# CHAPTER 3

# NEUTRAL DIFFERENCE EQUATION WITH NONLINEAR NEUTRAL TERM

## 3.1 Introduction

This chapter deals with the following second order neutral delay difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(x_n + p_n x_{n-k}^{\alpha}\right)\right)^{\lambda}\right) + q_n x_{n-\ell}^{\beta} = 0, \quad n \in \mathbb{N} \qquad (3.1.1)$$

where $k, \ell$ are positive integers, $\{a_n\}$ is a positive real sequence, $\{p_n\}$ and $\{q_n\}$ are nonnegative sequences and $\alpha$, $\lambda$ and $\beta$ are ratio of odd positive integers. Let $\theta = \max\{k, \ell\}$ . By a solution of equation (3.1.1) we mean a real sequence $\{x_n\}$ exists for all $n \geqslant -\theta$ and satisfies equation (3.1.1) for all $n \in \mathbb{N}$. It is easy to see that under the initial conditions

$$x_n = \phi_n, \quad n = -\theta, -\theta + 1, \ldots, 1 \tag{3.1.2}$$

equation (3.1.1) has a unique solution satisfying equation (3.1.2).

As indicated by Hale [30] and others, neutral equations having a nonlinearity in the neutral term arises in various applications. In [1, 2, 6], the authors considered equation (3.1.1) where $\alpha = \lambda = 1$, and either $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$ or $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$. Also in [38, 94, 97, 119] the authors considered equation (3.1.1) when $\alpha = 1$ and $\lambda \in (0, \infty)$ or $\lambda = 1$ and $\alpha \in (0, \infty)$ . This motivated us to consider the equation (3.1.1) which generalize some of the results obtained in the literature [1, 2, 6].

In Section 3.1 we establish necessary preliminary results and in Section 3.2 sufficient conditions for the oscillation of solutions of equation (3.1.1). In Section 3.3, we establish conditions for the existence of nonoscillatory solutions of equation (3.1.1). Finally in Section 3.4, we present some examples to illustrate the results obtained in the previous sections.

## 3.2 Oscillation Results

In this section we establish sufficient conditions for the oscillation of all solutions of equation (3.1.1). In the sequel we need the following conditions:

$(C_1)$ $\lim_{n\to\infty} \inf q_n > 0$;

$(C_2)$ $0 \leq p_n \leq 1$;

$(C_3)$ $-1 < -p \leq p_n \leq 0$, where $p > 0$ is a constant;

$(C_4)$ $\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{\frac{1}{\lambda}} = \infty$

We need the following results proved in [34, 45, 85, 122] for our subsequent discussion.

**Lemma 3.2.1.** *[122] Assume that $\phi_n \neq 0$ for many values of $n \in \mathbb{N}$. Then*

$$\Delta\omega_n + \phi_n \omega_{n-k}^{\gamma} = 0, n \in \mathbb{N}$$

*has an eventually positive solution if and only if the corresponding* *inequality*

$$\Delta\omega_n + \phi_n \omega^{\gamma}_{n-k} \leqslant 0, \quad n \in \mathbb{N}$$

*has an eventually positive solution.*

**Lemma 3.2.2.** *[34] If* $\omega_n > 0$, $\Delta\omega_n > 0$ *and* $\Delta\left(a_n(\Delta\omega_n)^{\lambda}\right) \leqslant 0$ *for all* $n \geqslant N \in \mathbb{N}$ *then* $\omega_n \geqslant \psi_n a_n^{\frac{1}{\lambda}} \Delta\omega_n$ *for all* $n \geqslant N$ *, where* $\psi_n = \sum\limits_{s=N}^{n-1} \frac{1}{a_s^{\frac{1}{\lambda}}}$.

**Lemma 3.2.3.** *[45] Assume that* $\phi_n \geqslant 0$ *for all* $n \in \mathbb{N}$ *and*

$$\liminf_{n\to\infty} \sum_{j=n-\ell}^{n-1} q_j > \left(\frac{\ell}{\ell+1}\right)^{\ell+1}.$$

*Then*

$$\Delta\omega_n + \phi_n \omega_{n-\ell} \leqslant 0$$

*has no eventually positive solution.*

**Lemma 3.2.4.** *[85] Assume that* $0 < \beta < \lambda$ *and* $\phi_n \geqslant 0$ *for all* $n \in \mathbb{N}$ *. Then every solution of*

$$\Delta\omega_n + \phi_n \omega^{\frac{\beta}{\lambda}}_{n-\ell} = 0$$

*oscillates if and only if*

$$\sum_{n=n_0}^{\infty} \phi_n = \infty .$$

**Lemma 3.2.5.** *[85] Assume that $\beta > \lambda$ and $\phi_n \geqslant 0$ for all $n \in \mathbb{N}$. Suppose further that there exists a $\delta > \frac{1}{\ell} \log \frac{\beta}{\lambda}$ such that*

$$\liminf_{n\to\infty} \phi_n \exp\left(-e^{\delta n}\right) > 0,$$

*then every solution of*

$$\Delta\omega_n + \phi_n \omega_{n-\ell}^{\frac{\beta}{\lambda}} = 0$$

*oscillates.*

**Theorem 3.2.6.** *Assume $\alpha \in [1, \infty)$ and $\frac{\beta}{\lambda} \in (0, \infty)$. If $(C_1)$ and $(C_2)$, $(C_4)$ hold and the difference equation*

$$\Delta\omega_n + q_n(1 - p_{n-\ell})^{\beta} \psi_{n-\ell}^{\beta} \omega_{n-\ell}^{\frac{\beta}{\lambda}} = 0 \tag{3.2.1}$$

*is oscillatory, then all solutions of equation (3.1.1) are oscillatory.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution equation (3.1.1) with $x_{n-\theta} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$. Setting $z_n = x_n + p_n x_{n-k}^{\alpha}$, we obtain $z_n > x_n > 0$ and

$$\Delta\left(a_n(\Delta z_n)^{\lambda}\right) = -q_n x_{n-\ell}^{\beta} < 0, \quad n \geqslant n_1 \geqslant n_0. \tag{3.2.2}$$

Then $\left\{a_n(\Delta z_n)^{\lambda}\right\}$ is strictly monotonic and $a_n\left(\Delta z_n\right)^{\lambda} > 0$. If $a_n\left(\Delta z_n\right)^{\lambda} \leqslant 0$ eventually then we have

$$a_n\left(\Delta z_n\right)^{\lambda} \leqslant a_{n_1}\left(\Delta z_{n_1}\right)^{\lambda} \leqslant 0$$

and

$$\Delta z_n \leqslant \frac{a_{n_1}^{\frac{1}{\lambda}}\left(\Delta z_{n_1}\right)}{a_n^{\frac{1}{\lambda}}}$$

for $n \geqslant n_2$. Summing the last inequality we obtain

$$z_n \leqslant z_{n_1} + a_{n_1}^{\frac{1}{\alpha}}\left(\Delta z_{n_1}\right)\sum_{s=n_1}^{n-1}\frac{1}{a_s^{\frac{1}{\lambda}}}.$$

The last inequality implies that $z_n \to -\infty$ as $n \to \infty$, a contradiction.

Next we claim that $\lim_{n\to\infty} x_n = 0$. Summing (3.2.2) from $n_2 > n_1 + \ell$ to $\infty$, we obtain

$$0 < a_{n_2}(\Delta z_{n_2})^{\lambda} - L = \sum_{n=n_2}^{\infty} q_n x_{n-\ell}^{\beta}$$

where $0 \leqslant L = \lim_{n\to\infty} a_n\left(\Delta z_n\right)^{\lambda} < \infty$ . Since $\sum_{n=n_2}^{\infty} q_n x_{n-\ell}^{\beta} < \infty$ , we have $\lim_{n\to\infty} q_n x_{n-\ell}^{\beta} = 0$, in view of $(C_1)$, we see that $\lim_{n\to\infty} x_n = 0$ holds. Therefore, there is a $n_3 \geqslant n_2$ such that

$$0 \leqslant x_n^{\alpha} \leqslant x_n, \quad n \geqslant n_3$$

or

$$0 \leqslant x_n^{\alpha-1} \leqslant 1, \quad n \geqslant n_3.$$

Then

$$(1-p_n)\, z_n \ \leq\ z_n - c_n z_{n-k}$$

$$= x_n + c_n x_{n-k}^{\alpha} - c_n x_{n-k} - c_n c_{n-k} x_{n-2k}^{\alpha}$$

$$\leq x_n + c_n x_{n-k}\left(x_{n-k}^{\alpha-1} - 1\right) \leq x_n,$$

for $n \geqslant n_4 \geqslant n_3 + 2k$. Since $\{z_n\}$ is positive and increasing, it follows from Lemma 3.2.2 and 3.2.4, we have

$$x_n \geqslant (1-p_n)\, z_n \geqslant (1-p_n)\, \psi_n \Big(a_n(\Delta z_n)^{\lambda}\Big)^{\frac{1}{\lambda}}, \quad n \geqslant n_4 \quad (3.2.3)$$

using (3.2.3) we find

$$q_n x_{n-\ell}^{\beta} \geq q_n \left(1-p_{n-\ell}\right)^{\beta} \psi_{n-\ell}^{\beta} \left(a_{n-\ell}(\Delta z_{n-\ell})^{\lambda}\right)^{\frac{\beta}{\lambda}}, n \geqslant n_4$$

and so from (3.1.1) we have

$$\Delta\left(a_n(\Delta z_n)^{\lambda}\right) + q_n(1-p_{n-\ell})^{\beta} \psi_{n-\ell}^{\beta}\Big(a_{n-\ell}(\Delta z_{n-\ell})^{\lambda}\Big)^{\frac{\beta}{\lambda}} \leqslant 0.$$

Thus, we see that $\left\{a_n(\Delta z_n)^{\lambda}\right\}$ is an eventually positive solution of

$$\Delta\omega_n + q_n(1-p_{n-\ell})^{\beta} \psi_{n-\ell}^{\beta} \omega_{n-\ell}^{\frac{\beta}{\lambda}} \leqslant 0.$$

Therefore by Lemma 3.2.1, the equation (3.2.1) has eventually positive solution, a contradiction. The proof is now complete. □

**Theorem 3.2.7.** *Assume conditions $(C_1)$, $(C_3)$ and $(C_4)$ hold. Then every solution of equation (3.1.1) either oscillates or tends to zero as $n \to \infty$.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1 which is not tending to zero, with $x_{n-\theta} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$ Setting $z_n = x_n + p_n x^{\alpha}_{n-k}$, we obtain $z_n \leqslant x_n$ and also inequality (3.2.2) for $n \geqslant n_1 \geqslant n_0$. Then, from the proof of Theorem 3.2.6 we have $a_n (\Delta z_n)^{\alpha} > 0$ for all $n \geq n_2 \geq n_1$.Summing (3.2.2) from $n_3 \geq n_2 + \ell$ to $\infty$, we obtain

$$0 < a_{n_3} (\Delta z_{n_3})^{\alpha} - L = \sum_{n=n_3}^{\infty} q_n x^{\beta}_{n-\ell}$$

where $0 \leq L = \lim_{n\to\infty} a_n (\Delta z_n)^{\alpha}$. Since $\sum_{n=n_3}^{\infty} q_n x^{\beta}_{n-\ell} < \infty$, we have $\lim_{n\to\infty} q_n x^{\beta}_{n-\ell} = 0$, inview of $(C_1)$, we see that $\lim_{n\to\infty} x_n = 0$ holds. This contradiction completes the proof. □

**Theorem 3.2.8.** *Assume that $\alpha \in [1, \infty)$ and $\lambda = \beta$. If conditions $(C_1)$, $(C_2)$ and $(C_4)$ hold, and*

$$\liminf_{n\to\infty} \sum_{j=n-\ell}^{n-1} q_j (1 - p_{j-\ell})^{\beta} \psi^{\beta}_{j-\ell} > \left(\frac{\ell}{\ell+1}\right)^{\ell+1} \tag{3.2.4}$$

*then all solutions of equation (3.1.1) are oscillatory.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1) with $x_{n-\theta} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$. Proceeding as in the proof of

Theorem 3.2.6 we obtain the inequality

$$\Delta\omega_n + (1 - p_{n-\ell})^\beta \, \psi_{n-\ell}^\beta q_n \omega_{n-\ell} \leq 0. \tag{3.2.5}$$

Then by Lemma 3.2.3 , the inequality (3.2.4) implies that (3.2.5) has no eventually positive solution, which is a contradiction. This completes the proof. □

**Theorem 3.2.9.** *Assume that* $\alpha \in [1, \infty)$ *and* $0 < \beta < \lambda$ *hold. If conditions* $(C_1)$ , $(C_2)$ *and* $(C_4)$ *hold, and*

$$\lim_{n\to\infty} \sum_{s=n_0}^{n} q_s \, (1 - p_{s-\ell})^\beta \, \psi_{s-\ell}^\beta = \infty \tag{3.2.6}$$

*then all solutions of equation (3.1.1) are oscillatory.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1) with $x_{n-\theta} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$ . Proceeding as in the proof of Theorem 3.2.6 we obtain the inequality

$$\Delta\omega_n + q_n \, (1 - p_{n-\ell})^\beta \, \psi_{n-\ell}^\beta \omega_{n-\ell}^{\frac{\beta}{\lambda}} \leq 0. \tag{3.2.7}$$

Then by Lemma 3.2.4 , the condition (3.2.6) implies that all solutions of equation (3.2.1) are oscillatory. Now the proof follows from Theorem 3.2.6. □

**Theorem 3.2.10.** *Assume that* $\alpha \in [1, \infty)$ *and* $\beta > \lambda$ *. If conditions* $(C_1), (C_2)$ *and* $(C_4)$ *hold, and there exists a constant* $\delta > \frac{1}{2} \log \frac{\beta}{\lambda}$ *such that*

$$\lim_{n \to \infty} q_n (1 - p_{n-\ell})^{\beta} \psi_{n-\ell}^{\beta} \exp\left(-e^{\delta n}\right) > 0 \qquad (3.2.8)$$

*then all solutions of equation (3.1.1) are oscillatory.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1) with $x_{n-\theta} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$ . Proceeding as in the proof of Theorem 3.2.6 we obtain (3.2.7) with $\beta > \lambda$. Then Lemma 3.2.5 implies that all solutions of equation (3.2.1) are oscillatory. Now the proof follows from Theorem 3.2.6. □

## 3.3 Existence of Nonoscillatory Solutions

In this section, we provide sufficient conditions for the existence of nonoscillatory solutions of equation of the form

$$\Delta\left(a_n \Delta\left(x_n - p x_{n-k}^{\alpha}\right)\right) + q_n x_{n-\ell}^{\beta} = 0, n \in \mathbb{N}. \qquad (3.3.1)$$

**Theorem 3.3.1.** *Assume that* $\alpha \geq 1, \beta > 1$ *and* $0 < p < 1$ *with*

$p(2\alpha - 1) < 1$. *If*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \left( \sum_{s=n_0}^{n-1} q_s \right) < \infty, n_0 \in \mathbb{N}, \tag{3.3.2}$$

*then equation (3.3.1) has a nonoscillatory solution.*

*Proof.* Choose $N \in \mathbb{N}$ sufficiently large so that we have from (3.3.2)

$$\max_{n \geq N} \left\{ \sum_{i=n}^{\infty} \frac{1}{a_i} \sum_{s=i}^{\infty} q_s, \beta \sum_{i=n}^{\infty} \frac{1}{a_i} \sum_{s=i}^{\infty} q_s \right\} \leq \left( \frac{1-p}{2} \right).$$

Consider the Banach space $B_N$ of all bounded real sequences $x = \{x_n\}, n \geq N$ with norm defined as $||x_n|| = \sup |x_n|, n \geq N$. We define a closed bounded, and convex subset $S$ of $B_N$ as

$$S = \left\{ x \in B_N : \frac{1-p}{2} \leq x_n \leq 1, n \geq N \right\}.$$

Define the operator $T : S \to B_N$ such that

$$(Tx)_n = (1-p) + p x_{n-k}^{\alpha} - \sum_{s=n}^{\infty} \frac{1}{a_s} \left( \sum_{j=s}^{\infty} q_j x_{j-\ell}^{\beta} \right), n \geq N.$$

Clearly $T$ is continuous and for any $x \in S$ and $n \geq N$, we have

$$\begin{aligned} (Tx)_n &\leq (1-p) + p x_{n-k}^{\alpha} \\ &\leq (1-p) + p \leq 1. \end{aligned}$$

Further

$$
\begin{aligned}
(Tx)_n &\geq (1-p) - \frac{1-p}{2} \\
&\geq \frac{1-p}{2}.
\end{aligned}
$$

Thus $TS \subset S$. Next we show that $T$ is a contraction mapping on $S$. For $x, y \in S$ and $n \geq N$

$$
|(Tx)_n - (Ty)_n| \leq p\left|x_{n-k}^{\alpha} - y_{n-k}^{\alpha}\right| + \sum_{i=n}^{\infty} \frac{1}{a_i}\left[\sum_{s=i}^{\infty} q_s \left|x_{s-\ell}^{\beta} - y_{s-\ell}^{\beta}\right|\right].
$$

Apply the Mean Value theorem to the function $f(u) = u^{\gamma}, \gamma \geq 1$ we have

$$
|x^{\gamma} - y^{\gamma}| \leq \gamma |x - y|, n \geq N, x, y \in S.
$$

Hence

$$
\begin{aligned}
\|Tx - Ty\| &\leq p\alpha \|x-y\| + \beta \|x-y\| \sum_{i=n}^{\infty} \frac{1}{a_i}\left(\sum_{s=i}^{\infty} q_s\right) \\
&\leq \left[p\alpha + \frac{1-p}{2}\right] \|x-y\|.
\end{aligned}
$$

Since $p(2\alpha - 1) < 1, T$ is a contraction mapping and so $T$ has a unique fixed point $x$ which is clearly a positive solution of equation (3.3.1). The proof is now complete. □

Our final result is for the case $0 < \beta < 1$.

**Theorem 3.3.2.** *Assume that* $\alpha \geq 1, 0 < \beta < 1$ *and* $0 < p < 1$ *with* $p\alpha + \beta < 1$. *If*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n_0}^{n-1} q_s < \infty, n_0 \in \mathbb{N}, \tag{3.3.3}$$

*then equation (3.3.1) has a nonoscillatory solution.*

*Proof.* Choose $N \in \mathbb{N}$ sufficiently large so that we have from (3.3.3)

$$\sum_{s=N}^{\infty} \frac{1}{a_s} \sum_{j=s}^{\infty} q_j < \frac{1 - p\alpha}{2}$$

Let $B_N$ be the Banach space considered in Theorem 3.3.2. We define a closed and bounded subset $S$ of $B_N$ as

$$S = \left\{ x \in B_N : \frac{1 - p\alpha}{2\alpha} \leq x_n \leq 1, n \geq N \right\}.$$

Define a operator $T : S \to B_N$ by

$$(Tx)_n = \frac{1 - p\alpha}{\alpha} + c x_{n-k}^{\alpha} - \sum_{i=n}^{\infty} \frac{1}{a_i} \left( \sum_{s=i}^{\infty} q_s x_{s-\ell}^{\beta} \right), n \geq N.$$

It is easy to see that $T$ is continuous , $TS \subset S$ and for $x, y$ and $n \geq N$ we have

$$\left|(Tx)_n - (Ty)_n\right| \leq c \left|x_{n-k}^{\alpha} - y_{n-k}^{\alpha}\right| + \sum_{i=n}^{\infty} \frac{1}{a_i} \left[ \sum_{s=i}^{\infty} q_s \left| x_{s-\ell}^{\beta} - y_{s-\ell}^{\beta} \right| \right].$$

Mean Value theorem applied to the function $f(t) = t^{\gamma}$,we see that for $x, y \in S$, we have

$$|y^{\gamma} - x^{\gamma}| \leq \gamma |y - x|, \text{if } \gamma \geq 1.$$

and

$$|y^{\gamma} - x^{\gamma}| \leq \gamma \frac{2\alpha}{1 - p\alpha} |y - x|, \text{if } 0 < \gamma < 1.$$

Hence

$$\|Tx - Ty\| \leq (p\alpha + \beta) \|x - y\|$$

and since $p\alpha + \beta < 1$, we see that $T$ is a contraction mapping on $S$ . Therefore $T$ has a unique fixed point $x$ which is clearly a nonoscillatory solution of equation (3.3.1) . The proof is now complete. □

## 3.4 Examples

In this section, we present some examples to illustrate the results.

**Example 3.4.1.** *Let $\alpha$ be an even integer and $\ell$ is an odd integer and $0 < \beta < \lambda$. Further more $0 < p < 1$, and $q, k > 0$. Consider* *the equation*

$$\Delta\left(\left(\Delta\left(x_n + p x_{n-k}^{\alpha}\right)\right)^{\lambda}\right) + q x_{n-\ell}^{\beta} = 0, n \geq 1. \tag{3.4.1}$$

*Since all conditions of Theorem 3.2.6 are satisfied, so we associate (3.4.1) with an equation*

$$\Delta\omega_n + q\left(1-p\right)^{\beta}\left(n-\ell\right)^{\beta}\omega_{n-\ell}^{\frac{\beta}{\lambda}} = 0, n \geq 1. \tag{3.4.2}$$

*But* $\sum\limits_{n=n_0}^{\infty}\left(n-\ell\right)^{\beta} = \infty$ *and hence every solution of equation (3.4.2) is oscillatory by Lemma 3.2.4 and so is every solution of equation (3.4.1).*

**Example 3.4.2.** *Let* $\alpha$ *be an even integer and* $\ell$ *is an odd integer and* $\beta > \lambda$. *Further* $0 < p < 1$, *and* $k > 0$ *and* $\delta > \frac{1}{\ell}\log\frac{\beta}{\lambda}$. *Consider the difference equation*

$$\Delta\left(\left(\Delta\left(x_n + p x_{n-k}^{\alpha}\right)\right)^{\lambda}\right) + \exp\left(e^{\delta n}\right) x_{n-\ell}^{\beta} = 0, n \geq 1. \tag{3.4.3}$$

*It is easy to see that all conditions of Theorem 3.2.10 are satisfied and therefore every solution of equation (3.4.3) is oscillatory.*

**Example 3.4.3.** *Consider the neutral difference equation*

$$\Delta\left(\left(\Delta\left(x_n + \frac{1}{4^n}x_{n-1}^{2}\right)\right)^{3}\right) + 1944 x_{n-1}^{3} = 0, n \geq 1. \tag{3.4.4}$$

*It is easy to see that all conditions of Theorem 3.2.8 are satisfied and hence all solutions of equation (3.4.4) are oscillatory.In fact,* $\{x_n\} = \{(-1)^n 2^n\}$ *is one such solution of equation (3.4.4).*

**Example 3.4.4.** *Consider the neutral difference equation*

$$\Delta\left(n\Delta\left(x_n - \frac{1}{4}x_{n-k}^2\right)\right) + \frac{1}{2^n}x_{n-\ell}^3 = 0, n \geq 1. \qquad (3.4.5)$$

*With* $a_n = n, p = \frac{1}{4}, \alpha = 2, \beta = 3$ *and* $q_n = \frac{1}{2^n}$, *we see that all conditions of Theorem 3.3.1 are satisfied and hence equation (3.4.5) has a nonoscillatory solution.*

**Example 3.4.5.** *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n - \frac{1}{8}x_{n-k}^2\right)\right) + \frac{1}{3^n}x_{n-\ell}^{\frac{1}{3}} = 0, n \geq 1. \qquad (3.4.6)$$

*With* $a_n = \frac{1}{n}, p = \frac{1}{8}, \alpha = 2, \beta = \frac{1}{3}$ *and* $q_n = \frac{1}{3^n}$, *we see that all conditions of Theorem 3.3.2 are satisfied and hence equation (3.4.6) has a nonoscillatory solution.*