = \max(1, \sup_{k,l} p_{k,l})$.

**Proof.** We prove the result for the space $l^2_\infty(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Let $(x_{k,l}^i)$ be any Cauchy sequence in $l^2_\infty(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Let $\epsilon > 0$ be given and for $t > 0$, choose $x_0$ be fixed such that $u_{k,l}M_{k,l} \left(\frac{tx_0}{2}\right) \geq 1$, then there exists a positive integer $n_0 \in \mathbb{N}$ such that $g(x_{k,l}^i - x_{k,l}^j) < \frac{\epsilon}{x_0 t}$, for all $i, j \geq n_0$. Using the definition of paranorm, we get

$$
\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) + \inf \left\{ \rho \frac{p_{k,l}}{\overline{\lambda}_l} : \sup_{k,l} (kl)^{-s} u_{k,l}M_{k,l} \left( q\left( \frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{\rho} \right) \right) \leq 1 \right\} < \frac{\epsilon}{x_0 t},
$$

for all $i, j \geq n_0$. \hfill (2.1.1)

Hence we have,

$$
\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) < \epsilon, \text{ for all } i, j \geq n_0.
$$

This implies that

$$
q(x_{k,l}^i - x_{k,l}^j) < \epsilon, \text{ for all } i, j \geq n_0 \text{ and } 1 \leq k, l \leq mn.
$$

Thus $(x_{k,l}^i)$ is a Cauchy sequence in $\mathbb{C}$ for $k, l = 1, 2, ..., mn$. Hence $(x_{k,l}^i)$ is convergent in $\mathbb{C}$ for $k, l = 1, 2, ..., mn$. Let \( \lim_{i \to \infty} x_{k,l}^i = x_{k,l} \), say for $k, l = 1, 2, ..., mn$. \hfill (2.1.2)

Again from equation (2.1.1) we have,

$$
\inf \left\{ \rho \frac{p_{k,l}}{\overline{\lambda}_l} : \sup_{k,l} (kl)^{-s} u_{k,l}M_{k,l} \left( q\left( \frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{\rho} \right) \right) \leq 1 \right\} < \epsilon, \text{ for all } i, j \geq n_0.
$$

Hence we get

$$
\sup_{k,l} (kl)^{-s} u_{k,l}M_{k,l} \left( q\left( \frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{\rho} \right) \right) \leq 1, \text{ for all } i, j \geq n_0.
$$

It follows that $(kl)^{-s} u_{k,l}M_{k,l} \left( q\left( \frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{g(x^i - x^j)} \right) \right) \leq 1$, for each $k, l \geq 1$ and for all $i, j \geq n_0$. For $t > 0$ with $(kl)^{-s} u_{k,l}M_{k,l} \left( \frac{tx_0}{2} \right) \geq 1$, we have

$$
(kl)^{-s} u_{k,l}M_{k,l} \left( q\left( \frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{g(x^i - x^j)} \right) \right) \leq (kl)^{-s} u_{k,l}M_{k,l} \left( \frac{tx_0}{2} \right).
$$

This implies that

$$q(\Delta_n^m x_{k,l}^i - \Delta_n^m x_{k,l}^j) < \frac{tx_0}{2} \frac{\epsilon}{tx_0} = \frac{\epsilon}{2}.
$$

Hence $q(\Delta_n^m x_{k,l}^i)$ is a Cauchy sequence in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. This implies that $q(\Delta_n^m x_{k,l}^i)$ is convergent in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. Let \( \lim_{i \to \infty} q(\Delta_n^m x_{k,l}^i) = y_{k,l} \) for each
Let $k, l \in \mathbb{N}$. Then for $k, l = 1$, we have
\[
\lim_{i \to \infty} q(\Delta^m_n x^{i}_{k,l}) = \lim_{i \to \infty} \sum_{v=0}^{m} (-1)^v \binom{m}{v} x^{i}_{1+nv,1+mv} = y_{1,1}.
\] (2.1.3)

We have by equations (2.1.2) and (2.1.3), $\lim_{i \to \infty} x^{i}_{mn+1} = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim_{i \to \infty} x^{i}_{k,l} = x_{k,l}$ exists for each $k, l \in \mathbb{N}$. Now we have for all $i, j \geq n_0$, 
\[
\sum_{k,l=1}^{mn} q(x^{i}_{k,l} - x^{j}_{k,l}) + \sup_{k,l} \rho \cdot u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n (x^{i}_{k,l} - x^{j}_{k,l})}{\rho} \right) \right) \leq 1 < \epsilon.
\]

This implies that
\[
\lim_{j \to \infty} \left\{ \sum_{k,l=1}^{mn} q(x^{i}_{k,l} - x^{j}_{k,l}) + \sup_{k,l} \rho \cdot u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n (x^{i}_{k,l} - x^{j}_{k,l})}{\rho} \right) \right) \leq 1 \right\} < \epsilon,
\]
for all $i \geq n_0$. Using the continuity of $(M_{k,l})$, we have
\[
\sum_{k,l=1}^{mn} q(x^{i}_{k,l} - x^{j}_{k,l}) + \sup_{k,l} \rho \cdot u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n (x^{i}_{k,l} - x^{j}_{k,l})}{\rho} \right) \right) \leq 1 < \epsilon,
\]
for all $i \geq n_0$. It follows that $(x^i - x) \in l_2^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$.

Since $x^i \in l_2^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$ and $l_2^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$ is a linear space, we have $x = x^i - (x^i - x) \in l_2^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$. This completes the proof. Similarly, we can prove that $c^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$ and $c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s)$ are complete paranormed spaces in view of the above proof.

**Theorem 2.1.4** If $m \geq 1$, then for all $0 < i \leq m$, $Z^2(\Delta^i_n, \mathcal{M}, u, p, q, s) \subset Z^2(\Delta^m_n, \mathcal{M}, u, p, q, s)$, where $Z^2 = c^2, c^2_0$ and $l_2^2$.

**Proof.** We shall prove it for only $c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s)$. Let $x = (x_{k,l}) \in c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s)$. Then
\[
P - \lim_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho} \right) \right) = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0.
\] (2.1.4)

Then from equation (2.1.4) we have
\[
P - \lim_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho} \right) \right) = 0,
\]
and
\[
P - \lim_{k,l} (kl)^{-s} u_{k,l} M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho} \right) \right) = 0.
\]
Now for
\[ \Delta^m_n x = (\Delta^m_n x_{k,l}) = (\Delta^{m-1}_n x_{k,l} - \Delta^{m-1}_n x_{k,l+1} - \Delta^{m-1}_n x_{k+1,l} + \Delta^{m-1}_n x_{k+1,l+1}), \]
we have
\[
(kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k,l}}{\rho} \right) + q \left( \frac{\Delta^{m-1}_n x_{k,l+1}}{\rho} \right) \right) \right]^{p_{k,l}}
+ q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) \right]^{p_{k,l}}
\leq K (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}}
+ [M \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) ]^{p_{k,l}}
+ [M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) ]^{p_{k,l}}
\leq K (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}}
+ K (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) \right]^{p_{k,l+1}}
+ K (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) \right]^{p_{k,l+1}}
+ K (kl)^{-s} u_{k,l} \left[ (kl)^{-s} M_{k,l} \left( q \left( \frac{\Delta^{m-1}_n x_{k+l+1}}{\rho} \right) \right) \right]^{p_{k,l+1}}.
\]

From this it follows that \( x = (x_{k,l}) \in c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s) \) and hence

\[ c^2_0(\Delta^{m-1}_n, \mathcal{M}, u, p, q, s) \subset c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s). \]

On applying the principle of induction, it follows that

\[ c^2_0(\Delta^i, \mathcal{M}, u, p, q, s) \subset c^2_0(\Delta^m_n, \mathcal{M}, u, p, q, s) \] for \( i = 0, 1, 2, \ldots, m - 1. \)

Similarly, we can prove the other cases.

**Theorem 2.1.5** (i) If \( 0 < \inf_{k,l} p_{k,l} \leq p_{k,l} < 1 \), then

\[ Z^2(\Delta^m_n, \mathcal{M}, u, p, q, s) \subset Z^2(\Delta^m_n, \mathcal{M}, u, q, s). \]

(ii) If \( 1 < p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty \), then

\[ Z^2(\Delta^m_n, \mathcal{M}, u, q, s) \subset Z^2(\Delta^m_n, \mathcal{M}, u, p, q, s), \]

where \( Z^2 = c^2, c^2_0 \) and \( l^2_\infty \).
Let $x = (x_{k,l}) \in l^2_{\infty}(\Delta^m_n, \mathcal{M}, u, p, q, s)$. Since $0 < \inf p_{k,l} \leq 1$, we have
\[
\sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right] \leq \sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}},
\]
and hence $x = (x_{k,l}) \in l^2_{\infty}(\Delta^m_n, \mathcal{M}, u, q, s)$.

(ii) Let $p_{k,l}$ for each $(k, l)$ and $\sup_{k,l} p_{k,l} < \infty$. Let $x = (x_{k,l}) \in l^2_{\infty}(\Delta^m_n, \mathcal{M}, u, q, s)$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $N$ such that
\[
\sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right] \leq \epsilon < 1,
\]
for all $k, l \in \mathbb{N}$. This implies that
\[
\sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq \sup_{k,l} (kl)^{-s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right].
\]
Thus $x = (x_{k,l}) \in l^2_{\infty}(\Delta^m_n, \mathcal{M}, u, p, q, s)$ and this completes the proof.

**Theorem 2.1.6** Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ be two sequences of Orlicz functions satisfying $\Delta_2$-condition. If $\beta = \lim_{t \to \infty} \frac{M''_{k,l}(t)}{t} \geq 1$, then
\[
Z^2 \left( \Delta^m_n, \mathcal{M}', u, p, q, s \right) = Z^2 \left( \Delta^m_n, \mathcal{M}'' \circ \mathcal{M}', u, p, q, s \right).
\]

**Proof.** We prove it for $Z^2 = c^2$ and the other cases will follow on applying similar techniques. Let $x = (x_{k,l}) \in c^2(\Delta^m_n, \mathcal{M}', u, p, q, s)$, then
\[
P = \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M'_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} = 0.
\]
Let $0 < \epsilon < 1$ and $\delta$ with $0 < \delta < 1$ such that $M''_{k,l}(t) < \epsilon$ for $0 \leq t < \delta$. Let
\[
y_{k,l} = M'_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l}}{\rho} \right) \right)
\]
and consider
\[
[M''_{k,l}(y_{k,l})]^{p_{k,l}} \left[ M'_{k,l}(y_{k,l}) \right]^{p_{k,l}} + [M''_{k,l}(y_{k,l})]^{p_{k,l}},
\]
where the first term is over $y_{k,l} \leq \delta$ and the second term is over $y_{k,l} > \delta$. From the first term in equation (2.1.5), we have
\[
(kl)^{-s} u_{k,l} [M''_{k,l}(y_{k,l})]^{p_{k,l}} < (kl)^{-s} u_{k,l} [M''_{k,l}(2)]^{p_{k,l}} \left[ \left( y_{k,l} \right) \right]^{p_{k,l}}.
\]
On the other hand, we use the fact that
\[
y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.
\]
Since \((M''_{k,l})\) is non-decreasing and convex, it follows that
\[
M''_{k,l}(y_{k,l}) - 1 + \frac{y_{k,l}}{\delta} < \frac{1}{2} M''_{k,l}(2) + \frac{1}{2} M''_{k,l}(\frac{2y_{k,l}}{\delta}).
\]
Since \((M''_{k,l})\) satisfies \(\Delta_2\)-condition, we have
\[
M''_{k,l}(y_{k,l}) < \frac{1}{2} K \frac{y_{k,l}}{\delta} M''_{k,l}(2) + \frac{1}{2} K \frac{y_{k,l}}{\delta} M''_{k,l}(2) = K \frac{y_{k,l}}{\delta} M''_{k,l}(2).
\]
Hence, from the second term in equation (2.1.5), it follows that
\[
(kl)^{-s} u_{k,l} [M''(y_{k,l})]^p_{k,l} \leq \max \left(1, (KM''_{k,l}(2)\delta^{-1})^H \right) (kl)^{-s} u_{k,l} [y_{k,l}]^p_{k,l}. \tag{2.1.7}
\]
By the equations (2.1.6) and (2.1.7), taking limit in the Pringsheim sense, we have
\[
x = (x_{k,l}) \in c^2(\Delta_n^m, \mathcal{M}'', u, p, q, s). \text{ Observe that in this part of the proof we did not need } \beta \geq 1. \text{ Now, let } \beta \geq 1 \text{ and } x = (x_{k,l}) \in c^2(\mathcal{M}', \Delta_n^m, u, p, q). \text{ Then, we have } M'_{k,l}(t) \geq \beta(t) \text{ for all } t \geq 0. \text{ It follows that } x = (x_{k,l}) \in c^2(\Delta_n^m, \mathcal{M}'', u, p, q, s) \text{ implies } x = (x_{k,l}) \in c^2(\Delta_n^m, \mathcal{M}', u, p, q, s). \text{ This implies } c^2(\Delta_n^m, \mathcal{M}', u, p, q, s) = c^2(\Delta_n^m, \mathcal{M}'', u, p, q, s).
\]

**Theorem 2.1.7** Let \(\mathcal{M}' = (M'_{k,l})\) and \(\mathcal{M}'' = (M''_{k,l})\) be two sequences of Orlicz functions, \(q, q_1\) and \(q_2\) be seminorms and \(s, s_1\) and \(s_2\) be positive real numbers. Then

1. \(Z^2(\Delta_n^m, \mathcal{M}', u, p, q, s) \cap Z^2(\Delta_n^m, \mathcal{M}'', u, p, q, s) \subset Z^2(\Delta_n^m, \mathcal{M}' + \mathcal{M}'', u, p, q, s),\)
2. \(Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1, s) \cap Z^2(\Delta_n^m, \mathcal{M}, u, p, q_2, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1 + q_2, s),\)
3. if \(q_1\) is stronger than \(q_2\), then \(Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q_2, s),\)
4. if \(s_1 \leq s_2\), then \(Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s_1) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s_2),\)

where \(Z^2 = c^2, c_0^2\) and \(l_\infty^2\).

**Proof.** (1) Let \(x = (x_{k,l}) \in c^2(\Delta_n^m, \mathcal{M}', u, p, q, s) \cap c^2(\Delta_n^m, \mathcal{M}'', u, p, q, s).\) Then
\[
P = \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M'_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho_1} \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_1 > 0,
\]
\[
P = \lim_{k,l} (kl)^{-s} u_{k,l} \left[ M''_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho_2} \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_2 > 0.
\]
Let \(\rho = \max(\rho_1, \rho_2).\) The result follows from the following inequality
\[
(kl)^{-s} \left[ (M'_{k,l} + M''_{k,l}) \left( q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right) \right]^{p_{k,l}} \leq K \left\{ (kl)^{-s} u_{k,l} \left[ M'_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho_1} \right) \right) \right]^{p_{k,l}} + (kl)^{-s} u_{k,l} \left[ M''_{k,l} \left( q \left( \frac{\Delta_n^m x_{k,l} - L}{\rho_2} \right) \right) \right]^{p_{k,l}} \right\}.
\]
The proofs of (2), (3) and (4) follows by same pattern.

**Theorem 2.1.8** If \( q_1 \equiv \text{(equivalent to)} \) \( q_2 \), then
\[
Z^2(\Delta^m_n; \mathcal{M}, u, p, q_1, s) = Z^2(\Delta^m_n; \mathcal{M}, u, p, q_2, s).
\]

**Proof.** It is easy to prove so we omit the details.

### 2.2 Double entire sequence spaces

In this section we make an attempt to study some interesting inclusion relations between double entire sequence spaces \( \Lambda^2_M(\Delta^m_n, u, p, q) \) and \( \Gamma^2_M(\Delta^m_n, u, p, q) \) defined by a sequence of Orlicz functions. The paranormed property and linearity of sequence spaces \( \Lambda^2_M(\Delta^m_n, u, p, q) \) and \( \Gamma^2_M(\Delta^m_n, u, p, q) \) are discussed in the beginning of this section.

Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers, \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers and \( X \) be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms \( q \). The symbol \( \Lambda^2(X) \), \( \Gamma^2(X) \) denotes the space of all double analytic and double entire sequences respectively defined over \( X \).

Now, we define the following sequence spaces in this section:

\[
\Lambda^2_M(\Delta^m_n, u, p, q) = \left\{ x \in \Lambda^2(X) : \sup_{r,s} \frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^m_n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\},
\]

\[
\Gamma^2_M(\Delta^m_n, u, p, q) = \left\{ x \in \Gamma^2(X) : \frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^m_n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty, \text{ for some } \rho > 0 \right\}.
\]

**Theorem 2.2.1** Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then the spaces \( \Gamma^2_M(\Delta^m_n, u, p, q) \) and \( \Lambda^2_M(\Delta^m_n, u, p, q) \) are linear spaces over the field of complex numbers \( \mathbb{C} \).
Proof. Let \( x, y \in \Gamma^2_M(\Delta^m_n, u, p, q) \) and \( \alpha, \beta \in \mathbb{C} \). In order to prove the result, we need to find some \( \rho_3 > 0 \) such that

\[
\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3} \right) \right) \right]^{p_{k,l}} \to 0 \quad \text{as} \quad r, s \to \infty. \tag{2.2.1}
\]

Since \( x, y \in \Gamma^2_M(\Delta^m_n, u, p, q) \), there exist some positive numbers \( \rho_1 \) and \( \rho_2 \) such that

\[
\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_1} \right) \right) \right]^{p_{k,l}} \to 0 \quad \text{as} \quad r, s \to \infty \tag{2.2.2}
\]

and

\[
\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_2} \right) \right) \right]^{p_{k,l}} \to 0 \quad \text{as} \quad r, s \to \infty. \tag{2.2.3}
\]

Since \( M = (M_{k,l}) \) is a non-decreasing convex function, \( q \) is a seminorm, \( \Delta^m_n \) is linear and so by using inequality (2.1), we have

\[
\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3} + \frac{\Delta^m_n (x_{k,l})}{\rho_3} \right) \right) \right]^{p_{k,l}}.
\]

Take \( \rho_3 > 0 \) such that \( \frac{1}{\rho_3} = \min \left\{ \frac{1}{|\alpha| \rho_1}, \frac{1}{|\beta| \rho_2} \right\} \)

\[
\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (\alpha x_{k,l})}{\rho_1} + \frac{\Delta^m_n (y_{k,l})}{\rho_2} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (x_{k,l})}{\rho_1} \right) \right) \right]^{p_{k,l}} + \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (y_{k,l})}{\rho_2} \right) \right) \right]^{p_{k,l}} \leq K \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (x_{k,l})}{\rho_1} \right) \right) \right]^{p_{k,l}} + K \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (y_{k,l})}{\rho_2} \right) \right) \right]^{p_{k,l}} \to 0 \quad \text{as} \quad r, s \to \infty.
\]
Hence
\[ \sum_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l} + \beta \Delta^m_n y_{k,l}}{\rho^3} \right) \right) \right]^{p_{k,l}} \to 0 \quad \text{as} \quad r, s \to \infty. \]

This proves that \( \Gamma^2_M(\Delta^m_n, u, p, q) \) is a linear space. Similarly, we can prove that \( \Lambda^2_M(\Delta^m_n, u, p, q) \) is a linear space.

**Theorem 2.2.2** Let \( M = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then the space \( \Gamma^2_M(\Delta^m_n, u, p, q) \) is a paranormed space with paranorm defined by

\[ g(x) = \inf \left\{ \rho \frac{p_{r,s}}{pr,s} : \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq 1, \; \rho > 0 \right\}, \]

where \( H = \max(1, \sup_{k,l} p_{k,l}) \).

**Proof.** Clearly \( g(x) \geq 0, \; g(x) = g(-x) \) and \( g(\theta) = 0 \), where \( \theta \) is the zero sequence of \( X \). For \((x_{k,l}), \; (y_{k,l}) \in \Gamma^2_M(\Delta^m_n, u, p, q)\), then there exist \( \rho_1, \; \rho_2 > 0 \) be such that

\[ \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho_1} \right) \right) \right]^{p_{k,l}} \leq 1 \]

and

\[ \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n y_{k,l}}{\rho_2} \right) \right) \right]^{p_{k,l}} \leq 1. \]

Suppose that \( \rho = \rho_1 + \rho_2 \), then

\[ \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n (x_{k,l} + y_{k,l})}{\rho} \right) \right) \right]^{p_{k,l}} \]

\[ \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho_1} \right) \right) \right]^{p_{k,l}} \]

\[ + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n y_{k,l}}{\rho_2} \right) \right) \right]^{p_{k,l}} \]

\[ \leq 1. \]

Hence

\[ g(x + y) \leq \inf \left\{ (\rho_1 + \rho_2) \frac{p_{r,s}}{pr,s} : \right. \]

\[ \left. \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\Delta^m_n x_{k,l}}{\rho_1 + \rho_2} \right) \right) \right]^{p_{k,l}} \leq 1, \; \rho_1, \; \rho_2 > 0, \; r, s \in \mathbb{N} \right\} \]
Thus we have

\[ g = \inf \left\{ (\rho_1)_{p,r} : \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \leq 1, \ \rho_1 > 0, \ r, s \in \mathbb{N} \right\} \]

+ \inf \left\{ (\rho_2)_{p,r} : \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \left( \frac{\Delta^m_n y_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \leq 1, \ \rho_2 > 0, \ r, s \in \mathbb{N} \right\}.

Thus we have \( g(x + y) \leq g(x) + g(y) \). Hence \( g \) satisfies the triangle inequality \( g(\lambda x) \)

\[ = \inf \left\{ (\rho)_{p,r} : \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \leq 1, \ \rho > 0, \ r, s \in \mathbb{N} \right\} \]

\[ = \inf \left\{ (t|\lambda)_{p,r} : \sup_{k,l \geq 1} u_{k,l} \left[ M_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \leq 1, \ t > 0, \ r, s \in \mathbb{N} \right\}, \]

where \( t = \frac{\rho}{|\lambda|} \). Hence \( \Gamma^2_{\mathcal{M}}(\Delta^m_n, u, p, q) \) is a paranormed space.

**Theorem 2.2.3** If \( \mathcal{M}' = (M'_{k,l}) \) and \( \mathcal{M}'' = (M''_{k,l}) \) be two sequences of Orlicz functions, then

\[ \Gamma^2_{\mathcal{M}'}(\Delta^m_n, u, p, q) \cap \Gamma^2_{\mathcal{M}''}(\Delta^m_n, u, p, q) \subseteq \Gamma^2_{\mathcal{M}' + \mathcal{M}''}(\Delta^m_n, u, p, q). \]

**Proof.** Let \( x \in \Gamma^2_{\mathcal{M}'}(\Delta^m_n, u, p, q) \cap \Gamma^2_{\mathcal{M}''}(\Delta^m_n, u, p, q) \). Then there exist \( \rho_1 \) and \( \rho_2 \) such that

\[ \frac{1}{(r, s)} \sum_{k,l = 1, 1}^{r,s} u_{k,l} \left[ M'_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty \quad (2.2.4) \]

and

\[ \frac{1}{(r, s)} \sum_{k,l = 1, 1}^{r,s} u_{k,l} \left[ M''_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty. \quad (2.2.5) \]

Let \( \rho = \min \left( \frac{1}{|\rho_1|}, \frac{1}{|\rho_2|} \right) \). Then we have

\[ \frac{1}{(r, s)} \sum_{k,l = 1, 1}^{r,s} u_{k,l} \left[ (M'_{k,l} + M''_{k,l}) \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \]

\[ \leq K \left[ \frac{1}{(r, s)} \sum_{k,l = 1, 1}^{r,s} u_{k,l} \left[ M'_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \right] + \]

\[ + K \left[ \frac{1}{(r, s)} \sum_{k,l = 1, 1}^{r,s} u_{k,l} \left[ M''_{k,l} \left( q \left( \left( \frac{\Delta^m_n x_{k,l}}{r_s} \right)^{\frac{1}{k+l}} \right) \right) \right]^{p_{k,l}} \right] \to 0 \text{ as } r, s \to \infty \]

by equations (2.2.4) and (2.2.5). Therefore \( x \in \Gamma^2_{\mathcal{M}' + \mathcal{M}''}(\Delta^m_n, u, p, q) \) and this completes the proof of the theorem.
**Theorem 2.2.4**  If \( m \geq 1 \). Then we have the following inclusions:

(i) \( \Gamma^2_M(\Delta^{m-1}_n, u, p, q) \subseteq \Gamma^2_M(\Delta^m_n, u, p, q) \),

(ii) \( \Lambda^2_M(\Delta^{m-1}_n, u, p, q) \subseteq \Lambda^2_M(\Delta^m_n, u, p, q) \).

**Proof.** (i) Let \( x \in \Gamma^2_M(\Delta^{m-1}_n, u, p, q) \). Then we have

\[
\frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l}(q \left( \frac{|\Delta^{m-1}_n x_{k,l}|^{1/(k+l)}}{\rho} \right)) \right] \rightarrow 0 \quad \text{as} \quad r, s \rightarrow \infty, \quad \text{for some} \quad \rho > 0.
\]

Since \( M = (M_{k,l}) \) is non-decreasing, convex function and \( q \) is a seminorm, we have

\[
\frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l}(q \left( \frac{|\Delta^{m-1}_n x_{k,l}|^{1/(k+l)}}{\rho} \right)) \right] \leq K \left\{ \frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l}(q \left( \frac{|\Delta^{m-1}_n x_{k,l}|^{1/(k+l)}}{\rho} \right)) \right] \right\} 
\]

\[
\rightarrow 0 \quad \text{as} \quad r, s \rightarrow \infty.
\]

Therefore

\[
\frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l}(q \left( \frac{|\Delta^{m-1}_n x_{k,l}|^{1/(k+l)}}{\rho} \right)) \right] \rightarrow 0 \quad \text{as} \quad r, s \rightarrow \infty. \quad \text{Hence} \quad x \in \Gamma^2_M(\Delta^m_n, u, p, q). \]

This completes the proof of (i). Similarly, we can prove (ii).

**Theorem 2.2.5**  If \( 0 \leq p_{k,l} \leq t_{k,l} \) and let \( \left\{ \frac{t_{k,l}}{p_{k,l}} \right\} \) be bounded, then

\( \Gamma^2_M(\Delta^m_n, u, t, q) \subseteq \Gamma^2_M(\Delta^m_n, u, p, q) \).

**Proof.** Let \( x \in \Gamma^2_M(\Delta^m_n, u, t, q) \). Then

\[
\frac{1}{(r,s)} \sum_{k,l=1}^{r,s} u_{k,l} \left[ M_{k,l}(q \left( \frac{|\Delta^m_n x_{k,l}|^{1/(k+l)}}{\rho} \right)) \right] \rightarrow 0 \quad \text{as} \quad r, s \rightarrow \infty. \quad (2.2.6)
\]
Let \( w_{k,l} = \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{t_{k,l}} \) and \((\lambda_{k,l}) = (\frac{p_{k,l}}{t_{k,l}})\). Since \( p_{k,l} \leq t_{k,l} \), we have \( 0 \leq \lambda_{k,l} \leq 1 \). Take \( 0 < \lambda < \lambda_{k,l} \). Define

\[ u_{k,l} = \begin{cases} w_{k,l}, & \text{if } w_{k,l} \geq 1 \\ 0, & \text{if } w_{k,l} < 1 \end{cases} \]

and

\[ v_{k,l} = \begin{cases} 0, & \text{if } w_{k,l} \geq 1 \\ w_{k,l}, & \text{if } w_{k,l} < 1 \end{cases} \]

\( w_{k,l} = u_{k,l} + v_{k,l} \), \( w_{k,l}^{\lambda} = \frac{\lambda_{k,l}}{t_{k,l}} \). It follows that \( u_{k,l}^{\lambda} \leq u_{k,l} \leq w_{k,l} \), \( v_{k,l}^{\lambda} \leq v_{k,l} \).

Since \( w_{k,l} = u_{k,l}^{\lambda} + v_{k,l} \), then \( w_{k,l}^{\lambda} \leq w_{k,l} + v_{k,l} \).

\[
\frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{\lambda_{k,l}} \leq \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{t_{k,l}}
\]

\[
\Rightarrow \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{p_{k,l}/t_{k,l}} \leq \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{t_{k,l}}
\]

\[
\Rightarrow \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{t_{k,l}}
\]

\[
\rightarrow 0 \text{ as } r,s \rightarrow \infty \text{ (by equation (2.6))}.
\]

Therefore

\[
\frac{1}{(r,s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\left| \Delta_{n}^{m} x_{k,l} \right|^{1+k}}{\rho} \right) \right) \right]^{p_{k,l}} \rightarrow 0 \text{ as } r,s \rightarrow \infty.
\]

Hence \( x \in \Gamma_{M}^{\mathcal{A}}(\Delta_{n}^{m}, u, p, q) \). From equation (2.6), we get

\[
\Gamma_{M}^{\mathcal{A}}(\Delta_{n}^{m}, u, t, q) \subset \Gamma_{M}^{\mathcal{A}}(\Delta_{n}^{m}, u, p, q).
\]
Theorem 2.2.6  (i) If $0 < \inf p_{k,l} \leq p_{k,l} \leq 1$, then $\Gamma^2_M(\Delta^n, u, p, q) \subset \Gamma^2_M(\Delta^n, u, q)$.

(ii) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then $\Gamma^2_M(\Delta^n, u, q) \subset \Gamma^2_M(\Delta^n, u, p, q)$.

Proof. (i) Let $x \in \Gamma^2_M(\Delta^n, u, p, q)$. Then

$$\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty. \quad (2.2.7)$$

Since $0 < \inf p_{k,l} \leq p_{k,l} \leq 1$,

$$\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty. \quad (2.2.8)$$

From equations (2.2.7) and (2.2.8), it follows that, $x \in \Gamma^2_M(\Delta^n, u, q)$. Thus

$$\Gamma^2_M(\Delta^n, u, p, q) \subset \Gamma^2_M(\Delta^n, u, q).$$

(ii) Let $p_{k,l} \geq 1$ for each $k, l$ and $\sup p_{k,l} < \infty$ and let $x \in \Gamma^2_M(\Delta^n, u, q)$. Then

$$\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty. \quad (2.2.9)$$

Since $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, we have

$$\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty.$$

This implies that $x \in \Gamma^2_M(\Delta^n, u, p, q)$. Therefore $\Gamma^2_M(\Delta^n, u, q) \subset \Gamma^2_M(\Delta^n, u, p, q)$.

Theorem 2.2.7  If $\frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta^n x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right] \leq |x_{k,l}|^{1/k+l}$, then $\Gamma^2 \subset \Gamma^2_M(\Delta^n, u, p, q)$.

Proof. Let $x \in \Gamma^2$. Then we have,

$$|x_{k,l}|^{1/k+l} \rightarrow 0 \text{ as } k, l \rightarrow \infty. \quad (2.2.10)$$
But \( \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq |x_{k,l}|^{1/k+l} \), by our assumption, implies that

\[ \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty \text{ by (2.2.9)}. \]

Then \( x \in \Gamma^2_\mathcal{M}(\Delta_n^m, u, p, q) \) and \( \Gamma^2 \subset \Gamma^2_\mathcal{M}(\Delta_n^m, u, p, q) \).

**Theorem 2.2.8** \( \Gamma^2_\mathcal{M}(\Delta_n^m, u, p, q) \) is a solid space.

**Proof.** Let \( (x_{k,l}) \in \Gamma^2_\mathcal{M}(\Delta_n^m, u, p, q) \), then

\[ \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty, \text{ for some } \rho > 0. \]

Let \( (\alpha_{k,l}) \) be a double sequence of scalars such that \( |\alpha_{k,l}| \leq 1 \) for all \( k, l \in \mathbb{N} \times \mathbb{N} \). Then we have

\[ \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{(r, s)} \sum_{k,l=1,1}^{r,s} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{|\Delta x_{k,l}|^{1/k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \to 0 \text{ as } r, s \to \infty \]

and this completes the proof.

**Corollary 2.2.9** \( \Gamma^2_\mathcal{M}(\Delta_n^m, u, p, q) \) is a monotone space.

**Proof.** It is obvious.

### 2.3 Chi double sequence spaces

This section presents a study of double chi and double analytic sequence spaces \( \chi^2_\mathcal{M}[\hat{c}, \Delta^m, u, p, q] \) and \( \Lambda^2_\mathcal{M}[\hat{c}, \Delta^m, u, p, q] \) defined by a sequence of Orlicz functions. It turns out that these spaces are linear and paranormed spaces. Some inclusion relations are reported in the end of this section.

Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Also, let \( (X, q) \) be a seminormed space over the field of complex numbers \( \mathbb{C} \) with the semi norm \( q \). We define the following sequence spaces in this section:
\[ \chi^2_M[\hat{c}, \Delta^m, u, p, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q\left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{1/(k+s+l)}}{\rho} \right) \right) \right] p_{k,l} = 0, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \right\} \]

and

\[ \Lambda^2_M[\hat{c}, \Delta^m, u, p, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \sup_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q\left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{1/(k+s+l)}}{\rho} \right) \right) \right] p_{k,l} < \infty, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \right\} . \]

If \( M(x) = x \), we get

\[ \chi^2[\hat{c}, \Delta^m, u, p, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q\left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{1/(k+s+l)}}{\rho} \right) \right) \right] p_{k,l} = 0, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \right\} \]

and

\[ \Lambda^2[\hat{c}, \Delta^m, u, p, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \sup_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q\left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{1/(k+s+l)}}{\rho} \right) \right) \right] p_{k,l} < \infty, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \right\} . \]

If \( p = (p_{k,l}) = 1 \) for all \( k, l \), we get

\[ \chi^2_M[\hat{c}, \Delta^m, u, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q\left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{1/(k+s+l)}}{\rho} \right) \right) \right] = 0, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \right\} \]

and
\[ \Lambda^2_M[\hat{c}, \Delta^m, u, q] = \]
\[ \left\{ x = (x_{k,l}) \in w^2(X) : \sup_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho} \right) \right) \right] < \infty, \]
\[ \text{uniformly in } s, \text{ for some } \rho > 0 \} \]

**Theorem 2.3.1** Let \( M = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then \( \chi_M^2[\hat{c}, \Delta^m, u, p, q] \) and \( \Lambda^2_M[\hat{c}, \Delta^m, u, p, q] \) are linear spaces over the field of complex numbers \( \mathbb{C} \).

**Proof.** Let \( x = (x_{k,l}) \in \chi_M^2[\hat{c}, \Delta^m, u, p, q] \) and \( \alpha, \beta \in \mathbb{C} \). Then there exist positive real numbers \( \rho_1, \rho_2 > 0 \) such that
\[ \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_1} \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_1 > 0 \]
and
\[ \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m y_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_2} \right) \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_2 > 0. \]
Let \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( M = (M_{k,l}) \) is non-decreasing, convex and therefore by using inequality (2.1), we have
\[ \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^{m(\alpha x_{k,s,l+s} + \beta y_{k,s,l+s})|)^{\frac{1}{\kappa_s + l_t}}}{\rho_3} \right) \right) \right]^{p_{k,l}} \]
\[ \leq \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_3} \right) \right) \right]^{p_{k,l}} \]
\[ + \left( \frac{((k + l)!|\Delta^m \beta y_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_3} \right)^{p_{k,l}} \]
\[ \leq K \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} \frac{1}{2^{k,l}} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_1} \right) \right) \right]^{p_{k,l}} \]
\[ + K \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} \frac{1}{2^{k,l}} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m y_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_2} \right) \right) \right]^{p_{k,l}} \]
\[ \leq K \lim_{\mu^\gamma \to \infty} \frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k,s,l+s}|)^{\frac{1}{\kappa_s + l_t}}}{\rho_1} \right) \right) \right]^{p_{k,l}} \]
Thus \( \alpha x + \beta y \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \). This proves that \( \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) is a linear space. Similarly, we can prove that \( \Lambda^2_M[\hat{c}, \Delta^m, u, p, q] \) is a linear space. This completes the proof of the theorem.

**Theorem 2.3.2** Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then \( \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) is a paranormed space with the paranorm

\[
g(x) = \inf \left\{ \rho^{\mu_\gamma H} : \sup_{\mu_\gamma \geq 1} \frac{1}{\mu_\gamma} \sum_{k,l=1}^{\mu_\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!! \Delta^m x_{k+s,l+s})}{\rho_1} \right) \right) \right]^{p_{k,l}} \leq 1, \rho > 0 \right\}, \text{ where } H = \max \left( 1, \sup_{k,l} p_{k,l} \right).
\]

**Proof.**

(i) Clearly \( g(x) \geq 0 \) for \( x = (x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \). Since \( \mathcal{M}(0) = 0 \), we get \( g(0) = 0 \).

(ii) \( g(-x) = g(x) \).

(iii) Let \( x = (x_{k,l}), \ y = (y_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \), then there exist positive number \( \rho_1, \rho_2 > 0 \) such that

\[
\lim_{\mu_\gamma \to \infty} \frac{1}{\mu_\gamma} \sum_{k,l=1}^{\mu_\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!! \Delta^m x_{k+s,l+s})}{\rho_1} \right) \right) \right]^{p_{k,l}} = 0
\]

and

\[
\lim_{\mu_\gamma \to \infty} \frac{1}{\mu_\gamma} \sum_{k,l=1}^{\mu_\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!! \Delta^m y_{k+s,l+s})}{\rho_2} \right) \right) \right]^{p_{k,l}} = 0.
\]

Let \( \rho = \rho_1 + \rho_2 \). Then by using Minkowski’s inequality, we have

\[
u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!! \Delta^m (x_{k+s,l+s} + y_{k+s,l+s}))}{\rho} \right) \right) \right]^{p_{k,l}} \leq \nu_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!! \Delta^m (x_{k+s,l+s}))}{\rho_1 + \rho_2} \right) \right) \right]^{p_{k,l}}
\]
\begin{align*}
&+ u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (y_{k+s,l+s}) ||^{\frac{1}{\rho_1 + \rho_2}}} \rho_1 + \rho_2 \right) \right) \right]^{p_{k,l}} \\
&\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (x_{k+s,l+s}) ||^{\frac{1}{\rho_1}}} \rho_1 \right) \right) \right]^{p_{k,l}} \\
&+ \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (y_{k+s,l+s}) ||^{\frac{1}{\rho_2}}} \rho_2 \right) \right) \right]^{p_{k,l}}
\end{align*}

and thus

\[ g(x + y) = \inf \left\{ \rho_1, \rho_2 \right\}^{p_{\mu, \gamma}/H} : \sup_{\mu \geq 1} \frac{1}{\mu} \sum_{k,l=1}^{\mu} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (x_{k+s,l+s} + y_{k+s,l+s}) ||^{\frac{1}{\rho_1 + \rho_2}}} \rho_1 + \rho_2 \right) \right) \right]^{p_{k,l}} \leq \inf \left\{ \rho_1, \rho_2 \right\}^{p_{\mu, \gamma}/H} : \sup_{\mu \geq 1} \frac{1}{\mu} \sum_{k,l=1}^{\mu} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (x_{k+s,l+s}) ||^{\frac{1}{\rho_1}}} \rho_1 \right) \right) \right]^{p_{k,l}} + \inf \left\{ \rho_1, \rho_2 \right\}^{p_{\mu, \gamma}/H} : \sup_{\mu \geq 1} \frac{1}{\mu} \sum_{k,l=1}^{\mu} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (y_{k+s,l+s}) ||^{\frac{1}{\rho_2}}} \rho_2 \right) \right) \right]^{p_{k,l}}. \]

Now, let \( \lambda \in \mathbb{C} \), then the continuity of the product follows from the following inequality

\[ g(\lambda x) = \inf \left\{ \rho_1, \rho_2 \right\}^{p_{\mu, \gamma}/H} : \sup_{\mu \geq 1} \frac{1}{\mu} \sum_{k,l=1}^{\mu} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m \lambda x_{k+s,l+s} ||^{\frac{1}{\rho_1 + \rho_2}}} \rho_1 + \rho_2 \right) \right) \right]^{p_{k,l}} = \inf \left\{ \lambda \left\| \rho_1, \rho_2 \right\}^{p_{\mu, \gamma}/H} : \sup_{\mu \geq 1} \frac{1}{\mu} \sum_{k,l=1}^{\mu} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k + l) || \Delta^m (y_{k+s,l+s}) ||^{\frac{1}{\rho_2}}} \rho_2 \right) \right) \right]^{p_{k,l}}, \]

where \( r = \frac{\rho_1}{|\lambda|} \). This completes the proof of the theorem.

**Theorem 2.3.3** Let \( \mathcal{M} = (M_{k,l}) \) and \( \mathcal{T} = (T_{k,l}) \) be two sequences of Orlicz functions. Then

\[ \chi_{\mathcal{M}}^2 [\hat{c}, \Delta^m, u, p, q] \cap \chi_{\mathcal{T}}^2 [\hat{c}, \Delta^m, u, p, q] \subseteq \chi_{\mathcal{M} + \mathcal{T}}^2 [\hat{c}, \Delta^m, u, p, q]. \]

**Proof.** It is easy to prove, so we omit the details.

**Theorem 2.3.4** Let \( \mathcal{M} = (M_{k,l}) \) and \( \mathcal{T} = (T_{k,l}) \) be two sequences of Orlicz functions and let \( q_1 \) and \( q_2 \) be two seminorms on \( X \), we have
(i) $\chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_1] \cap \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_2] \subseteq \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_1 + q_2],$

(ii) if $q_1$ is stronger than $q_2$ then $\chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_1] \subseteq \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_2],$

(iii) if $q_1$ is equivalent to $q_2$ then $\chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_1] = \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q_2].$

**Proof.** It is trivial, so we omit it.

**Theorem 2.3.5**

(i) Let $0 \leq p_{k,l} \leq r_{k,l}$ and $\left\{ \frac{r_{k,l}}{p_{k,l}} \right\}$ be bounded. Then

$$\chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, r, q] \subset \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q].$$

(ii) If $u_1 \leq u_2$ implies $\chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u_1, p, q] \subset \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u_2, p, q].$

**Proof.**

(i) Let $x = (x_{k,l}) \in \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, r, q]$. Then

$$\lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{q_s + t_s}}}{r} \right) \right) \right]_{r_{k,l}} = 0.$$

Let

$$t_{k,l} = \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{q_s + t_s}}}{r} \right) \right) \right]_{r_{k,l}}$$

and $(\lambda_{k,l}) = \left( \frac{p_{k,l}}{r_{k,l}} \right)$. Since $p_{k,l} \leq r_{k,l}$, we have $0 \leq \lambda_{k,l} \leq 1$. Take $0 < \lambda < \lambda_{k,l}$.

Define $u_{k,l} = t_{k,l} (t_{k,l} \geq 1)$; $u_{k,l} = 0 (t_{k,l} < 1)$; and $v_{k,l} = 0 (t_{k,l} \geq 1)$; $v_{k,l} = t_{k,l} (t_{k,l} < 1)$; $t_{k,l} = u_{k,l} + v_{k,l}$; $t_{\lambda,k,l} \leq v_{\lambda,k,l}$.

Now it follows that

$$u_{\lambda,k,l} \leq t_{k,l} \text{ and } v_{\lambda,k,l} \leq v_{k,l}$$ (2.3.1)

i.e $t_{\lambda,k,l} = u_{\lambda,k,l} + v_{\lambda,k,l}$, $t_{\lambda,k,l} \leq t_{k,l} + v_{k,l}$ by equation (2.3.1). Thus

$$\lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{q_s + t_s}}}{r} \right) \right) \right]_{r_{k,l}} \lambda_{k,l}$$

$$\leq \lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{q_s + t_s}}}{r} \right) \right) \right]_{r_{k,l}}$$
⇒ \lim_{\mu \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{\frac{p_{k,l}}{r_{k,l}}} \\
\leq \lim_{\mu \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{r_{k,l}} \\
\Rightarrow \lim_{\mu \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{r_{k,l}} \\
= \lim_{\mu \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{r_{k,l}} = 0,
we have
\\
\lim_{\mu \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{p_{k,l}} = 0.

Hence \( x = (x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \). This proves that
\[
\chi^2_M[\hat{c}, \Delta^m, u, r, q] \subseteq \chi^2_M[\hat{c}, \Delta^m, u, p, q].
\]

(ii) The proof is easy, so omitted.

**Theorem 2.3.6** Let \( M = (M_{k,l}) \) be a sequence of Orlicz functions, \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be a sequence of strictly positive real numbers. Then the following statements are equivalent:

(i) \( \Lambda^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q] \),

(ii) \( \chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2_M[\hat{c}, \Delta^m, u, p, q] \),

(iii) \( \sup_{\mu} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!!(\Delta^m x_{k+s,l+s})^{\frac{1}{\lambda(x+l)}})}{\rho} \right) \right) \right]^{p_{k,l}} < \infty. \)

**Proof.** (i) \( \Rightarrow \) (ii) is obvious.
(ii) ⇒ (iii) Let $\chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$. Suppose that (iii) is not satisfied. Then for some $\rho > 0$, we have

$$\sup_{\mu \gamma} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k+l)!\Delta^m x_{k+l,s+l})^{\frac{1}{k_s+l_s}}}{\rho} \right) \right) \right]^{p_{k,l}} = \infty$$

and therefore there is sequence $(\mu_i \gamma_i)$ of positive integers such that

$$\frac{1}{\mu_i \gamma_i} \sum_{k,l=1}^{\mu_i \gamma_i} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{i-1}{\rho} \right) \right) \right]^{p_{k,l}} > i, \ i = 1, 2, \ldots.$$  \hspace{1cm} (2.3.2)

Define $x = (x_{k,l})$ by

$$\left( (k+l)!\Delta^m x_{k,l} \right)^{\frac{1}{k+l}} = \begin{cases} i^{-1}, & \text{if } 1 \leq k, l \leq \mu_i \gamma_i, \ i = 1, 2, \ldots \\ 0, & \text{if } k > \mu_i, \ l > \gamma_i \end{cases}$$

Then $x \in \chi^2[\hat{c}, \Delta^m, u, p, q]$ but by equation (2.3.2), $x \notin \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$ which contradicts (ii). Hence (iii) must hold.

(iii) ⇒ (i) Let (iii) satisfied and $x = (x_{k,l}) \in \Lambda^2[\hat{c}, \Delta^m, u, p, q]$. Suppose that $x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q]$. Then

$$\sup_{(\mu \gamma)} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k+l)!\Delta^m x_{k+l,s+l})^{\frac{1}{k_s+l_s}}}{\rho} \right) \right) \right]^{p_{k,l}} = \infty. \hspace{1cm} (2.3.3)$$

Let $t = ((k+l)!\Delta^m x_{k+l,s+l})^{\frac{1}{k_s+l_s}}$ for each $k, l$ and fixed $s$, then by equation (2.3.3)

$$\sup_{(\mu \gamma)} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t}{\rho} \right) \right) \right]^{p_{k,l}} = \infty,$$

which contradicts (iii). Hence (i) must hold. This completes the proof.

**Theorem 2.3.7** Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Then the following statements are equivalent:

(i) $\chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q]$,

(ii) $\chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$,

(iii) $\inf_{\mu \gamma} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t}{\rho} \right) \right) \right]^{p_{k,l}} > 0, \ (t, \rho > 0).$
where \( t = ((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{s^2 + r^2}} \).

**Proof.** \((i) \Rightarrow (ii)\) is obvious.

\((ii) \Rightarrow (iii)\) Let \( \chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2[\hat{c}, \Delta^m, u, p, q] \). Suppose that \((iii)\) is not satisfied. Then for some \( \rho > 0 \), we have

\[
\inf_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{k,l=1}^{\mu\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t}{\rho} \right) \right) \right]^{p_{k,l}} = 0 \quad (t, \rho > 0). \tag{2.3.4}
\]

We can choose an index sequence \((\mu_i\gamma_i)\) of positive integers such that

\[
\frac{1}{\mu_i\gamma_i} \sum_{k,l=1}^{\mu_i\gamma_i} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{i}{\rho} \right) \right) \right]^{p_{k,l}} > i^{-1}, \quad i = 1, 2, \ldots.
\]

Define \( x = (x_{k,l}) \) by

\[
\left( (k + l)!|\Delta^m x_{k,l}| \right)^{\frac{1}{\mu\gamma}} = \begin{cases} 
1, & \text{if } 1 \leq k, l \leq \mu_i\gamma_i, \quad i = 1, 2, \ldots \\
0, & \text{if } k, l > \mu_i\gamma_i
\end{cases}
\]

Thus by equation (2.3.4) \( x \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) but \( x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q] \) which contradicts \((ii)\). Hence \((iii)\) must hold.

\((iii) \Rightarrow (i)\) Let \((iii)\) satisfied and \( x = (x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \), then

\[
\lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{k,l=1}^{\mu\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{s^2 + r^2}})}{\rho} \right) \right]^{p_{k,l}} = 0, \quad \text{uniformly in } s. \tag{2.3.5}
\]

Suppose that \( x \notin \chi^2[\hat{c}, \Delta^m, u, p, q] \). Then for some number \( \epsilon_0 > 0 \) and index \( \mu_0\gamma_0 \), we have \( (k_s + l_s)!|\Delta^m x_{k_s+l_s}|^{\frac{1}{s^2 + r^2}} \geq \epsilon_0 \), for some \( s > s' \) and \( 1 \leq k, l \leq \mu_0\gamma_0 \). Therefore,

\[
u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\epsilon_0}{\rho} \right) \right) \right]^{p_{k,l}} \leq u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k_s + l_s)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{s^2 + r^2}})}{\rho} \right) \right]^{p_{k,l}}
\]

and consequently by equation (2.3.5). Hence

\[
\lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{k,l=1}^{\mu\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\epsilon_0}{\rho} \right) \right) \right]^{p_{k,l}} = 0,
\]

which contradicts \((iii)\). Hence \( \chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q] \). This completes the proof.

**Theorem 2.3.8** Let \( 1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty \). The inclusion

\[
\Lambda^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q]
\]
hold if and only if

\[
\frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t}{\rho} \right) \right) \right]^{p_{k,l}} = \infty \quad (t, \rho > 0). \tag{2.3.6}
\]

**Proof.** Let \( \Lambda^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q] \). Suppose that equation (2.3.6) does not hold. Therefore there is a number \( t_0 > 0 \) and an index sequence \((\mu_i \gamma_i)\) such that

\[
\frac{1}{\mu_i \gamma_i} \sum_{k,l=1}^{\mu_i \gamma_i} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t_0}{\rho} \right) \right) \right]^{p_{k,l}} \leq N < \infty, \quad i = 1, 2, \ldots. \tag{2.3.7}
\]

Define the sequence \( x = (x_{k,l}) \) by

\[
(x_{k,l}) = \begin{cases} 
t_0, & \text{if } 1 \leq k, l \leq \mu_i \gamma_i, \ i = 1, 2, \ldots \\
0, & \text{if } k, l > \mu_i \gamma_i
\end{cases}
\]

Thus by equation (2.3.7), \( x \in \Lambda^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q] \), but \( x \notin \chi^2[\hat{c}, \Delta^m, u, p, q] \). Hence (2.3.6) must hold.

Conversely, let equation (2.3.6) hold. If \( x \in \Lambda^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q] \), then for each \( s \) and \( \mu^\gamma \)

\[
\frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k+l)!|\Delta^m x_{k+s,l+s}|^{\frac{1}{k+s+l}}}{} \right) \right) \right]^{p_{k,l}} \leq N < \infty. \tag{2.3.8}
\]

Suppose that \( x \notin \chi^2[\hat{c}, \Delta^m, u, p, q] \). Then for some number \( \epsilon_0 > 0 \) there is a number \( s_0 \) and index \( \mu_0 \gamma_0 \)

\[
((k+l)!|\Delta^m x_{k+s,l+s}|)^{\frac{1}{k+s+l}} \geq \epsilon_0, \quad \text{for } s \geq s_0.
\]

Therefore

\[
\begin{align*}
&u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\epsilon_0}{\rho} \right) \right) \right]^{p_{k,l}} \leq u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k+l)!|\Delta^m x_{k+s,l+s}|^{\frac{1}{k+s+l}}}{} \right) \right) \right]^{p_{k,l}} \\
&\text{and hence for each } k, l \text{ and } s, \text{ we get}
\end{align*}
\]

\[
\frac{1}{\mu^\gamma} \sum_{k,l=1}^{\mu^\gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{\epsilon_0}{\rho} \right) \right) \right]^{p_{k,l}} \leq N < \infty,
\]

for some \( N > 0 \), clearly equation (2.3.8) contradicts equation (2.3.6). Hence \( \Lambda^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q] \). This completes the proof.

**Theorem 2.3.9** Let \( 1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty \). The inclusion

\[
\Lambda^2[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2_{\mathcal{M}}[\hat{c}, \Delta^m, u, p, q]
\]
hold if and only if

\[
\lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t}{\rho} \right) \right) \right]^{p_{k,l}} = 0 \quad (t, \rho > 0). \quad (2.3.9)
\]

**Proof.** Let \( \Lambda^2[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2_M[\hat{c}, \Delta^m, u, p, q] \). Suppose that equation (2.3.9) does not hold. Therefore there is a number \( t_0 > 0 \).

\[
\lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma_i} \sum_{k,l=1}^{\mu \gamma_i} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{t_0}{\rho} \right) \right) \right]^{p_{k,l}} = L \neq 0. \quad (2.3.10)
\]

Define the sequence \( x = (x_{k,l}) \) by

\[
((k+l)!x_{k,l})^\frac{1}{k+l} = t_0 \sum_{v=0}^{k,l-\eta} (-1)^v \left( \gamma + (k,l) - v - 1 \over (k,l) - v \right)
\]

for \( k, l = 1, 2, \cdots \). Thus by equation (2.3.10), \( x \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \), but \( x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q] \). Hence equation (2.3.9) must hold.

Conversely, let equation (2.3.9) hold and \( x \in \Lambda^2[\hat{c}, \Delta^m, u, p, q] \), then for every \( m, n \) and \( s \)

\[
((k+l)!|\Delta^m x_{k+s,l+s}|)^\frac{1}{k+l} \leq N < \infty.
\]

Therefore

\[
u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k+l)!|\Delta^m x_{k+s,l+s}|^\frac{1}{k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq u_{k,l} \left[ M_{k,l} \left( \frac{N}{\rho} \right) \right]^{p_{k,l}}
\]

and

\[
\frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{(k+l)!|\Delta^m x_{k+s,l+s}|^\frac{1}{k+l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( \frac{N}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{by equation (2.3.9)}.
\]

Hence \( x = (x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \). This completes the proof.

**Theorem 2.3.10** The space \( \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) is solid.

**Proof.** Let \( x = (x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) and \( (\alpha_{k,l}) \) be a sequence of scalars such that \( |\alpha_{k,l}|^\frac{1}{k+l} \leq 1 \) for all \( k, l \in \mathbb{N} \). Then
\[
\lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|^{1/\rho})}{\rho} \right) \right) \right]^{p_{k,l}} \\
\leq \lim_{\mu \gamma \to \infty} \frac{1}{\mu \gamma} \sum_{k,l=1}^{\mu \gamma} u_{k,l} \left[ M_{k,l} \left( q \left( \frac{((k + l)!|\Delta^m x_{k+s,l+s}|^{1/\rho})}{\rho} \right) \right) \right]^{p_{k,l}}
\]

for all \( k, l \in \mathbb{N} \). Hence \((\alpha_{k,l}x_{k,l}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) for all sequences of scalars \( \alpha_{k,l} \) with \( |\alpha_{k,l}| \leq 1 \) for all \( k, l \in \mathbb{N} \) whenever \( x_{k,l} \in \chi^2_M[\hat{c}, \Delta^m, u, p, q] \).

**Theorem 2.3.11** The space \( \chi^2_M[\hat{c}, \Delta^m, u, p, q] \) is monotone.

**Proof.** It is obvious.