Chapter 5

Third Order Neutral Differential Equation with Mixed Nonlinearities

5.1 Introduction

This chapter deals with the third order neutral differential equation with mixed nonlinearities of the form

\[ a(t)[x(t) + c(t)x(t - \tau)]'' + p(t)x^{\beta}(t - \sigma) = e(t) + q(t)x^{\gamma}(t - \sigma), \quad t \geq t_0 \]  \quad (5.1.1)

where \( t_0 > 0, \ a, c \in C^3([0, \infty), (0, \infty)), \ p, q \in C([0, \infty), [0, \infty]), \ e \in C([0, \infty), \mathbb{R}), \ \tau, \sigma \)

are nonnegative constants and \( \beta, \gamma \) are ratio of odd positive integers.

By a solution of equation (5.1.1), we mean a continuous function \( x(t) \) satisfying equation (5.1.1), which is defined for \( t \geq t_0 - T \), and satisfies \( \sup\{|x(t)| : t \geq t_1\} > 0 \) for all \( t_1 \geq t_0 > 0 \), where \( T = \max\{\tau, \sigma\} \).

In recent years, the oscillation theory and asymptotic behavior of differential equations and their applications have been investigated and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs \([3, 12, 29, 50, 59, 64]\) and papers \([9, 17, 14, 13, 16, 21, 48, 52]\).

Regarding third order differential equations, Hanan [53] studied the third order linear differential equation

\[ x''''(t) + p(t)x(t) = 0, \ t \geq t_0, \]  \quad (5.1.2)

and established some sufficient conditions for oscillation and nonoscillation of equation (5.1.2).

Saker [83] examined the oscillation of the self-adjoint nonlinear delay differential
chapter 5. third order neutral differential equation with mixed nonlinearities

equation

\[
(c(t)[a(t)x'(t)]')' + q(t)f(x(t - \sigma)) = 0, \ t \geq t_0
\]  

(5.1.3)

using the Riccati transformation techniques.

Graef et al. [42] obtained criteria for all solutions of

\[
(a(t)[b(t)(y(t) + py(t - \sigma))]')' + q(t)f(y(t - \sigma)) = 0, \ t \geq t_0
\]  

(5.1.4)

to be oscillatory when \( \int_{t_0}^{\infty} \frac{1}{a(t)} dt = \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty \).

Baculíková and Džurina [12] obtained some sufficient conditions which ensure that all nonoscillatory solutions of

\[
[a(t)([x(t) + p(t)x(\delta(t))]')']' + q(t)x'(\tau(t)) = 0, \ t \geq t_0
\]  

(5.1.5)

tend to zero when \( t \to \infty \).

It is easy to see that all the above mentioned equations (5.1.2) - (5.1.5) contain only one nonlinearity and the problem of determining oscillation and nonoscillation of solutions of differential equation with mixed nonlinearities received less attention, see for example [3, 6], and the references cited therein. However, to study the oscillatory behavior of the forced differential equation (5.1.1) with mixed nonlinearities, the known techniques either do not work or impose severe restrictions on the forcing term \( e(t) \). Motivated by these observations, in this chapter, we shall provide some easily verifiable sufficient conditions for the oscillation of all solutions of equation (5.1.1).

In Section 5.2, we establish some sufficient conditions for oscillation of solutions of equation (5.1.1). Examples are provided in Section 5.3.

5.2 Oscillation Results

In this section, we obtain some new oscillation criteria for the equation (5.1.1)

We begin with the following lemma given in [54].
5.2. Oscillation Results

Lemma 5.1 If $X$ and $Y$ are nonnegative, then

(I) $X^\lambda - \lambda X Y^{\lambda-1} + (\lambda - 1) Y^\lambda \geq 0$ for all $\lambda \geq 1$;

(II) $X^\mu - \mu X Y^{\mu-1} - (1 - \mu) Y^\mu \leq 0$ for all $0 < \mu < 1$.

In the above inequalities, equality holds if and only if $X=Y$.

Theorem 5.2 Let $\gamma = 1$ and $\beta > 1$. If

$$\lim \inf_{t \to \infty} \int_{t_0}^{\infty} \left[ \frac{k_1}{a(s)} + \frac{k_2 a(s)}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( e(\xi) + A(\xi) \right) d\xi d\eta \right] ds = -\infty,$$

and

$$\lim \sup_{t \to \infty} \int_{t_0}^{\infty} \left[ \frac{k_1}{a(s)} + \frac{k_2 a(s)}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( e(\xi) - A(\xi) \right) d\xi d\eta \right] ds = \infty$$

hold for all $t \geq t_0 > 0$ and all $k_1, k_2 \in \mathbb{R}$, where

$$A(\xi) = (\beta - 1) \beta^{1/\beta} q^{\frac{1}{\beta-1}}(\xi) p^{1/\beta}(\xi),$$

then all solutions of equation (5.1.1) are oscillatory.

Proof. Suppose that $x(t)$ is an eventually positive solution of equation (5.1.1). Set $z(t) = x(t) + c(t)x(t-\tau)$. Then $z(t) > 0$ for all $t \geq t_0 > 0$. From equation (5.1.1), we have

$$(a(t)z'(t))'' = c(t) + q(t)x(t-\sigma) - p(t)x^\beta(t-\sigma).$$

Let $X = p^{\beta}(t)x(t-\sigma)$ and $Y = \left( \frac{1}{\beta} q(p^{-\beta}(t)) \right)^{\frac{1}{\beta-1}}$. Then by Lemma 5.1(I), we have

$$p(t)x^\beta(t-\sigma) - q(t)x(t-\sigma) \geq (1 - \beta) \beta^{1/\beta} q^{\frac{1}{\beta-1}}(t)p^{1/\beta}(t).$$

Using the inequality (5.2.4) in equation (5.2.3), we obtain

$$(a(t)z'(t))'' \leq c(t) + (\beta - 1) \beta^{1/\beta} q^{\frac{1}{\beta-1}}(t)p^{1/\beta}(t), t \geq t_0.$$

Integrating both sides of (5.2.5) from $t_0$ to $t$, we obtain

$$(a(t)z'(t))' \leq (a(t_0)z'(t_0))' + \int_{t_0}^{t} \left( e(s) + A(s) \right) ds.$$
Again integrating, we obtain
\[ z'(t) \leq \frac{a(t_0)z'(t_0)}{a(t)} + \frac{(a(t_0)z'(t_0))' (t - t_0)}{a(t)} + \frac{1}{a(t)} \int_{t_0}^{t} \int_{t_0}^{s} \left( e(u) + A(u) \right) du ds. \]

Once again integrating the last inequality, we obtain
\[ z(t) \leq z(t_0) + \int_{t_0}^{t} \left[ \frac{a(t_0)z'(t_0)}{a(s)} + \frac{(a(t_0)z'(t_0))' (s - t_0)}{a(s)} \right] ds + \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( e(\xi) + A(\xi) \right) d\xi du ds. \]

Taking lim inf as \( t \to \infty \) in the last inequality, we obtain a contradiction to \( z(t) > 0 \) for all \( t \geq t_0 \) because of (5.2.1). The proof is similar when \( z(t) \) is eventually negative and hence the proof is complete.

From Theorem 5.2, we obtain the following corollary.

**Corollary 5.1** Assume that \( \gamma = 1 \) and \( \beta > 1 \). Further assume
\[ \int_{t_0}^{\infty} \frac{t}{a(t)} dt < \infty, \] (5.2.6)
and
\[ \int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{t} \int_{t_0}^{s} q^{\frac{\beta}{\gamma - 1}}(\xi) p^{\frac{1}{\gamma - 1}}(\xi) d\xi ds dt < \infty \]
hold. If
\[ \lim_{t \to \infty} \inf \int_{t_0}^{t} \left[ \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} e(\xi) d\xi du \right] ds = -\infty, \] (5.2.7)
and
\[ \lim_{t \to \infty} \sup \int_{t_0}^{t} \left[ \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} e(\xi) d\xi du \right] ds = +\infty, \] (5.2.8)
then every solution of equation (5.1.1) is oscillatory.

**Theorem 5.3** Let \( 0 < \gamma < 1 \) and \( \beta = 1 \). If
\[ \lim_{t \to \infty} \inf \int_{t_0}^{t} \left[ \frac{k_1}{a(s)} + \frac{k_2 s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( e(\xi) + B(\xi) \right) d\xi du \right] ds = -\infty \] (5.2.9)
and
\[ \lim_{t \to \infty} \sup \int_{t_0}^{t} \left[ \frac{k_1}{a(s)} + \frac{k_2 s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( e(\xi) - B(\xi) \right) d\xi du \right] ds = \infty \]
5.2. Oscillation Results

hold for all \( t \geq t_0 \) and all \( k_1, k_2 \in \mathbb{R} \), where

\[
B(\xi) = (1 - \gamma)^{\frac{1}{\gamma^2}} p^{\frac{1}{\gamma^2 - 1}}(\xi)q^{\frac{1}{\gamma^2}}(\xi),
\]  
(5.2.10)

then all solutions of equation (5.1.1) are oscillatory.

**Proof.** Suppose that \( x(t) \) is an eventually positive solution of equation (5.1.1). Let \( z(t) = x(t) + c(t)x(t - \tau) \). Then \( z(t) > 0 \) for all \( t \geq t_0 > 0 \). From equation (5.1.1), we have

\[
(a(t)z'(t))'' = e(t) + q(t)x(t - \sigma) - p(t)x(t - \sigma).
\]  
(5.2.11)

Take \( X = q^{\frac{1}{\gamma}}(t)x(t - \sigma) \) and \( Y = \left(\frac{1}{\gamma}p(t)q^{-\frac{1}{\gamma}}(t)\right)^{\frac{1}{\gamma^2 - 1}} \). Then by Lemma 5.1(II), we obtain

\[
q(t)x(t - \sigma) - p(t)x(t - \sigma) \leq (1 - \gamma)^{\frac{1}{\gamma^2}} p^{\frac{1}{\gamma^2 - 1}}(t)q^{\frac{1}{\gamma^2}}(t).
\]  
(5.2.12)

Using the inequality (5.2.12) in equation (5.2.11), we obtain

\[
(a(t)z'(t))'' \leq e(t) + (1 - \gamma)^{\frac{1}{\gamma^2}} p^{\frac{1}{\gamma^2 - 1}}(t)q^{\frac{1}{\gamma^2}}(t), \quad t \geq t_0.
\]  
(5.2.13)

The rest of the proof is similar to that of Theorem 5.2 and hence omitted. \( \square \)

From Theorem 5.3, we have the following corollary.

**Corollary 5.2** Assume that \( 0 < \gamma < 1 \) and \( \beta = 1 \). If

\[
\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{s} p^{\frac{1}{\gamma^2 - 1}}(\xi)q^{\frac{1}{\gamma^2}}(\xi) d\xi ds dt < \infty
\]

and conditions (5.2.6)-(5.2.8) hold then every solution of equation (5.1.1) is oscillatory.

**Theorem 5.4** Let \( 0 < \gamma < 1 \) and \( \beta > 1 \). If there exists a positive function \( b(t) \) such that

\[
\liminf_{t \to \infty} \int_{t_0}^{t} \left[ \frac{k_1}{a(s)} + \frac{k_2s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( e(\xi) + D(\xi) \right) d\xi d\xi du \right] ds = -\infty
\]  
(5.2.14)

and

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left[ \frac{k_1}{a(s)} + \frac{k_2s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( e(\xi) - D(\xi) \right) d\xi d\xi du \right] ds = +\infty
\]
hold for all $t \geq t_0$ and all $k_1, k_2 \in \mathbb{R}$, where
\[ D(\xi) = (\beta - 1)\beta^{\beta + 1}(\xi)p^{1/\beta}(\xi) + (1 - \gamma)\gamma^{1/\gamma}b^{1/\gamma}(\xi)q^{1/\gamma}(\xi), \] (5.2.15)
then every solution of equation (5.1.1) is oscillatory.

**Proof.** Let $x(t)$ be an eventually positive solution of equation (5.1.1). Set $z(t) = x(t) + c(t)x(t - \tau)$. Then $z(t) > 0$ for all $t \geq t_0 > 0$ and from equation (5.1.1), we have
\[
(a(t)z'(t))'' = e(t) + \left[ b(t)x(t - \sigma) - p(t)x^\beta(t - \sigma) \right] + \left[ q(t)x^\gamma(t - \sigma) - b(t)x(t - \sigma) \right].
\] (5.2.16)
The rest of the proof is similar to that of Theorems 5.2 and 5.3 and hence omitted. \(\square\)

**Corollary 5.3** Let $0 < \gamma < 1$, $\beta > 1$ and (5.2.6) hold. Moreover assume that there exists a positive function $b(t)$ such that
\[
\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{t} \int_{t_0}^{s} b^{1/\gamma}(\xi)p^{1/\beta}(\xi)d\xi ds dt < \infty
\]
and
\[
\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{t} \int_{t_0}^{s} b^{1/\gamma}(\xi)q^{1/\gamma}(\xi)d\xi ds dt < \infty
\]
hold. If (5.2.7) and (5.2.8) hold then every solution of equation (5.1.1) is oscillatory.

**Proof.** The proof follows from Theorem 5.4. \(\square\)

**Theorem 5.5** Let $\gamma = 1$ and $\beta > 1$. If
\[
\int_{t_0}^{\infty} \left[ \frac{k_1}{a(s)} + \frac{k_2 s^k}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( |e(\xi)| + |A(\xi)| \right) d\xi du \right] ds < \infty
\] (5.2.17)
with $A(\xi)$ defined as in (5.2.2) and for all positive real constants $k_1, k_2$, then all nonoscillatory solutions of equation (5.1.1) are bounded.

**Proof.** As in the proof of Theorem 5.2, we obtain
\[
|x(t)| \leq |z(t)| \leq |z(t_0)| + \int_{t_0}^{t} \left[ \frac{a(t_0)|z'(t_0)|}{a(s)} + \left| \frac{a(t_0)x'(t_0)}{a(s)} \right| (s - t_0) \right] + \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( |e(\xi)| + |A(\xi)| \right) d\xi du \right] ds.
\]
The conclusion now follows from the condition (5.2.17). \(\square\)
5.2. Oscillation Results

**Theorem 5.6** Let $0 < \gamma < 1$ and $\beta = 1$. If

\[
\int_{t_0}^{\infty} \left[ \frac{k_1}{a(s)} + \frac{k_2 s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( |e(\xi)| + |B(\xi)| \right) d\xi du \right] ds < \infty \tag{5.2.18}
\]

with $B(\xi)$ defined as in (5.2.10) and for all positive real constants $k_1, k_2$, then all nonoscillatory solutions of equation (5.1.1) are bounded.

**Proof.** As in the proof of Theorem 5.3, we obtain

\[
|z(t)| \leq |z(t_0)| + \int_{t_0}^{t} \left[ \frac{a(t_0)}{a(s)} |z'(t_0)| + \frac{|(a(t_0)z'(t_0))'(s - t_0)|}{a(s)} \right] ds
\]

\[
+ \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( |e(\xi)| + |B(\xi)| \right) d\xi du ds.
\]

The conclusion now follows from the condition (5.2.18). \qed

**Theorem 5.7** Let $0 < \gamma < 1$ and $\beta > 1$. If there exists a positive function $b(t)$ such that for all $k \in \mathbb{R}$

\[
\int_{t_0}^{\infty} \left[ \frac{k_1}{a(s)} + \frac{k_2 s}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} \left( |e(\xi)| + |D(\xi)| \right) d\xi du \right] ds < \infty
\]

where $D(\xi)$ is as defined in (5.2.15) and for all positive real constants $k_1, k_2$, then all nonoscillatory solutions of equation (5.1.1) are bounded.

**Proof.** As in the proof of Theorem 5.4, we obtain

\[
|z(t)| \leq |z(t_0)| + \int_{t_0}^{t} \left[ \frac{a(t_0)}{a(s)} |z'(t_0)| + \frac{|(a(t_0)z'(t_0))'(s - t_0)|}{a(s)} \right] ds
\]

\[
+ \frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \left( |e(\xi)| + |D(\xi)| \right) d\xi du ds
\]

and then the conclusion follows from the hypothesis. \qed

Next we consider the following differential equation of the form

\[
z''(t) + b(t)z(t) + q(t)x^\gamma(t - \sigma) = p(t)x^\beta(t - \sigma) + e(t) \tag{5.2.19}
\]

where $z(t) = x(t) + c(t)x(t - \tau)$, $b \in C([0, \infty), \mathbb{R})$ and $p, q, e, \beta, \gamma, \sigma, \tau$ are as in equation (5.1.1).
Theorem 5.8 Suppose that there exists a function \( H \in C^1(D, \mathbb{R}) \), where
\( D = \{(t, s) : t \geq s \geq t_0\} \) such that
\[(i) \quad H(t, t) = 0 \text{ for all } t \geq t_0, \quad H(t, s) > 0 \text{ for } t > s \geq t_0, \]
\[(ii) \quad H \text{ has a continuous and nonpositive partial derivative on } D \text{ with respect to the second variable such that} \]
\[\frac{\partial H(t, s)}{\partial s} = -h_1(t, s), \quad \frac{\partial h_1(t, s)}{\partial s} = -h_2(t, s), \quad \frac{\partial h_2(t, s)}{\partial s} = -h_3(t, s), \]
\[h_1(t, t) = h_2(t, t) = 0 \text{ for } t \geq t_0, \]
\[0 \leq \lim_{t \to \infty} h_i(t, t_0) < \infty, \quad i = 1, 2, \quad (5.2.20) \]
and
\[b(s)H(t, s) + h_3(t, s) < 0 \text{ for } t > s \geq t_0. \quad (5.2.21)\]
If \( \beta > 1, \ 0 < \gamma < 1 \) and for \( t \geq t_0, \)
\[\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s)e(s) - Q(t, s) \right) ds = +\infty \quad (5.2.22) \]
and
\[\lim_{t \to \infty} \inf \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s)e(s) - Q(t, s) \right) ds = -\infty \quad (5.2.23) \]
where
\[Q(t, s) = (1 - \gamma)^{\frac{1}{1-\gamma}} (h_3(t, s))^{\frac{1}{\gamma-1}} \left( H(t, s)p(s) \right)^{\frac{1}{\gamma}} \]
\[+ \left( \beta - 1 \right) \beta^{\frac{\beta}{\beta-1}} (h_3(t, s))^{\frac{\beta}{\beta-1}} \left( H(t, s)p(s) \right)^{\frac{1}{\gamma}} \]
then every solution of equation (5.2.19) is oscillatory.

Proof. Let \( x(t) \) be an eventually positive solution of equation (5.2.19) and so \( z(t) \) is eventually positive. Multiplying equation (5.2.19) by \( H(t, s) \) for \( t > s \geq t_0 \) and then
integrating from \( t_0 \) to \( t \) and using (5.2.21), we obtain
\[
\int_{t_0}^{t} H(t, s)e(s)\,ds = \int_{t_0}^{t} H(t, s)z''(s)\,ds + \int_{t_0}^{t} H(t, s)b(s)z(s)\,ds \\
+ \int_{t_0}^{t} H(t, s)q(s)x'(s - \sigma)\,ds - \int_{t_0}^{t} H(t, s)p(s)x(s - \sigma)\,ds \\
= -H(t, t_0)z''(t_0) - h_1(t, t_0)z'(t_0) - h_2(t, t_0)z(t_0) \\
+ \int_{t_0}^{t} h_3(t, s)z(s)\,ds + \int_{t_0}^{t} H(t, s)b(s)z(s)\,ds \\
+ \int_{t_0}^{t} H(t, s)q(s)x'(s - \sigma)\,ds - \int_{t_0}^{t} H(t, s)p(s)x(s - \sigma)\,ds \\
< -H(t, t_0)z''(t_0) - h_1(t, t_0)z'(t_0) - h_2(t, t_0)z(t_0) \\
+ \int_{t_0}^{t} \left[ H(t, s)q(s)x'(s - \sigma) - h_3(t, s)x(s - \sigma) \right] ds \\
+ \int_{t_0}^{t} \left[ h_3(t, s)x(s - \sigma) - H(t, s)p(s)x(s - \sigma) \right] ds. \tag{5.2.24}
\]

In view of (5.2.20), there exists a constant \( K \) such that for \( t \geq t_0 \),
\[
-H(t, t_0)\left[ z''(t_0) + \frac{h_1(t, t_0)}{H(t, t_0)}z'(t_0) + \frac{h_2(t, t_0)}{H(t, t_0)}z(t_0) \right] \leq KH(t, t_0).
\]

Therefore (5.2.24) becomes
\[
\int_{t_0}^{t} H(t, s)e(s)\,ds \leq KH(t, t_0) + \int_{t_0}^{t} \left[ H(t, s)q(s)x'(s - \sigma) - h_3(t, s)x(s - \sigma) \right] ds \\
+ \int_{t_0}^{t} \left[ h_3(t, s)x(s - \sigma) - H(t, s)p(s)x(s - \sigma) \right] ds. \tag{5.2.25}
\]

Taking \( X_1 = \left( H(t, s)q(s) \right)^{\frac{1}{\gamma}} x(s - \sigma), \ Y_1 = \left( \frac{1}{\gamma} h_3(t, s) \left( H(t, s)q(s) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma - 1}} \) and using Lemma 5.1, we obtain
\[
H(t, s)q(s)x'(s - \sigma) - h_3(t, s)x(s - \sigma) \\
\leq (1 - \gamma)\gamma^{\frac{1}{\gamma - 1}} (h_3(t, s))^{\frac{1}{\gamma - 1}} \left( H(t, s)q(s) \right)^{\frac{1}{\gamma - 1}}. \tag{5.2.26}
\]

Similarly by taking \( X_2 = \left( H(t, s)p(s) \right)^{\frac{1}{\beta}} x(s - \sigma), Y_2 = \left( \frac{1}{\beta} h_3(t, s) \left( H(t, s)p(s) \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\beta - 1}} \) and using Lemma 5.1, we obtain
\[
h_3(t, s)x(s - \sigma) - H(t, s)p(s)x(s - \sigma) \\
\leq (\beta - 1)\beta^{\frac{1}{\beta - 1}} (h_3(t, s))^{\frac{1}{\beta - 1}} \left( H(t, s)p(s) \right)^{\frac{1}{\beta - 1}}. \tag{5.2.27}
\]
Substituting (5.2.26) and (5.2.27) in (5.2.25), we obtain

\[ \int_{t_0}^{t} H(t, s)c(s)\,ds \leq KH(t, t_0) + \int_{t_0}^{t} \left(1 - \gamma\right)\gamma^{\frac{1}{1-\gamma}}(h_3(t, s))^{\frac{1}{1-\gamma}} \left(H(t, s)q(s)\right)^{\frac{1}{1\gamma}} \,ds + \int_{t_0}^{t} (\beta - 1)\beta^{\frac{1}{1-\beta}}(h_3(t, s))^{\frac{1}{\beta-1}} \left(H(t, s)p(s)\right)^{\frac{1}{1-\beta}} \,ds. \]

or

\[ \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s)c(s) - Q(t, s) \right) \,ds \leq K. \]

Taking \( \lim \sup \) as \( t \to \infty \) in the last inequality, we obtain a contradiction to condition (5.2.22). If \( x(t) \) is eventually negative, then proceeding as above leads to a contradiction with the condition (5.2.23). This completes the proof. \( \square \)

Next we consider the following differential equation of the form

\[ z''(t) + b(t)z(t) + q(t)x'(t - \tau) = p(t)x(t - \sigma) + e(t) \]  \hspace{1cm} (5.2.28)

where \( z(t) = x(t) + c(t)x(t - \tau) \), \( b \in C([0, \infty), \mathbb{R}) \) and \( p, q, e, \gamma, \sigma, \tau \) are as in equation (5.1.1).

**Theorem 5.9** Let the function \( H(t, s) \) be as in Theorem 5.8 such that conditions (5.2.20) and (5.2.21) hold. If \( 0 < \gamma < 1 \) and (5.2.22), (5.2.23) hold with

\[ Q(t, s) = (1 - \gamma)\gamma^{\frac{1}{1-\gamma}}p(s)\gamma^{\frac{1}{1-\gamma}}q(s)^{\frac{1}{1-\gamma}} \left(H(t, s)\right)^{\frac{1}{1-\gamma}}, \]

then every solution of equation (5.2.28) is oscillatory.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (5.2.28) and so \( z(t) \) is eventually positive. Proceeding as in Theorem 5.8, we obtain

\[ \int_{t_0}^{t} H(t, s)c(s)\,ds \leq KH(t, t_0) + \int_{t_0}^{t} \left[ q(s)H(t, s)x'(s - \sigma) - H(t, s)p(s)x(s - \sigma) \right] \,ds. \]

Take \( X = \left(H(t, s)q(s)\right)^{\frac{1}{2}} x(s - \sigma) \), \( Y = \left(\frac{1}{2}p(s)H(t, s)\right)^{\frac{1}{2}} \left(H(t, s)q(s)\right)^{\frac{1}{2-1}} \) and the rest of the proof is similar to that of Theorem 5.8 and hence omitted. \( \square \)
Our next equation is of the form

$$z''(t) + b(t)x(t) + q(t)x(t - \sigma) = p(t)x(t - \sigma) + e(t)$$  \hfill (5.2.29)

where \(z(t) = x(t) + c(t)x(t - \tau), \ b \in C([0, \infty), \mathbb{R})\) and \(p, q, e, \gamma, \sigma, \tau\) are as in equation (5.1.1).

**Theorem 5.10** Let the function \(H(t, s)\) be as in Theorem 5.8 such that

$$p(s)H(t, s) - h_3(t, s)c(s) \geq 0 \text{ for } t > s \geq t_0$$  \hfill (5.2.30)

and condition (5.2.20) hold. If \(0 < \gamma < 1\) and (5.2.22),(5.2.23) hold with

$$Q(t, s) = (1 - \gamma)\frac{\gamma}{\gamma - 1} \left( p(s)H(t, s) - h_3(t, s)c(s) \right)^{\frac{1}{\gamma - 1}} \left( H(t, s)\frac{q(s)}{r(s)} \right)^{\frac{1}{\gamma - 1}},$$

then every solution of equation (5.2.29) is oscillatory.

**Proof.** Let \(x(t)\) be an eventually positive solution of equation (5.2.29) and so \(z(t)\) is eventually positive. Proceeding as in Theorem 5.8 and using (5.2.30), we obtain

$$\int_{t_0}^{t} H(t, s)e(s)ds \leq KH(t, t_0)$$

$$+ \int_{t_0}^{t} \left[ q(s)H(t, s)x^{\gamma}(s - \sigma) - (p(s)H(t, s) - h_3(t, s)c(s))x(s - \sigma) \right] ds.$$

Take

$$X = \left( H(t, s)q(s) \right)^{\frac{1}{\gamma}} x(s - \sigma), \ Y = \left( \frac{1}{\gamma}[p(s)H(t, s) - h_3(t, s)c(s)](H(t, s)q(s))^{\frac{1}{\gamma}} \right)^{\frac{1}{1 - \gamma}}$$

and the rest of the proof is similar to that of Theorem 5.8 and hence omitted. \hfill \Box

**Remark 5.1** Theorems 5.9 and 5.10 are not valid when \(p(t) \equiv 0\).

Next we consider the following differential equation of the form

$$z''(t) + b(t)x(t) = p(t)x^{\beta}(t - \sigma) + e(t)$$  \hfill (5.2.31)

where \(z(t) = x(t) + c(t)x(t - \tau), \ b \in C([0, \infty), \mathbb{R})\) and \(p, e, \beta, \sigma, \tau\) are as in equation (5.1.1).
**Theorem 5.11** Let the function $H(t, s)$ be as in Theorem 5.8 such that conditions (5.2.20), (5.2.21) hold. If $\beta > 1$ and (5.2.22), (5.2.23) hold with

$$Q(t, s) = (\beta - 1)\beta^{\frac{2}{1-\beta}}(h_3(t, s)c(s))^{\frac{2}{\beta-1}} \left( H(t, s)p(s) \right)^{\frac{1}{\beta-1}},$$

then every solution of equation (5.2.31) is oscillatory.

**Proof.** Let $x(t)$ be an eventually positive solution of equation (5.2.31) and so $z(t)$ is eventually positive. Proceeding as in Theorem 5.8, we obtain

$$\int_{t_0}^{t} H(t, s)e(s)ds \leq KH(t, t_0) + \int_{t_0}^{t} \left[ h_3(t, s)c(s)x(s - \sigma) - H(t, s)p(s)x^\beta(s - \sigma) \right]ds.$$ 

Take $X = \left( H(t, s)p(s) \right)^{\frac{1}{\beta}} x(s - \sigma)$, $Y = \left( \frac{1}{\beta} h_3(t, s)c(s) \left( H(t, s)p(s) \right)^{\frac{1}{\beta-1}} \right)^{\frac{1}{\beta-1}}$ and the rest of the proof is similar to that of Theorem 5.8 and hence omitted. \(\square\)

Finally we consider the following third order differential equation of the form

$$z'''(t) - p(t)z(t) - q(t)z(t-\tau) = e(t) \quad \text{(5.2.32)}$$

where $z(t) = x(t) + c(t)x(t-\tau)$ and $p, q, e, \sigma, \tau$ are as in equation (5.1.1).

**Theorem 5.12** Let the function $H(t, s)$ be as in Theorem 5.8 such that condition (5.2.20) hold, and

$$q(s)H(t, s) - h_3(t, s)c(s) \geq 0, \quad \text{(5.2.33)}$$

$$p(s)H(t, s) - h_3(t, s) \geq 0 \text{ for } t > s \geq t_0. \quad \text{(5.2.34)}$$

If

$$\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)e(s)ds = +\infty \quad \text{(5.2.35)}$$

and

$$\lim_{t \to \infty} \inf \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)e(s)ds = -\infty \quad \text{(5.2.36)}$$

then every solution of equation (5.2.32) is oscillatory.
5.3. Examples

Proof. Let \( x(t) \) be an eventually positive solution of equation (5.2.32). Proceeding as in Theorem 5.8, we obtain

\[
\int_{t_0}^t H(t, s)e(s)ds \leq KH(t, t_0) + \int_{t_0}^t \left[h_3(t, s)c(s) - q(s)H(t, s)\right]x(s - \tau)ds
\]

\[
+ \int_{t_0}^t \left[h_3(t, s) - H(t, s)p(s)\right]x(s)ds.
\]

Applying conditions (5.2.33) and (5.2.34) in the above inequality, we obtain

\[
\int_{t_0}^t H(t, s)e(s)ds \leq KH(t, t_0).
\]

Taking \( \lim \sup \) as \( t \to \infty \) in the above inequality, we obtain a contradiction to condition (5.2.35). If \( x(t) \) is eventually negative, then proceeding as above leads to a contradiction with the condition (5.2.36). This completes the proof. \( \square \)

5.3 Examples

In this section, we provide some examples to illustrate our results.

Example 5.1 Consider the differential equation

\[
\left(t^3 \left(x(t) + x(t - 1)\right)\right)'' + t^2x^3(t - 1) = (18t^2 - t^4)\cos t + (6t - 9t^3)\sin t + t^3 x(t - 1) \tag{5.3.1}
\]

where \( t \geq 1 \). Here \( \beta = 3, \gamma = 1, a(t) = t^3, c(t) = 1, p(t) = t^2, q(t) = t^3, e(t) = (18t^2 - t^4)\cos t + (6t - 9t^3)\sin t, \tau = \sigma = 1 \). It is easy to see that all conditions of Corollary 5.1 are satisfied and hence every solution of equation (5.3.1) is oscillatory.

Example 5.2 Consider the differential equation

\[
\left(e^\gamma \left(x(t) + 2x(t - 1)\right)\right)'' + e^{2t}x(t - 2) = -e^{2t}\cos t - 7e^{2t}\sin t + e^{-2t}x^3(t - 2) \tag{5.3.2}
\]

where \( t \geq 0 \). Here \( \beta = 1, \gamma = \frac{1}{3}, a(t) = e^\gamma, c(t) = 2, p(t) = e^{2t}, q(t) = e^{-2t}, e(t) = -e^{2t}\cos t - 7e^{2t}\sin t, \tau = 1, \sigma = 2 \). It is easy to see that all conditions of Corollary 5.2 are satisfied and hence every solution of equation (5.3.2) is oscillatory.
Example 5.3 Consider the differential equation

\[
\left( t^4 \left( x(t) + x(t-1) \right) \right)'' + t^2 x^3(t-1) = (28t^3 - t^5) \cos t + (12t^2 - 11t^4) \sin t + t^{\frac{3}{2}} x^{\frac{1}{2}}(t-1) \tag{5.3.3}
\]

where \( t \geq 1 \). Here \( \beta = 3, \gamma = \frac{1}{3}, a(t) = t^4, c(t) = 1, p(t) = t^2, q(t) = t^{\frac{1}{3}}, e(t) = (28t^3 - t^5) \cos t + (12t^2 - 11t^4) \sin t, \tau = \sigma = 1 \). If we choose \( b(t) = t^{\frac{3}{2}} \) it is easy to verify that all conditions of Corollary 5.3 are satisfied and hence every solution of equation (5.3.3) is oscillatory.

Remark 5.2 Theorems 5.2 to 5.7 can be extended to the more general equation of the form

\[
\left[ a(t) \left( x(t) + c(t)x(t-\tau) \right) \right]'' + p(t)x^\alpha(t-\sigma) = e(t) + q(t)x^\gamma(t-\sigma), \quad t \geq t_0.
\]

Further note that the results of this chapter are not applicable when \( e(t) \equiv 0 \). Also in our results, we have used the condition \( \int_{t_0}^{\infty} \frac{1}{a(t)} \, dt < \infty \) instead of \( \int_{t_0}^{\infty} \frac{1}{a(t)} \, dt = \infty \). So our results are essentially new.

We conclude this chapter with the following remark.

Remark 5.3 It would be interesting to extend the results of this chapter to \( n \)th order neutral differential equations with mixed nonlinearities.