CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Sequence spaces

So far as we know in several branches of analysis, for instances, the structural theory of topological vector spaces, shauder basis theory, summability theory and theory of functions, the study of sequence spaces occupies a very prominent position. The impact and importance of this study can be appreciated when one sees the construction of numerous examples locally convex spaces obtained as a consequence of the dual structure displayed by several pairs of distinct sequence spaces, thus reflecting in depth the distinguishing structural features of the spaces in question. Besides these distinct sequence spaces endowed with different polar topologies provide an excellent source to vector space pathalogists for the introduction on locally convex spaces to several new and penetrating notions implicit in the theory of Banach spaces. Apart from this the theory of sequence spaces is a powerful tool for obtaining positive results concerning shauder basis and their associated types. Moreover, the theory of sequence spaces has made remarkable advances in enveloping summability theory via unified techniques effecting matrix transformation from one sequence space into another.

The study of Orlicz and Lorentz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg ([Lin 70], [Lin 73]) got interested in Orlicz spaces in connection with finding Banach spaces with symmetric shauder basis having complementary subspaces isomorphic to $c_0$ or $l_p (1 \leq p < \infty)$. Subsequently, Lindenstrauss and Tzafriri [LiT 71] studied these Orlicz sequence spaces in more details and solved many important and interesting structural problem in Banach spaces ([LiT 71], [LiT 72], [LiT 73]). In the meantime, Woo [Woo 73] generalized the concept of Orlicz sequence spaces to modular sequence spaces and this lead him to sharpen some of the results of Lindberg and of Lindenstrauss and Tzafriri (see [Woo 75]). The Orlicz sequence spaces are the special cases of Orlicz spaces which were introduced in [Orl 32] and extensively studied in [KrR 61]. Orlicz spaces find a number of useful application in the theory of non-linear integral equations, whereas the Orlicz sequence spaces are generalization of $l^p$-spaces, $L^p$-spaces find themselves enveloped in Orlicz spaces. For deeper results in Orlicz sequence spaces and modular sequence spaces one can refer to ([LiT 73], [LiT 77], [Woo 73]) and several references therein.

The sequence spaces are generalized in several direction by several mathematicians.
Some have studied single sequence spaces (see [PrC 94], [Esi 97], [EsE 2000], [Mur 83], [Sav 04]), while some of them have studied double sequence spaces for more details (see [AlB 05], [Bas 09], [Esi 11], [Mur 04], [SaP 11], [Tri 03]). The initial work on double sequences found in Bromwich [Bro 65]. Later on it was studied by Hardy [Har 17], Moricz [Mor 91], Moricz and Rhoades [MoR 88], Tripathy ([Tri 03], [Tri 04]), Basarir and Sonalcan [Bas 99] and many others. Hardy [Har 17] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [Zel 01] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [MuE 03] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Subsequently, Mursaleen [Mur 04] and Mursaleen and Edely [MuE 04] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M$-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the $M$-core of $x$. More recently, Altay and Basar [AlB 05] have defined the spaces $BS$, $BS(t)$, $CS_p$, $CS_{bp}$, $CS_r$ and $BV$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $M_u$, $M_u(t)$, $C_p$, $C_{bp}$, $C_r$ and $L_u$, respectively and also examined some properties of these sequence spaces and determined the $\alpha$-duals of the spaces $BS$, $BV$, $CS_{bp}$ and the $\beta(v)$-duals of the spaces $CS_{bp}$ and $CS_r$ of double series. Now, Basar and Sever [BaS 09] have introduced the Banach space $L_q$ of double sequences corresponding to the well known space $\ell_q$ of single sequences and examined some properties of the space $L_q$. By the convergence of a double sequence we mean the convergence of the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [Pri 1900]. We shall write more briefly as $P$-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number $M$ such that $|x_{k,l}| < M$ for all $k$ and $l$.

**Orlicz function**: An Orlicz function $M : [0, \infty) \to [0, \infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$.

Let $w$ be the space of all real or complex sequences $x = (x_k)$. Lindenstrauss and Tzafriri [LiT 71] used the idea of Orlicz function to define the following sequence
space,

$$\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \rho > 0 \right\},$$

known as an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$ 

It was shown in [LiT 71] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p(p \geq 1)$. An Orlicz function $M$ satisfies $\Delta_2$-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_{0}^{x} \eta(t)dt,$$

where $\eta$ is known as the kernel of $M$, is a right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

**Musielak-Orlicz function**: A sequence $M = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [Mal 89], [Mus 83]). A sequence $N = (N_k)$ defined by

$$N_k(v) = \sup \{|v|u - M_k(u) : u \geq 0\}, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $M$. For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ and its subspace $h_M$ are defined as follows:

$$t_M = \left\{ x \in w : I_M(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_M = \left\{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \right\},$$

where $I_M$ is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_M.$$

We consider $t_M$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \left\{ \frac{1}{k} \left( 1 + I_M(kx) \right) : k > 0 \right\}.$$
A Musielak-Orlicz function \((M_k)\) is said to satisfy \(\Delta_2\)-condition if there exist constants \(a, K > 0\) and a sequence \(c = (c_k)_{k=1}^{\infty} \in \ell^1_+\) (the positive cone of \(\ell^1\)) such that the inequality
\[
M_k(2u) \leq KM_k(u) + c_k
\]
holds for all \(k \in \mathbb{N}\) and \(u \in R_+\), whenever \(M_k(u) \leq a\).

**Difference sequence spaces**: The notion of difference sequence spaces was introduced by Kızmaz [Kız 81], who studied the difference sequence spaces \(l_\infty(\Delta), c(\Delta)\) and \(c_0(\Delta)\). The notion was further generalized by Et and Çolak [EtÇ 97] by introducing the spaces \(l_\infty(\Delta^n), c(\Delta^n)\) and \(c_0(\Delta^n)\). Let \(m, n\) be non-negative integers, then for \(Z = l_\infty, c, c_0\) we have sequence spaces
\[
Z(\Delta^m_n) = \{x = (x_k) \in w : (\Delta^m_n x_k) \in Z\},
\]
where \(\Delta^m_n x = (\Delta^m_n x_k) = (\Delta^{m-1}_n x_k - \Delta^{m-1}_n x_{k+1})\) and \(\Delta^0_n x_k = x_k\) for all \(k \in \mathbb{N}\), which is equivalent to the following binomial representation
\[
\Delta^m_n x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.
\]
Taking \(n = 1\), we get the spaces which were studied by Et and Çolak [EtÇ 97]. Taking \(m = n = 1\), we get the spaces which were introduced and studied by Kızmaz [Kız 81].

**Paranormed spaces**: Let \(X\) be a linear metric space. A function \(p : X \to \mathbb{R}\) is called paranorm, if
\begin{enumerate}
  \item \(p(x) \geq 0\), for all \(x \in X\),
  \item \(p(-x) = p(x)\), for all \(x \in X\),
  \item \(p(x + y) \leq p(x) + p(y)\), for all \(x, y \in X\),
  \item if \((\lambda_n)\) is a sequence of scalars with \(\lambda_n \to \lambda\) as \(n \to \infty\) and \((x_n)\) is a sequence of vectors with \(p(x_n - x) \to 0\) as \(n \to \infty\), then \(p(\lambda_n x_n - \lambda x) \to 0\) as \(n \to \infty\).
\end{enumerate}

A paranorm \(p\) for which \(p(x) = 0\) implies \(x = 0\) is called total paranorm and the pair \((X, p)\) is called a total paranormed space. For any two paranorms \(p\) and \(q\), \(p\) is called stronger than \(q\) if whenever \((x_n)\) is a sequence such that \(p(x_n) \to 0\) as \(n \to \infty\), then also \(q(x_n) \to 0\) as \(n \to \infty\). If \(p\) is stronger than \(q\), then \(q\) is said to be weaker than \(p\). If \(p\) is stronger than \(q\) and \(q\) is stronger than \(p\), then \(p\) and \(q\) are called equivalent. If \(p\) is stronger than \(q\), but \(p\) and \(q\) are not equivalent, then \(p\) is said to be strictly stronger than \(q\). It is easy to see that every totally paranormed space is a linear metric space and the converse is also true. The metric of any linear metric space is given by some total paranorm (see [Wil 84], Theorem 10.4.2, P-183).
2-normed spaces: The concept of 2-normed spaces was initially developed by Gähler [Gäh 65] in the mid of 1960’s, as an interesting linear space generalization of a normed linear space which was subsequently studied by many others (see [Mur 10], [MuA 11]) and references therein. Recently, a lot of activities have been started to study summability and related topics in these linear spaces (see [RFC 01], [SGSG 07]).

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-norm on $X$ is a function $||.|| : X \times X \rightarrow \mathbb{R}$ which satisfies the following:

1. $||x, y|| = 0$ if and only if $x$ and $y$ are linearly dependent,
2. $||x, y|| = ||y, x||$,
3. $||\alpha x, y|| = |\alpha||x, y||$, $\alpha \in \mathbb{R}$,
4. $||x, y + z|| \leq ||x, y|| + ||x, z||$, for all $x, y, z \in X$.

The pair $(X, ||.||)$ is then called a 2-normed space (see [GuM 01]). For example, we may take $X = \mathbb{R}^2$ equipped with 2-norm as $||x, y||$ is the area of the parallelogram spanned by the vectors $x$ and $y$ which may be given explicitly by the formula

$$||x_1, x_2|| = \text{abs} \left( \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right).$$

Then, clearly $(X, ||.||)$ is a 2-normed space.

$n$-normed spaces: The concept of $n$-normed spaces was initially developed by Misiak [Mis 89]. Since then, many others have studied this concept and obtained various results (see [Gun 01], [Gun 01], [GuM 01]).

Let $n \in \mathbb{N}$ and $X$ be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is a field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$. A real valued function $||.||, \cdots, ||$ on $X^n$ satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha||x_1, x_2, \cdots, x_n||$, for any $\alpha \in \mathbb{R}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a $n$-norm on $X$, and the pair $(X, ||.||, \cdots, ||)$ is called a $n$-normed space.

For example, we may take $X = \mathbb{R}^n$ being equipped with the $n$-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the $n$-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$, which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$
where $E$ denotes the Euclidean norm and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, ||\cdot, \cdot||)$ be a $d$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in $X$. Then the following function $||\cdot, \cdot, \cdot||_\infty$ on $X^{n-1}$ defined by

$$
||x_1, x_2, \ldots, x_{n-1}||_\infty = \max \{|\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \ldots, n\} 
$$

is known as an $(n-1)$-norm on $X$ with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence $(x_k)$ in an $n$-normed space $(X, ||\cdot, \cdot, \cdot||)$ is said to converge to some $L \in X$ in the $n$-norm if

$$
\lim_{k \to \infty} ||x_k - L, z_1, \ldots, z_{n-1}|| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.
$$

A sequence $(x_k)$ in an $n$-normed space $(X, ||\cdot, \cdot, \cdot||)$ is said to be Cauchy with respect to the $n$-norm if

$$
\lim_{k,p \to \infty} ||x_k - x_p, z_1, \ldots, z_{n-1}|| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.
$$

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be a $n$-Banach space.

**Ideal Convergence :** In many branches of science and engineering we deal with different kinds of sequences and series and when we deal with these it is important to check their convergence. The idea of statistical convergence was introduced by Fast [Fas 51] and since then several generalizations and application of this concept have been investigated by various authors (see [Kum 07], [MuA 11], [MuM 09], [MME 10], [MuE 03], [MuE 09], [TrH 11]) and references therein. One of its generalizations is the ideal convergence or $I$-convergence which was introduced by Kastyrko et. al [KSW 2000] and studied by Balcerzak et. al [BDK 07], Komisarski [Kom 08] and Das et. al [DKWM 08].

Now we shall give some definitions which are used in second and third chapter of this thesis.

**Definition 1.1.1 :** Let $K$ be a subset of $N$ the set of natural numbers. Then the asymptotic density of $K$ denoted by $\delta(K)$ is defined as

$$
\delta(K) = \lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : k \in K \right\} \right|,
$$

where the vertical bars denote the cardinality of the enclosed set.

**Definition 1.1.2 :** A sequence $x = (x_k)$ is said to be statistically convergent to the
number $l$ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n : |x_k - l| \geq \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_{n} \frac{1}{n} \left| \{k \leq n : |x_k - l| \geq \epsilon\} \right| = 0.$$ 

In this case we write st-lim $x = l$.

**Definition 1.1.3**: If $X$ is a non-empty set then a family of subsets of $X$ is called an ideal $I$ of $X$ if and only if

1. $\phi \in I$,
2. $A, B \in I$ implies $A \cup B \in I$,
3. for each $A \in I$ and $B \subset A$, we have $B \in I$.

$I$ is called non-trivial ideal if $X \in I \neq \phi$.

**Definition 1.1.4**: A non-trivial ideal $I$ of $X$ is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons i.e. $\{x\} \in I$ for each $x \in X$.

**Definition 1.1.5**: A sequence space $E$ is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.

**Definition 1.1.6**: A double sequence $(x_{k,l})$ for all $k, l \in \mathbb{N}$ is a double infinite array of real or complex numbers.

**Definition 1.1.7**: Let $w^2$ denote the space of all complex double sequences $x = (x_{k,l})$. The space consisting of all those sequences $x$ in $w^2$ such that $M_{k,l}\left(\frac{|x_{k,l}|^{1/k+l}}{\rho}\right) \to 0$ as $k, l \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by $\Gamma^2_{\mathcal{M}}$ and is known as double Orlicz space of entire sequences. The space $\Gamma^2_{\mathcal{M}}$ is a metric space with the metric $d(x, y) = \sup_{k,l} M_{k,l}\left(\frac{|x_{k,l} - y_{k,l}|^{1/k+l}}{\rho}\right)$ for all $x = (x_{k,l})$ and $y = (y_{k,l}) \in \Gamma^2_{\mathcal{M}}$.

**Definition 1.1.8**: The space consisting of all those sequences $x \in w^2$ such that $\left(\sup_{k,l} M_{k,l}\left(\frac{|x_{k,l}|^{1/k+l}}{\rho}\right)\right) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by $\Lambda^2_{\mathcal{M}}$ and is known as double Orlicz space of analytic sequences.

**Definition 1.1.9**: A sequence $x = (x_{k,l})$ is called a double gai sequence if $(k + l)!|x_{k,l}|^{1/k+l} \to 0$ as $k, l \to \infty$. The vector space of all double gai sequences will be denoted by $\chi^2$. 

De la Vallee-Poussin means: Let $\Lambda = (\lambda_r)$ be a non-decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{r+1} \leq \lambda_r + 1$. The generalized de la vallee-Poussin mean is defined by $t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k$, where $I_r = [r - \lambda_r + 1, r]$.

A sequence $x = (x_k)$ is said to be $(V, \lambda)$-summable to a number $l$ if $t_r(x) \to l$ as $r \to \infty$ (see [Lei 65]). Savaş [Sav 04] introduce the following sequence spaces

$$[v, \lambda]_0 = \{ x = (x_k) : \lim_{r} \frac{1}{\lambda_r} \sum_{k \in I_r} |x_k| = 0 \},$$

$$[v, \lambda] = \{ x = (x_k) : \lim_{r} \frac{1}{\lambda_r} \sum_{k \in I_r} |x_k - le| = 0, \text{ for some } l \in \mathbb{C} \}$$

and

$$[v, \lambda]_{\infty} = \{ x = (x_k) : \sup_{r} \frac{1}{\lambda_r} \sum_{k \in I_r} |x_k| < \infty \}.$$

For the above sets of sequences one can say these are strongly summable to zero, strongly summable and strongly bounded respectively by the de la vallee-Poussin method. In the special case where $\lambda_r = r$ for $r = 1, 2, 3, \ldots$, the sets $[V, \lambda]_0$, $[V, \lambda]$ and $[V, \lambda]_{\infty}$ reduce to the sets $w_0$, $w$ and $w_{\infty}$ introduced and studied by Maddox [Mad 89]. Parashar and Choudhary [PaC 94] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function $M$, which generalized the well known Orlicz sequence space $l_M$ and strongly summable sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_{\infty}$. It may be noted that the spaces of strongly summable sequences were discussed by Maddox [Mad 89].

Lacunary Sequences: A sequence $x = (x_k) \in \ell_{\infty}$ is said to be almost convergent if all Banach limits of $(x_k)$ coincide. In [Mad 67] it was shown that

$$\hat{c} = \{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \}.$$

In ([Mad 67], [Mad 68]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number $L$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r), r = 0, 1, 2, \ldots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio $i_r/i_{r-1}$ will be
denoted by $q_r$. The space $N_\theta$ of lacunary strongly convergent sequences was defined by Freedman et. al. [FSR 78] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

1.2 Composition operators

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial \mathbb{D}$ its boundary and $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Given a holomorphic self-map $\varphi$ of $\mathbb{D}$ and $f \in H(\mathbb{D})$, we set

$$C_\varphi f = f \circ \varphi$$

and we call the resulting operator on $H(\mathbb{D})$, the composition operator induced by the symbol $\varphi$. These operators have been studied in depth on Hardy spaces and have gained increasing recognition during the last three decades, mainly due to the fact that they provide, just as, for example, Hankel and Toeplitz operators, ways and means to link classical function theory to functional analysis and operator theory. We first recall the Hardy spaces and then describe a survey of composition operators on them. For $0 < p < \infty$, the Hardy space $H^p$ is the space of all analytic functions $f$ on the open unit disk $\mathbb{D}$ for which

$$||f||_{H^p} = \lim_{r \to 1^-} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$

For $1 \leq p < \infty$, $H^p$ becomes a Banach space under the norm $|| \cdot ||_{H^p}$, whereas for $p = \infty$, $H^\infty$ is a Banach algebra under the supremum norm

$$||f||_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

For every $f \in H^p$ ($1 \leq p < \infty$), the radial limit

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

exists $m$-almost everywhere, where $m$ is the normalized Lebesgue measure on the unit circle $\partial \mathbb{D}$ and $f^* \in L^p(m)$, the Banach space of all complex-valued measurable functions $f$ on $\partial \mathbb{D}$ such that

$$||f||_{L^p} = \left\{ \int_0^{2\pi} |f(e^{i\theta})|^p dm(\theta) \right\}^{1/p} < \infty.$$

Furthermore, for $p = 2$, $H^2$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})} d\theta.$$
For detailed study of Hardy spaces (see [Hof 62], [Dur 70], [Gar 81] and [Rud 87]). J. V. Ryff [Ryf 66] was the first mathematician who initiated the study of composition operators on Hardy spaces although in a different terminology. He actually exploited the idea of Littlewood [Lit 25], who employed the term subordinate to deal with such type of operators and established that if $f$ is subordinate to $F$, then
\[ \int_0^{2\pi} |f(re^{i\theta})|^p dm(\theta) \leq \int_0^{2\pi} |F(re^{i\theta})|^p dm(\theta), \quad 0 < r < 1. \]
If $f$ and $F$ are analytic in $\mathbb{D}$ and if there exists a function $\phi$ satisfying Schwartz’s Lemma such that $f(z) = F(\phi(z))$, then $f$ is subordinate to $F$. Ryff in his paper [Ryf 66] proved that if $f \in H^p$ and $\phi$ is analytic self-map of $\mathbb{D}$, then $f \circ \phi \in H^p$ and
\[ ||f \circ \phi||_{H^p} \leq \left( \frac{1 + ||\phi(0)||}{1 - ||\phi(0)||} \right)^{1/p} ||f||_{H^p}. \]
In 1969, H. J. Schwartz [Sch 69] wrote first Ph.D. thesis on composition operators on $H^p$ in which he studied composition operators in details. He calculated several upper and lower estimates for the norm of $C_\phi$. By using orthonormal vectors $e_n$ defined by $e_n(z) = z^n$ for $n \in \mathbb{N}$, he was able to give a characterization for a bounded linear operator on $H^p$ to be a composition operator. In fact, he proved that a non-zero operator $A$ on $H^p$ is a composition operator if and only if $Ae_n = (Ae_1)^n$, $n = 0, 1, 2, \ldots$. It has also been observed that $C_\phi$ is invertible if and only if $\phi$ is a conformal automorphism of the unit disk. Normal and compact composition operators have been characterized. He also noted that if $C_\phi$ is compact, then $|\phi^*| < 1$ a.e. on the unit circle. In other words, $C_\phi$ is not compact whenever the set \{|$\phi^*| = 1$\} has positive measure. Further, he also observed that this condition is not sufficient by showing that the composition operator induced by
\[ \phi(z) = \frac{1 + z}{2} \]
is not compact, even though $\phi$ maps only a single point of the unit circle onto the unit circle, $\phi(1) = 1$. This work was continued by Shapiro and Taylor [ShT 73] who proved that if $C_\phi$ is compact on some Hardy space $H^p$ for some $p$ ($0 < p < \infty$), then it is compact on all of the Hardy spaces $H^p$ ($0 < p < \infty$). They also introduced the concept of angular derivative and proved that $C_\phi$ is not compact, whenever $\phi$ has an angular derivative at some point of the unit circle. Non-existence of the angular derivative condition is not a sufficient condition for compactness of $C_\phi$ on Hardy spaces $H^p$ in general. However, the angular derivative condition does characterize the compactness of $C_\phi$ on $H^p$ if the inducing map is univalent. They also provide several necessary and sufficient conditions for the compactness of $C_\phi$ on $H^p$. They
noted that the condition
\[ \int_{\partial D} \frac{1}{1 - |\varphi^*(\zeta)|} dm(\zeta) < \infty, \]
established by Schwartz as a sufficient condition for compactness of \( C_\varphi \), turns out to be a necessary and sufficient condition for \( C_\varphi \) to be a Hilbert-Schmidt composition operator on \( H^2 \). The connection with Hilbert-Schmidt operators and the fact that \( C_\varphi \) is Hilbert-Schmidt whenever \( \varphi(D) \) lie in an inscribed polygon comes from Shapiro and Taylor [ShT 73], where it is shown that such operators actually lie in every Schatten \( p \)-class. Examples of compact composition operators that were not Hilbert-Schmidt were also introduced in [ShT 73]. Cowen ([Cow 83], [Cow 88]) explored various properties of composition operators on Hardy spaces. He was able to compute essential norm estimate and essential spectra of certain nice composition operators on \( H^p \). MacCluer [Mac 85] brought Carleson measure conditions to study composition operators on \( H^p(\mathbb{B}^n) \). Finally, J. H. Shapiro [Sha 87] was able to discover the connection between the essential norm of a composition operator on the Hardy space \( H^2 \) and the Nevanlinna counting function for \( \varphi \), and obtained the general expression
\[ \|C_\varphi\|_e^2 = \limsup_{|w| \to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}}, \]
where by the essential norm of \( C_\varphi \), we mean its distance, in the operator norm, from the space of compact operators on \( H^2 \). In particular, he proved that \( C_\varphi \) is compact on \( H^2 \) if and only if
\[ N_\varphi(w) = o\left( \log \frac{1}{|w|} \right) \quad \text{as} \quad |w| \to 1, \]
thus providing a complete function theoretic characterization of compact composition operators in terms of inducing map’s Nevanlinna counting function \( N_\varphi \).

Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \) and \( \psi \in H(\mathbb{D}) \). Then we can define a linear operator
\[ W_{\psi, \varphi} f(z) = \psi(z) f(\varphi(z)) \]
for \( f \in H(\mathbb{D}) \) and \( z \in \mathbb{D} \), called a weighted composition operator. We can regard this operator as a generalization of a multiplication operator \( M_\psi \) induced by \( \psi \) and a composition operator \( C_\varphi \) induced by \( \varphi \), where \( M_\psi f(z) = \psi(z) f(z) \) and \( C_\varphi f(z) = f(\varphi(z)) \). In fact, \( W_{\psi, \varphi} = M_\psi C_\varphi \). When \( \psi = 1 \), we just have the composition operator \( C_\varphi \) and when \( \varphi(z) = z \) we have the multiplication operator \( M_\psi \).

Motivated by the desire to understand the influence of geometric properties of \( \varphi \) on composition operators, various authors have composed \( C_\varphi \) with multiplication and differentiation operators.
Weighted composition operators appear naturally in different contexts. For example, Singh and Sharma [SiS 79] related the boundedness of composition operators on Hardy space of the upper half-plane with the boundedness of weighted composition operators on the Hardy space of the open unit disk $\mathbb{D}$. Weighted composition operators also played an important role in the study of compact composition operators on Hardy spaces and Bergman spaces of unbounded domains (see for example [ShS 03] and [Mat 99] for more details). Isometries in many Banach spaces of analytic functions are just weighted composition operators, for example see [For 64].

Recently, several authors have studied weighted composition operators on different spaces of analytic functions. For example, one can refer to ([Att 92], [CoH 04]) for study of these operators on Hardy spaces, ([Kam 79], [OhT 01]) for disk algebra, ([OSZ 03], [OhZ 01]) for Bloch-type spaces and ([CuZ 04], [Mat 99]) for weighted Bergman spaces.

Let $\eta_a(z) = (a - z)/(1 - \bar{a}z)$, $a, z \in \mathbb{D}$, that is, the involutive automorphism of $\mathbb{D}$ interchanging points $a$ and 0. It is well known that

$$\frac{1 - |\eta_a(z)|^2}{1 - |z|^2} = \frac{1 - |a|^2}{1 - \bar{a}z} = |\eta'_a(z)|.$$ 

Also let the Green function in $\mathbb{D}$ with logarithmic singularity at $a$ is given by

$$g(z, a) = \log \left| \frac{1 - \bar{a}z}{a - z} \right| = \log \frac{1}{|\eta_a(z)|}.$$ 

We recall some spaces of holomorphic functions and some of their properties.

**Weighted Bergman spaces** : Let

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$$

be the normalized area measure on $\mathbb{D}$. For each $\alpha \in (-1, \infty)$, we set $d\nu_{\alpha}(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, $z \in \mathbb{D}$. Then $d\nu_{\alpha}$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$, let $L^p_{\alpha}$ denotes the weighted Lebesgue space which contains measurable functions $f$ on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^p d\nu_{\alpha}(z) < \infty$. Also denote by $A^p_{\alpha} = L^p_{\alpha} \cap H(\mathbb{D})$ the weighted Bergman space with the norm defined as

$$\|f\|_{A^p_{\alpha}} = \left( \int_{\mathbb{D}} |f(z)|^p d\nu_{\alpha}(z) \right)^{1/p} < \infty.$$ 

Note that $\|f\|_{A^p_{\alpha}}$ is a norm only if $1 \leq p < \infty$. When $0 < p < 1$, $A^p_{\alpha}$ is an F-space with respect to the translation invariant metric defined by $d^p_{\alpha}(f, g) = \|f - g\|_{A^p_{\alpha}}$. 

Bergman-Privalov spaces: Let $\alpha \in (-1, \infty)$ and $p \geq -1$. The weighted Bergman-Privalov space $AN_{p,\alpha} = AN_{p,\alpha}(D)$ consists of all $f \in H(D)$ such that
\[
\|f\|_{AN_{p,\alpha}} = \int_D \ln (1 + |f(z)|) dA_\alpha(z) < \infty.
\]
It is easy to see that the function $\|\cdot\|_{AN_{p,\alpha}}$ is not a norm on $AN_{p,\alpha}$, however $d_{AN_{p,\alpha}}(f,g) = \|f-g\|_{AN_{p,\alpha}}$ defines a translation invariant metric on $AN_{p,\alpha}$ and the metric turns $AN_{p,\alpha}$ into a complete metric space.

$\alpha$-Bloch spaces: A function $f$ holomorphic in $D$ is in $\alpha$-Bloch space $B^\alpha$ if
\[
\sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty
\]
and in the little $\alpha$-Bloch Space $B^\alpha_0$ if $\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0$.
For $f \in B^\alpha$ define
\[
\|f\|_{B^\alpha} = |f(0)| + \sup_{z \in D \cap B} (1 - |z|^2)^\alpha |f'(z)|.
\]
With this norm $B^\alpha$ is a Banach space and the little $\alpha$-Bloch Space $B^\alpha_0$ is a closed subspace of the $\alpha$-Bloch Space. Note that $B^1 = B$, the usual Bloch space.

Weighted type spaces: A positive continuous function $\omega$ on $[0,1)$ is called normal weight if
1. $\omega$ is a radial weight, that is $\omega(z) = \omega(|z|)$ for every $z \in D$.
2. there exist positive numbers $s$ and $t$, $0 < s < t$ such that
\[
\frac{\omega(r)}{(1-r)^s} \to 0, \quad \frac{\omega(r)}{(1-r)^t} \to \infty, \quad \text{as} \quad r \to 1^-.
\]
We also consider the weighted Bloch type spaces and little weighted Bloch type spaces of holomorphic functions:
(i) Weighted Bloch type spaces: A function $f$ holomorphic in $D$ is in weighted Bloch type space if $\|f\|_\omega = \sup_{z \in D} \omega(|z|) |f'(z)| < \infty$. For $f \in B_\omega$ define
\[
B_\omega = |f(0)| + \sup_{z \in D} \omega(|z|) |f'(z)|.
\]
With this norm $B_\omega$ is a Banach space.
(ii) Little weighted Bloch type spaces:
\[
B_{\omega,0} = \{ f \in H(D) : \lim_{|z| \to 1} \omega(|z|) |f'(z)| = 0 \}.
\]

Weighted Bergman spaces of controlled growth: For any $\alpha > 0$, the weighted Bergman spaces of controlled growth $A^{-\alpha}_\infty$ consists of analytic functions $f$ in $D$ such that $|f|_{A^{-\alpha}_\infty} = \sup \{(1 - |z|^2)^\alpha |f(z)| : z \in D\} < \infty$. Each $A^{-\alpha}_\infty$ is a non-separable Banach space with the norm defined above and contains all bounded analytic functions on $D$. The closure in $A^{-\alpha}_\infty$ of the set of polynomials will be denoted by $A^{-\alpha}_{\infty,0}$,
which is a separable Banach space and consists of exactly those functions $f$ in $A_{-1}^\infty$ with $\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f(z)| = 0$.

**Logarithmic $\alpha$-growth spaces** : We write $\mathcal{LA}_{-\alpha}^\infty$ for logarithmic $\alpha$-growth space of holomorphic functions $f$ on $\mathbb{D}$ for which

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \log \frac{2}{(1 - |z|^2)} |f(z)| < \infty.$$ 

Likewise, we write $\mathcal{LA}_{-\alpha,0}^\infty$ for logarithmic little $\alpha$-growth space of holomorphic functions $f$ on $\mathbb{D}$ for which

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha \log \frac{2}{(1 - |z|^2)} |f(z)| = 0.$$ 

**Logarithmic $\alpha$-Bloch spaces** : We write $\mathcal{LB}_{-\alpha}^\infty$ for logarithmic $\alpha$-Bloch space of holomorphic functions $f$ on $\mathbb{D}$ for which

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \log \frac{2}{(1 - |z|^2)} |f'(z)| < \infty.$$ 

Likewise we write $\mathcal{LB}_{-\alpha,0}^\infty$ for logarithmic little $\alpha$-Bloch space of holomorphic functions $f$ on $\mathbb{D}$ for which

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha \log \frac{2}{(1 - |z|^2)} |f'(z)| = 0.$$ 

**Zygmund spaces** : Denote by $\mathcal{Z}$ the class of all $f \in H(\mathbb{D}) \cap C(\mathbb{D})$ such that

$$||f|| = \sup \frac{f(e^{i(\theta+\delta)}) + f(e^{i(\theta-\delta)}) - 2f(e^{i\theta})}{\delta} < \infty,$$

where sup is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $\delta > 0$. By Theorem 5.3 of [Dur 70] and the closed graph theorem, we have $f \in \mathcal{Z}$ if and only if $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < \infty$. Moreover, $\mathcal{Z}$ is a Banach space under the norm $|| \cdot ||_\mathcal{Z}$, where

$$||f||_\mathcal{Z} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f''(z)|.$$  \hspace{1cm} (1.2.1)

The space $\mathcal{Z}$ is known as Zygmund space. The little Zygmund space of the open unit disk $\mathbb{D}$, denoted by $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$ consisting of functions $f$ with $\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f''(z)| = 0$.

With the norm $|| \cdot ||_\mathcal{Z}$ the Zygmund space $\mathcal{Z}$ is a Banach space and the little Zygmund space $\mathcal{Z}_0$ is a closed subspace of the Zygmund space $\mathcal{Z}$. 

\[ F(p, q, s) \text{ and } F_0(p, q, s) \text{ spaces} : \] For \( 0 < p < \infty \), \( -2 < q < \infty \) and \( 0 \leq s < \infty \), \( F(p, q, s) \) and \( F_0(p, q, s) \) are defined as the set of \( f \in H(\mathbb{D}) \) for which

\[
M(f) = \left\{ \sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^p (1 - |z|^2)^q (1 - |\eta_a(z)|^2)^s dA(z) \right\}^{1/p} < \infty
\]

and

\[
\lim_{|a| \to 1} \int_{D} |f'(z)|^p (1 - |z|^2)^q (1 - |\eta_a(z)|^2)^s dA(z) = 0, \quad 0 < s < \infty,
\]

respectively. The spaces \( F(p, q, s) \) and \( F_0(p, q, s) \) were introduced by R. Zhao in [Zha 96], are known as general family of function spaces. For \( 1 \leq p < \infty \), \( F(p, q, s) \) is a Banach space with respect to the norm

\[
||f||_{F(p, q, s)} = |f(0)| + M(f),
\]

and \( F_0(p, q, s) \) is a closed subspace of \( F(p, q, s) \). The importance of these spaces stems from the fact that for appropriate parameter values of \( p, q \) and \( s \) they coincide with several classical function spaces.

For example, \( F(2, 1, 0) \) is the Hardy space \( H^2 \), \( F(p, p + \alpha, 0), \ (\alpha > -1) \) is the Bergman space \( \mathcal{A}_\alpha \), \( F(p, q, s) = \mathcal{B}^{2+q}_{2+q} \) and \( F_0(p, q, s) = \mathcal{B}^{2+q}_{2+q} \) for \( s > 1 \), \( F(p, q, s) \subset \mathcal{B}^{2+q}_{2+q} \) and \( F_0(p, q, s) \subset \mathcal{B}^{2+q}_{2+q} \) for \( 0 < s \leq 1 \), \( F(2, 0, p) = Q_p \) and \( F_0(2, 0, p) = Q_{p,0} \), and \( F(2, 0, 1) = \text{BMOA} \), the space of analytic functions with bounded mean oscillation and \( F_0(2, 0, 1) = \text{VMOA} \), the space of analytic functions with vanishing mean oscillation. If \( q + s \leq -1 \), then \( F(p, q, s) \) is the space of constant functions.

\( Q_K(p, q) \) spaces : Let \( 0 < p < \infty \), \( -2 < q < \infty \) and \( K : [0, \infty] \to [0, \infty] \) a non-decreasing continuous function. A function \( f \in H(\mathbb{D}) \) is in \( Q_K(p, q) \) if

\[
M(f) = \left\{ \sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^p (1 - |z|^2)^q K(\rho(z, a)) dA(z) \right\}^{1/p} < \infty.
\]

Throughout this thesis, we assume that

\[
\int_0^1 (1 - r^2)^q K(-\log r) rdr < \infty,
\]

(1.2.3)

since otherwise \( Q_K(p, q) \) consists only of constant functions (see [WhZ 06]). For \( 1 \leq p < \infty \), \( Q_K(p, q) \) is a Banach space with respect to the norm \( ||f||_{Q_K(p,q)} = |f(0)| + M(f) \). If \( K(x) = x^s, \ s \geq 0 \), then the spaces \( Q_K(p, q) \) reduces to \( F(p, q, s) \).

Finally, we recall some spaces of holomorphic functions with complex order. Let \( A \) be the class of all functions \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]
which are holomorphic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( f \) in \( \mathcal{A} \) is said to be starlike function of complex order \( \gamma (\gamma \in \mathbb{C} \setminus \{0\}) \) and type \( \alpha, \ (0 \leq \alpha < 1) \) if and only if

\[
Re \left\{ 1 + \frac{1}{\gamma} \left[ \frac{zf'(z)}{f(z)} - 1 \right] \right\} > \alpha \quad (z \in U).
\] (1.2.4)

We denote by \( S_\alpha^*(\gamma) \) the class of all such functions. Also a function \( f \) in \( \mathcal{A} \) is said to be convex of complex order \( \gamma (\gamma \in \mathbb{C} \setminus \{0\}) \) and type \( \alpha, \ (0 \leq \alpha < 1) \) if and only if

\[
Re \left\{ 1 + \frac{1}{\gamma} \left[ \frac{zf''(z)}{f'(z)} \right] \right\} > \alpha \quad (z \in U).
\] (1.2.5)

We denote by \( C_\alpha(\gamma) \) the class of all such functions. The classes \( S_0^*(\gamma) \) and \( C_0(\gamma) \) were introduced by Nasr and Aouf in ([NaA 85] and Wiatrowski [Wia 71]). While the classes \( S_\alpha^*(\gamma) \) and \( C_\alpha(\gamma) \) were defined and studied by Frasin in [Fra 06]. Note that the classes \( S_\alpha^*(\gamma) \) and \( C_\alpha(\gamma) \) are generalization of the classes \( \lambda \)-spiral like functions of order \( \alpha \) which was introduced by Libera [Lib 67]. The class \( S_\alpha^*(\cos \lambda e^{-i\lambda}) \) (\( |\lambda| < \frac{\pi}{2}, \ 0 \leq \alpha < 1 \)) of \( \lambda \)-Roberton functions of order \( \alpha \) was introduced by Chichra in [Chi 75].

The differentiation operator denoted by \( D \) and is defined by \( \text{D}f = f' \), \( f \in H(\mathbb{D}) \). Studying products of concrete linear operators between spaces of holomorphic functions attracted some attention recently, (see [HiP 05], [LiS 08], [MaM 95], [OSZ 03], [Sha 09], [ShS 11], [Ste 08], [StS 11]) and the related references therein. The products of composition, multiplication and differentiation operators can be defined in the following six ways

\[
(M_\psi C_\varphi D f)(z) = \psi(z)f'(\varphi(z)),
\]
\[
(M_\psi D C_\varphi f)(z) = \psi(z)\varphi'(z)f'(\varphi(z)),
\]
\[
(C_\varphi M_\psi D f)(z) = \psi(\varphi(z))f'(\varphi(z)),
\] (1.2.6)
\[
(DM_\psi C_\varphi f)(z) = \psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z)),
\]
\[
(C_\varphi DM_\psi f)(z) = \psi'(\varphi(z))f(\varphi(z)) + \psi(\varphi(z))f'(\varphi(z))
\]
\[
(DC_\varphi M_\psi f)(z) = \psi'(\varphi(z))\varphi'(z)f(\varphi(z)) + \psi(\varphi(z))\varphi'(z)f'(\varphi(z))
\]

for \( z \in \mathbb{D} \) and \( f \in H(\mathbb{D}) \). Note that the operator \( M_\psi C_\varphi D \) induces many known operators. If \( \psi(z) = 1 \), then \( M_\psi C_\varphi D = C_\varphi D \), while when \( \psi(z) = \varphi'(z) \), then we get the operator \( DC_\varphi \). These two operators have been studied in [HiP 05]. If we put \( \varphi(z) = z \), then \( M_\psi C_\varphi D = M_\psi D \), that is, the product of differentiation operator and multiplication operator. Also note that \( M_\psi DC_\varphi = M_\psi \varphi' C_\varphi D \) and \( C_\varphi M_\psi D = M_\psi \varphi C_\varphi D \). Thus the corresponding characterizations of boundedness
and compactness of $M_{\psi}DC_\varphi$ and $C_\varphi M_{\psi}D$ can be obtained by replacing $\psi$, respectively by $\psi'\phi$ and $\psi \circ \varphi$ in the results stated for $M_{\psi}C_\varphi D$. In order to treat operators in equation (1.2.6) in a unified manner, we introduce the following operator

$$T^n_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z)f^{(n)}(\varphi(z)) + \psi_2(z)f^{(n+1)}(\varphi(z)), \quad f \in H(\mathbb{D}),$$

(1.2.7)

where $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi$ a holomorphic self-map of $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. It is clear that composition operator, multiplication operator, differentiation operator and all products of composition, multiplication and differentiation operators in equation (1.2.7) can be obtained from the operator $T^0_{\psi_1,\psi_2,\varphi}$ by fixing $\psi_1$ and $\psi_2$. More specifically we have $C_\varphi = T^1_{0,0,\varphi}, \quad M_\psi = T^1_{\psi,0,z}, \quad D = T^1_{1,0,z} = T^0_{0,1,z}, \quad M_\psi C_\varphi = T^0_{\psi,0,\varphi}, \quad C_\varphi D = T^1_{0,1,\varphi}, \quad C_\varphi M_\psi = T^0_{\psi \circ \varphi,0,\varphi}, \quad DC_\varphi = T^0_{0,\varphi',\varphi} = T^1_{\psi',0,\varphi}, \quad M_\psi D = T^0_{0,\psi,z} = T^1_{1,\psi,0} = T^0_{\psi',\psi,0}, \quad DM_\psi = T^0_{0,\psi,\psi',z}, \quad M_\psi C_\varphi D = T^0_{0,\psi,\psi',\varphi}, \quad M_\psi DC_\varphi = T^0_{0,\psi',\psi,\varphi}, \quad C_\varphi DM_\psi = T^0_{0,\psi',\psi,\varphi}, \quad DC_\varphi M_\psi = T^0_{(\psi',\psi,\varphi)\circ \varphi, \psi, \varphi, \varphi} \quad \text{and} \quad D^n_{\psi,\varphi} = T^0_{\psi,\varphi,\varphi}.$

For $i \in \{1,2,\cdots,n\}$, let $\alpha_i \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers and $f, \varphi \in \mathcal{A}$ such that $\varphi(U) \subset U$. We define composition type operators as follows:

$$F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \left((f_1 \circ \varphi)(z)\right)^{\alpha_1} \left((f_2 \circ \varphi)(z)\right)^{\alpha_2} \cdots \left((f_n \circ \varphi)(z)\right)^{\alpha_n},$$

(1.2.8)

$$G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \left((f_1 \circ \varphi)(z)\right)^{\alpha_1} \left((f_2 \circ \varphi)(z)\right)^{\alpha_2} \cdots \left((f_n \circ \varphi)(z)\right)^{\alpha_n},$$

(1.2.9)

$$H_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \int_0^z \left((f_1 \circ \varphi)(\zeta)\right)^{\alpha_1} \cdots \left((f_n \circ \varphi)(\zeta)\right)^{\alpha_n} g'(\zeta) d\zeta$$

(1.2.10)

and

$$I_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \int_0^z \left((f_1 \circ \varphi)'(\zeta)\right)^{\alpha_1} \cdots \left((f_n \circ \varphi)'(\zeta)\right)^{\alpha_n} g(\zeta)\ z)\) d\zeta$$

(1.2.11)

The composition type operators $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}$, $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}$, $H_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}$ and $I_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}$ are generalizations of some well known operators, defined respectively, as

$$F_{1,\varphi,\alpha}(z) = \left((f_1 \circ \varphi)(z)\right)^{\alpha},$$

$$G_{1,\varphi,\alpha}(z) = \left((f_1 \circ \varphi)(z)\right)^{\alpha},$$

$$H_{1,\varphi,g,\alpha}(z) = \int_0^z \left((f_1 \circ \varphi)(\zeta)\right)^{\alpha} g'(\zeta) d\zeta$$

and

$$I_{1,\varphi,g,\alpha}(z) = \int_0^z \left((f_1 \circ \varphi)'(\zeta)\right)^{\alpha} g(\zeta)\ z)\) d\zeta.$$

The operator $F_\varphi = F_{1,\varphi,1}$ is known as composition operator defined as

$$F_\varphi(z) = f_1 \circ \varphi, \quad f_1 \in \mathcal{A}.$$
The operator $H_{1,\varphi,g,1}$ induced by $g$ and $\varphi$ defined as

$$H_{1,\varphi,g,1}(z) = \int_0^z f_1(\varphi(\zeta))dg(\zeta) = \int_0^z f_1(\varphi(\zeta))g'(\zeta)d\zeta = \int_0^1 f_1(\varphi(tz)) z g'(tz)dt$$

can be viewed as a generalization of the Riemann-Stieltjes operator $T_g$ induced by $g$, defined by

$$T_gf(z) = \int_0^z f(\zeta)dg(\zeta) = \int_0^1 f(tz)zg'(tz)dt, \ z \in \mathbb{D}.$$  

C. Pommerenke [Pom 77] initiated the study of Riemann-Stieltjes operator on the Hardy space $H^2$, where he showed that $T_g$ is bounded on $H^2$ if and only if $g$ is in $BMOA$. This was extended to other Hardy spaces $H^p$, $1 \leq p < \infty$, in [AIC 01] and [AlS 95], where compactness of $T_g$ on $H^p$ and Schatten class membership of $T_g$ on $H^2$ was also completely characterized in terms of the symbol $g$. The operator $I_{1,\varphi,g,1}$ induced by $g$ and $\varphi$, defined as

$$I_{1,\varphi,g,1}(z) = \int_0^z f_1'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)d\zeta, \ z \in \mathbb{D}.$$  

The operator $I_{1,\varphi,g,1}$ is the generalization of the operator $J_g$, recently defined by Yoneda in [Yon 04] as

$$J_gf(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta, \ z \in \mathbb{D}.$$  

Also, if $g(z) = \varphi'(z)$, then $I_{1,\varphi,g,1}$ reduces to difference of the composition operator $F_\varphi$ and the point evaluation map, defined as

$$I_{1,\varphi,\varphi',1}f_1(z) = (f_1 \circ \varphi)(z) - f_1(\varphi(0)).$$  

These operators have gained increasing attention during the last three decades, mainly due to the fact that they provide ways and means to link classical function theory to functional analysis and operator theory. Recently, several authors have studied Riemann-Stieltjes type operators on different spaces of holomorphic functions. For example, one can refer to ([AlS 95], [Rät 07], [Sha 09], [SaS 11], [SiZ 99], [Xia 97], [Xia 04], [Yon 04]) and the related references therein for the study of these operators on different spaces of holomorphic functions.

1.3 Structure of work

The main purpose of the present thesis is to carry out the study of some new sequence spaces and composition operators. This thesis consists of five chapters. All the chapters are further divided into sections. The chapter I consists of only three sections. In the first section we record in brief a historical development and
preliminaries for sequence spaces, whereas in the second section we present some historical background and preliminary material for composition operators on Hardy spaces and recall some spaces of analytic function between which we study composition operators in this thesis. The section third of this chapter contain structure of work. The chapter II is composed of three sections. In this chapter we initiate a study of double sequence spaces defined by a sequence of Orlicz functions. The first section contain some results on the sequence spaces $c^2(\Delta_m, \mathcal{M}, u, p, q, s)$, $c_0^2(\Delta_m, \mathcal{M}, u, p, q, s)$ and $l_\infty^2(\Delta_m, \mathcal{M}, u, p, q, s)$. The sequence spaces $\Lambda^2_{\mathcal{M}}(\Delta_m, u, p, q)$ and $\Gamma^2_{\mathcal{M}}(\Delta_m, u, p, q)$ are defined and studied in the second section of this chapter. This section provides various inclusion relations and topological properties of these spaces. These results makes our study of sequence spaces as most significant. The third section is devoted to a study of double chi and double analytic sequence spaces induced by the sequence of Orlicz functions. Some properties like linearity, paranormed and some inclusion relations are reported in this section. It is also proved that these sequence spaces are solid and monotone. The chapter III is devoted to a study of some sequence spaces over 2-normed spaces and $n$-normed spaces. This chapter is divided into five sections. The first section considers the sequence spaces $w_\sigma^0[\mathcal{M}, p, ||., ||]_\theta$, $w_\sigma[\mathcal{M}, p, ||., ||]_\theta$ and $w_\sigma^\infty[\mathcal{M}, p, ||., ||]_\theta$ by using invariant mean and lacunary sequences. We also make an effort to study some topological properties of these sequence spaces. A characterization for the inclusion relations reported in the end of this section. The difference sequence spaces $[\hat{c}, \mathcal{M}, p, ||., ||](\Delta^m)$, $[\hat{c}, \mathcal{M}, p, ||., ||](\Delta^m)$ and $[\hat{c}, \mathcal{M}, p, ||., ||](\Delta^m)$ over 2-normed spaces are studied in the second section of this chapter. We also make an effort to study some inclusion relation and topological properties of these sequence spaces. The third section of this chapter deals with sequence spaces over 2-normed spaces using ideal convergence. The property like linearity, paranormed, monotonicity, solidness and some inclusion relations are studied in this section. A study of sequence spaces defined by a Musielak-Orlicz function over $n$-normed spaces is made in the fourth section of this chapter. In this section we first define some sequence spaces and then study their topological properties and inclusion relation between these spaces. The main aim of fifth section is to study difference sequence spaces $[c, \mathcal{M}, p, ||., ||, \cdots, ||]_\theta^0(\Delta^m)$, $[c, \mathcal{M}, p, ||., ||, \cdots, ||]_\theta^0(\Delta^m)$ and $[c, \mathcal{M}, p, ||., ||, \cdots, ||]_\theta^0(\Delta^m)$ over $n$-normed spaces defined by a Musielak-Orlicz function. It is very interesting that if sup $[M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[c, \mathcal{M}, p, ||., ||, \cdots, ||]_\theta^0(\Delta^m) \subset [c, \mathcal{M}, p, ||., ||, \cdots, ||]_\theta^0(\Delta^m)$. The chapter IV include generalized weighted composition operators between some spaces of holomorphic functions. This chapter contains three sections. The main purpose of the first section is to explore a necessary and sufficient condition for
boundedness and compactness of the generalized weighted composition operator $D_{\varphi,\psi}^n$ on $Q_K(p,q)$ spaces to Bloch type spaces. The upper and lower bounds for norm of $D_{\varphi,\psi}^n$ from $Q_K(p,q)$ spaces to Bloch type spaces are also computed. The second section of this chapter studies product of multiplication, composition and differentiation operators from $H^\infty$ to weighted Bloch spaces. The third section concerns with the composition type operator on some classes of holomorphic functions with complex order. In this section we have studied interesting properties of composition type operators between spaces of analytic function of complex order. In the last chapter, we consider weighted composition operators $W_{\psi,\varphi}$, products of differentiation and composition operator from weighted Bergman-Privalov spaces. The results are divided into two sections. The section I includes the boundedness and compactness of weighted composition operators $W_{\psi,\varphi}$ from weighted Bergman-Privalov spaces to Zygmund spaces. We also prove in this section that every bounded weighted composition operator $W_{\psi,\varphi}$ from $\mathcal{AN}_{p,\alpha}$ to $\mathcal{Z}$ or $\mathcal{Z}_0$ is compact. The second section is devoted to a study of product of composition and differentiation operator $C_{\varphi}D$ and $DC_{\varphi}$ from weighted Bergman-Privalov spaces to Zygmund spaces. We note that every bounded $W_{\psi,\varphi}$ or bounded $C_{\varphi}D$ or $DC_{\varphi}$ from Bergman-Privalov spaces to Zygmund spaces is compact also.