CHAPTER 2

UNSTABLE NEUTRAL TYPE DIFFERENCE EQUATION

2.1 Introduction

This chapter deals with the oscillatory behavior of solutions of the neutral type difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(x_n+px^{\alpha}_{n-k}\right)\right)^{\lambda}\right)-q_nf\left(x_{n-\ell}\right)=0, n\in\mathbb{N} \quad (2.1.1)$$

where $\{a_n\}$ is a positive real sequence, $\{q_n\}$ is a nonnegative real sequence, p is a real number,k and ℓ are positive integers, α and λ are ratio of odd positive integers and $f:\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, \ell\}$. By a solution of equation (2.1.1) we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\theta$ and satisfies equation (2.1.1) for all $n \in \mathbb{N}$. It is easy to see that under the initial conditions

$$x_n = \phi_n, n = -\theta, -\theta + 1, \cdots, 1 \tag{2.1.2}$$

equation (2.1.1) has a unique solution satisfying (2.1.2).

The oscillatory and asymptotic behavior of solutions of particular form of equation (2.1.1) has been investigated by several authors. See for example [1, 2, 6], and the references cited there in. Infact most of the results established for the equation (2.1.1) are for the case $a_n = 1$ and $\alpha = \lambda = 1$ or $\alpha = 1$ or $\lambda = 1$. Motivated by this observation in this chapter consider equation (2.1.1) and discuss the oscillatory behavior of solutions of equation (2.1.1).

The plan of the chapter is as follows: In Section 2.2 we present sufficient conditions for the oscillation of all bounded solutions of equation (2.1.1), and in Section 2.3 we establish similar results for unbounded solutions of equation (2.1.1). In Section 2.4, we present some examples to illustrate the results obtained in the previous sections.

2.2 Bounded Oscillation

In [89], and [95], the authors proved that there always exists an unbounded nonoscillatory solutions for the equation of the form (2.1.1). However in this section we establish conditions for the oscillation of all bounded solutions of equation (2.1.1).

Theorem 2.2.1. *Assume that*

(C_1) $-1 < p \leq 0$;

(C_2) $\dfrac{f(u)}{u^{\beta}} \geq M > 0 \, for \, u \neq 0$;

(C_3) $\sum\limits_{n=0}^{\infty} \dfrac{1}{a_n^{\frac{1}{\lambda}}} = \infty.$

If

$$\lambda \geq \beta \text{ and } \limsup_{n\to\infty} \sum_{s=n-\ell}^{n} \left(\frac{1}{a_s} \sum_{t=s}^{n-1} q_t \right)^{\frac{1}{\lambda}} = \infty, n \in \mathbb{N}, \quad (2.2.1)$$

then every bounded solution of equation (2.1.1) is oscillatory.

Proof. Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive bounded solution of equation (2.1.1). Define

$$z_n = x_n + p x_{n-k}^{\alpha}. \tag{2.2.2}$$

From equation (2.1.1), we have $\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq 0$ for all large $n \in \mathbb{N}$. If $a_n\left(\Delta z_n\right)^{\lambda} \geq 0$ eventually, then by (C_3), $\lim_{n\to\infty} z_n = \infty$, a contradiction to the boundedness of $\{x_n\}$. Therefore $a_n\left(\Delta z_n\right)^{\lambda} < 0$ and since λ is a ratio of odd positive integers and $\{a_n\}$ is eventually positive we may take $\Delta z_n < 0$ for all $n \geq n_0 \in \mathbb{N}$. Now we have two possibilities for $\{ z_n\}$.

(a) $z_n > 0$ for $n \geq n_1 \geq n_0$;

(b) $z_n < 0$ for $n \geq n_1$.

In case (a), equation (2.1.1) can be written as

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq M q_n x_{n-\ell}^{\beta}$$

using (2.2.2), we obtain

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq M q_n z_{n-\ell}^{\beta}. \tag{2.2.3}$$

Summing (2.2.3) from s to $n-1$ yields,

$$a_n(\Delta z_n)^\lambda - a_s(\Delta z_s)^\lambda \geq M\sum_{t=s}^{n-1} q_t z_{t-\ell}^{\beta}$$

or

$$-\Delta z_s \geq z_{n-\ell}^{\frac{\beta}{\lambda}} \frac{M^{\frac{1}{\lambda}}}{a_s^{\frac{1}{\lambda}}}\left(\sum_{t=s}^{n-1} q_t\right)^{\frac{1}{\lambda}}.$$

Summing the last inequality in s from $n-\ell$ to n, we see that

$$-z_{n+1} + z_{n-\ell} \geq z_{n-\ell}^{\frac{\beta}{\lambda}} M^{\frac{1}{\lambda}} \sum_{s=n-\ell}^{n}\left(\frac{1}{a_s}\sum_{t=s}^{n-1} q_t\right)^{\frac{1}{\lambda}} \tag{2.2.4}$$

which contradicts (2.2.1) since $\{z_n\}$ is bounded.For the case (b), we obtain

$$x_n < -p x_{n-k}^{\alpha} < (-p)^{\alpha+1} x_{n-2k}^{\alpha^2} < \cdots < (-p)^{\alpha+j} x_{n-jk}^{\alpha^j}$$

for $n \geq n_1 + jk$ and we are led to that $\lim\limits_{n\to\infty} x_n = 0$. Hence $\lim\limits_{n\to\infty} z_n = 0$ which is again a contradiction. This completes the proof of the theorem. □

Remark 2.2.1. *If $\alpha = 1$, then Theorem 2.2.1 reduces to Theorem 1 of Thandapani, Pandian and Balasubramaian [95]. Also if $\alpha = \lambda = 1$; and $a_n \equiv 1$ then Theorem 2.2.1 reduces to Theorem 4.1 of Lalli and Zhang [48].*

Theorem 2.2.2. *In addition to* (C_2) *and* (C_3) *assume that* $p = -1$. *If* $\lambda \geq \beta$ *and condition (2.2.1) holds, then every bounded solution of equation (2.1.1) is oscillatory.*

Proof. Assume that $\{x_n\}$ is an eventually positive solution of equation (2.1.1). Proceeding as in the proof of Theorem 2.2.1, we see that there are two possibilities for $\{z_n\}$:

(a) $z_n > 0, \Delta z_n < 0, \Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq 0$ for $n \geq n_1 \in \mathbb{N}$;

(b) $z_n < 0, \Delta z_n < 0, \Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq 0$ for $n \geq n_1 \in \mathbb{N}$.

In case (a), we are led to (2.2.4) which contradicts condition (2.2.1). In case (b), we have $\lim_{n\to\infty} z_n = -\delta$ where $\delta > 0$ is a finite number. So there is an integer $n_2 \in \mathbb{N}$ such that $-\delta < z_n < -\frac{\delta}{2}$ for $n \geq n_2$. Hence

$$-\delta < x_n - x_{n-k}^{\alpha} < -\frac{\delta}{2}, n \geq n_2.$$

Then

$$x_{n-k} > \left(\frac{\delta}{2}\right)^{\frac{1}{\alpha}}$$

or

$$x_{n-\ell}^{\beta} > \left(\frac{\delta}{2}\right)^{\frac{\beta}{\alpha}}$$

and from equation (2.1.1) we obtain

$$\Delta\left(a_n\left(\Delta z_n\right)^{\alpha}\right) \geq M q_n x_{n-\ell}^{\beta} \geq M\left(\frac{\delta}{2}\right)^{\frac{\beta}{\alpha}} q_n.$$

The rest proof is similar to that of Theorem 2.2.1 and hence the details are omitted. □

In the following, we present oscillatory criteria for equation (2.1.1) when $p < -1$.

Theorem 2.2.3. *In addition to conditions* (C_2) *and* (C_3) *assume that* $p < -1$. *If* $\lambda \geq \beta$ *and*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{a_n}\sum_{s=n}^{\infty} q_s\right)^{\frac{1}{\lambda}} = \infty, n_0 \in \mathbb{N}, \tag{2.2.5}$$

then every bounded solution of equation (2.1.1) is oscillatory.

Proof. Assume without loss of generality, that $\{x_n\}$ is a bounded solution of equation (2.1.1) and z_n is defined by (2.2.1). Thus there are two possibilities for $\{z_n\}$ as in Theorem 2.2.2. In case (a), we have $x_n > -p x_{n-k}^{\alpha}$ for $n \geq n_1 \in \mathbb{N}$ and there exists a constant $\delta > 0$ such that $x_n \geq \delta$ for $n \geq n_1$. Hence from equation (2.1.1), we have

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq M \delta^{\beta} q_n, n \geq n_1.$$

In case (b), there exists a finite number $\eta > 0$ such that $\lim_{n\to\infty} z_n = -\eta$. Then there exists an integer $n_2 \geq n_1$ such that $-\eta < z_n < -\frac{\eta}{2}$ for $n \geq n_2$, that is,

$$-\eta < x_n + px_{n-k}^{\alpha} < -\frac{\eta}{2}$$

for $n \geq n_2$. Hence $x_{n-k} > \left(-\frac{\eta}{2p}\right)^{\frac{1}{\alpha}}$,n$\geq n_2$. From equation (2.1.1), we have

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq M\left(-\frac{\eta}{2p}\right)^{\frac{\beta}{\alpha}} q_n, n \geq n_2.$$

Thus, in both the cases we are led to the inequality

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq Bq_n, n \geq n_3 \geq n_2, \qquad (2.2.6)$$

where B is a constant. Summing (2.2.6) from n to N for $N > n_3$, we have

$$a_{N+1}\left(\Delta z_{N+1}\right)^{\lambda} - a_n\left(\Delta z_n\right)^{\lambda} \geq B\sum_{s=n}^{N} q_s, n_3 \leq n < N.$$

Hence

$$-a_n\left(\Delta z_n\right)^{\lambda} \geq B\sum_{s=n}^{N} q_s; N > n;$$

which implies that $\sum\limits_{n=n_0}^{\infty} q_n < \infty$, and so

$$-\Delta z_n \geq \left(\frac{B}{a_n} \sum_{s=n}^{\infty} q_s \right)^{\frac{1}{\lambda}}.$$

Summing the last inequality from n to $N-1$ for $N-1 > n$, we have

$$z_n \geq z_N + B^{\frac{1}{\lambda}} \sum_{s=n}^{N-1} \left(\frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right)^{\frac{1}{\lambda}}$$

or

$$z_{n_0} \geq B^{\frac{1}{\lambda}} \sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{t=n}^{\infty} q_t \right)^{\frac{1}{\lambda}}$$

which contradicts condition (2.2.5). This completes the proof of the theorem. □

Remark 2.2.2. *If* $\alpha = 1$ *then Theorem 2.2.3 reduces to Theorem 4 of Thandapani, Pandian and Balasubramanian [95]. Further, if* $\lambda = \alpha = 1$, $a_n \equiv 1$, *then Theorem 2.2.1 reduces to Theorem 4.2 of Lalli and Zhang [48].*

Our final result in this section deals with the case $p > 0$.

Theorem 2.2.4. *In addition to* (C_2) *and* (C_3) *assume* $p > 0$ *and*

$\ell \geq k$. *If* $\min(\lambda - \beta, \alpha\lambda - \beta) \geq 0$ *and*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t \right)^{\frac{1}{\lambda}} = \infty, n_0 \in \mathbb{N}, \tag{2.2.7}$$

then every bounded solution of equation (2.1.1) is oscillatory.

Proof. Assume without loss of generality that $\{x_n\}$ is a bounded positive solution of equation (2.1.1) and z_n is defined by (2.2.1). Since $p > 0$, then by Theorem 2.2.1 we have case (a) only. Since $\{z_n\}$ is positive and decreasing, we may assume without loss of generality $\{x_n\}$ is decreasing and from (2.2.1) we obtain

$$\begin{aligned} z_n &\leq x_{n-k} + p x_{n-k}^{\alpha} \leq x_{n-k}^{\alpha}\left(x_{n-k}^{1-\alpha} + p\right) \\ &\leq x_{n-k}^{\alpha}\left(M_1^{1-\alpha} + p\right) \text{ if } 0 < \alpha \leq 1 \end{aligned}$$

and

$$\begin{aligned} z_n &\leq x_{n-k}\left(1 + p x_{n-k}^{\alpha-1}\right) \\ &\leq x_{n-k}\left(1 + p M_1^{\alpha-1}\right) \text{ if } \alpha \geq 1, \end{aligned}$$

where M_1 is the upper bound of $\{x_n\}$. Hence

$$x_n \geq \begin{cases} \dfrac{z_{n+k}^{\frac{1}{\alpha}}}{M_2}, & \text{if } 0 < \alpha \leq 1 \\ \dfrac{z_{n+k}}{M_3} & \text{if } \alpha \geq 1. \end{cases}$$

Therefore from equation (2.1.1) we obtain

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq \frac{M}{M_3} q_n z_{n+k-\ell}^{\beta}, \text{ if } \alpha \geq 1 \tag{2.2.8}$$

and

$$\Delta\left(a_n\left(\Delta z_n\right)^{\lambda}\right) \geq \frac{M}{M_2} q_n z_{n+k-\ell}^{\frac{\beta}{\alpha}}, \text{ if } 0 < \alpha \leq 1. \tag{2.2.9}$$

Summing (2.2.8) from n to s, we obtain

$$a_{n+1}\left(\Delta z_{n+1}\right)^{\lambda} - a_s\left(\Delta z_s\right)^{\lambda} \geq \frac{M}{M_3} \sum_{t=s}^{n} q_t z_{s+k-\ell}^{\beta}$$

or

$$\begin{aligned} -\Delta z_s &\geq \left(\frac{M}{M_3}\right)^{\frac{1}{\lambda}} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t z_{s+k-\ell}^{\beta}\right)^{\frac{1}{\lambda}} \\ &\geq \left(\frac{M}{M_3}\right)^{\frac{1}{\lambda}} z_{n+k-\ell}^{\frac{\beta}{\lambda}} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Summing the last inequality in s from N to $n\,(n \geq N + \ell - k)$, we obtain

$$-z_{n+1} + z_N \geq \left(\frac{M}{M_3}\right)^{\frac{1}{\lambda}} z_N^{\frac{\beta}{\lambda}} \sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t\right)^{\frac{1}{\lambda}}. \tag{2.2.10}$$

Similarly from (2.2.9) we obtain

$$-z_{n+1} + z_N \geq \left(\frac{M}{M_2}\right)^{\frac{1}{\lambda}} z_N^{\frac{\beta}{\lambda\alpha}} \sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t\right)^{\frac{1}{\lambda}}. \tag{2.2.11}$$

Since $\min(\lambda-\beta, \alpha\lambda-\beta) \geq 0$ we get from (2.2.10) and (2.2.11)

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{t=s}^{n} q_t \right)^{\frac{1}{\lambda}} < \infty$$

a contradiction to (2.2.7). This completes the proof. □

2.3 Unbounded Oscillation

In this section, we present sufficient conditions for the oscillation of unbounded solutions of equation of the form

$$\Delta\left(a_n\left(\Delta\left(x_n + p x_{n-k}\right)\right)^{\lambda}\right) - q_n f\left(x_{n+\ell}\right) = 0, n \in \mathbb{N}. \quad (2.3.1)$$

Theorem 2.3.1. *In addition to* (C_2), *assume that* $\lambda \leq \beta$, *and*

$$0 \leq p < 1 \quad (2.3.2)$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n}^{n+\ell-1} \left(\frac{1}{a_s} \sum_{t=n}^{s-1} q_t \right)^{\frac{1}{\lambda}} = \infty \quad (2.3.3)$$

hold. Then every unbounded solution of equation (2.3.1) oscillates.

Proof. Assume for the sake of contradiction that equation (2.3.1) has an eventually positive unbounded solution $\{x_n\}$. Then $x_n > 0$

for $n \geq n_1 \in \mathbb{N}$. Let z_n be defined by (2.2.1). Then $z_n > 0$ for $n \geq n_1$ and $\{z_n\}$ is unbounded and $\Delta\left(a_n(\Delta z_n)^\lambda\right) \geq 0$ for $n \geq n_1$. Thus $\{\Delta z_n\}$ is of constant sign and $\Delta z_n > 0$ for all $n \geq n_2 \geq n_1$ since $\{z_n\}$ is unbounded. From (2.2.1) and inview of z_n is increasing we have

$$x_{n+\ell} \geq (1-p)\, z_{n+\ell} \tag{2.3.4}$$

for $n \geq n_2$. From (2.3.1) and (2.3.4), we obtain

$$\Delta\left(a_n(\Delta z_n)^\lambda\right) \geq M(1-p)^\beta q_n z_{n+\ell}^\beta, n \geq n_2.$$

Now summing the last inequality from n to $s-1$, we obtain

$$a_s(\Delta z_s)^\lambda - a_n(\Delta z_n)^\lambda \geq M(1-p)^\beta \sum_{t=n}^{s-1} q_t z_{s+1}^\beta$$

or

$$\Delta z_s \geq M^{\frac{1}{\lambda}}(1-p)^{\frac{\beta}{\lambda}} z_{n+\ell}^{\frac{\beta}{\lambda}} \left(\frac{1}{a_s}\sum_{t=n}^{s-1} q_t\right)^{\frac{1}{\lambda}}.$$

Again summing the last inequality in s from n to $n+\ell-1$, we obtain

$$z_{n+\ell} - z_n \geq M^{\frac{1}{\lambda}}(1-p)^{\frac{\beta}{\lambda}} z_{n+\ell}^{\frac{\beta}{\lambda}} \sum_{s=n}^{n+\ell-1}\left(\frac{1}{a_s}\sum_{t=n}^{s-1} q_t\right)^{\frac{1}{\lambda}}.$$

Hence

$$z_{n+\ell}^{\frac{\beta}{\lambda}}\left(M^{\frac{1}{\lambda}}(1-p)^{\frac{\beta}{\lambda}}\sum_{s=n}^{n+\ell-1}\left(\frac{1}{a_s}\sum_{t=n}^{s-1}q_t\right)-z_{n+\ell}^{1-\frac{\beta}{\lambda}}\right)\leq 0$$

which contradicts condition (2.3.3). This completes the proof. □

Remark 2.3.1. *If $p=0$ then Theorem 2.3.1 reduces to a result in Wong and Agarwal [99].*

Next we consider the case $\lambda=\beta=1$ in the equation (2.3.1), and discuss the oscillation of all unbounded solutions of equation (2.3.1) by relaxing the condition (2.3.2).

Theorem 2.3.2. *In addition to condition (C_2), assume that $\lambda=\beta=1$ and $p\geq 0$. If $\left\{\frac{a_n}{a_{n-k}}\right\}$ is nondecreasing and*

$$\limsup_{n\to\infty}\sum_{j=n}^{n-k+\ell}\frac{1}{a_j}\left[\sum_{s=n}^{j-1}\left(q_s+pq_{s-k}\frac{a_s}{a_{s-k}}\right)\right]>\frac{(1+p)^2}{M}\qquad(2.3.5)$$

then every unbounded solution of equation (2.3.1) is oscillatory.

Proof. Assume without loss of generality that equation (2.3.1) has an eventually positive unbounded solution $\{x_n\}$. Then $z_n>0$ for all $n\geq n_1\geq n_0+k$, $\{z_n\}$ is unbounded, and from equation (2.3.1), we have

$$\Delta\left(a_n\Delta z_n\right)\geq Mq_nx_{n+\ell}.$$

Thus $\Delta(a_n \Delta z_n) \geq 0$ and this implies that Δz_n is of constant sign. But if we take $\Delta z_n < 0$, then $\{z_n\}$ would be bounded. Therefore $\Delta z_n > 0$ for $n \geq n_2 \geq n_1$. Let

$$y_n = z_n + p z_{n-k}$$

then $y_n > 0$ and $\Delta y_n > 0$ for all large n and further

$$\begin{aligned}
\Delta(a_n \Delta y_n) &= \Delta(a_n \Delta z_n) + p\Delta(a_n \Delta z_{n-k}) \\
&\geq M q_n y_{n+\ell} + p\frac{a_n}{a_{n-k}} M q_{n-k} y_{n-k+\ell} \\
&\geq \frac{M}{(1+p)^2}\left(q_n + p q_{n-k}\frac{a_n}{a_{n-k}}\right) y_{n-k+\ell}.
\end{aligned}$$

Hence $\{y_n\}$ is a positive solution of the inequality

$$\Delta(a_n \Delta y_n) - A_n y_{n-k+\ell} \geq 0, n \geq n_2, \tag{2.3.6}$$

where $A_n = \dfrac{M}{(1+p)^2}\left(q_n + p q_{n-k}\dfrac{a_n}{a_{n-k}}\right)$. Summing (2.3.6) from n to $j-1$ we have

$$a_j \Delta y_j - a_n \Delta y_n \geq \sum_{s=n}^{j-1} A_s y_{s-k+\ell}.$$

Now summing in j from n to $n-k+\ell$, we are led to

$$y_{n-k+\ell} - y_n \geq y_{n-k+\ell}\left(\sum_{j=n}^{n-k+1} \frac{1}{a_j} \sum_{s=n}^{j-1} A_s\right)$$

where we have used the monotonicity of $\{y_n\}$. Hence

$$y_{n-k+\ell}\left[\sum_{j=n}^{n-k+\ell}\left(\frac{1}{a_j}\sum_{s=n}^{j-1}A_s\right)-1\right]\leq 0$$

which contradicts the condition (2.3.5). This completes the proof.

□

In our next theorem we consider the case when the condition (2.3.5) is not satisfied.

Theorem 2.3.3. *In addition to* (C_2) *assume that* $\lambda = \beta = 1$ *and* $p > 0$. *Further assume that* $a_n \equiv 1$ *and that there exists a sequence and a positive integer such that*

$$b_n > 0, \Delta b_n \leq 0, n \in \mathbb{N} \tag{2.3.7}$$

and

$$\ell \geq 2m \tag{2.3.8}$$

$$q_n^* \geq \frac{1+p}{M} b_{n+1} b_{n+m} \geq 0 \tag{2.3.9}$$

where $q_n^* = \min\{q_n, q_{n-k}\}$. *If the difference inequality*

$$\Delta u_n - b_{n+m} u_{n+m} \geq 0 \tag{2.3.10}$$

has no eventually positive solution then all unbounded solutions of equation (2.3.1) are oscillatory.

Proof. Assume that $x_n > 0$ is an unbounded solution of equation (2.3.1). Let z_n and y_n be the same as defined in Theorem 2.2.4. Then proceeding as in the proof of Theorem 2.3.2, we obtain

$$\Delta^2 y_n - \frac{Mq_n^*}{(1+p)} y_{n+\ell} \geq 0 \tag{2.3.11}$$

where $\{y_n\}$ is a positive solution for $n \geq n_2 \in \mathbb{N}$, we put

$$d_n = \Delta y_n + b_n y_{n+m}$$

then $\{d_n\}$ is positive and

$$\Delta d_n - b_{n+1} d_{n+m} = \Delta^2 y_n + \Delta b_n y_{n+m} - b_{n+1} b_{n+m} y_{n+2m}.$$

Hence, in view of (2.3.7) - (2.3.9) and (2.3.11), one obtains for all large n.

$$\begin{aligned}\Delta d_n - \frac{\Delta b_n}{b_n} d_n - b_{n+1} d_{n+m} &\geq \Delta^2 y_n - \frac{\Delta b_n}{b_n} \Delta z_n - b_{n+1} b_{n+m} y_{n+2m} \\ &\geq \Delta^2 y_n - b_{n+1} b_{n+m} y_{n+2m} \\ &\geq \Delta^2 y_n - \frac{Mq_n^*}{1+p} y_{n+\ell} \geq 0.\end{aligned}$$

Setting

$$d_n = b_n V_n$$

we conclude that $\{V_n\}$ is a positive solution of (2.3.10). This contradiction completes the proof. □

Corollary 2.3.4. *Assume that all hypothesis of Theorem 2.3.3 hold except (2.3.10). If*

$$\limsup_{n\to\infty}\left[\frac{1}{m}\sum_{s=n}^{n+m} b_{s+m}\right] > \frac{m^m}{(m+1)^{m+1}} \tag{2.3.12}$$

then equation (2.3.1) does not allow unbounded nonoscillatory solution.

Proof. It is known that [2] condition (2.3.12) is sufficient for (2.3.11) to have no eventually positive solutions. Hence the assertion follows from Theorem 2.3.3. □

2.4 Examples

In this section we present some examples to illustrate the theorems obtained in Sections 2.2 and 2.3.

Example 2.4.1. *Consider the difference equation*

$$\Delta\left(\frac{1}{n^3}\left(\Delta\left(x_n - \frac{1}{2}x_{n-2}^3\right)\right)^3\right) - \frac{c}{n}x_{n-\ell}^{\frac{1}{3}}\left(1 + x_{n-1}^2\right) = 0, \tag{2.4.1}$$

where $n \geq 1$. *Here* $p = -\frac{1}{2}, a_n = \frac{1}{n^3}, q_n = \frac{c}{n}, f(u) = u^{\frac{1}{3}}\left(1+u^2\right), \alpha = 3, \lambda = 3$. *If we take* $\beta = \frac{1}{3}$ *then* $M = 1$ *and* *it is easy to see that all conditions of Theorem 2.2.1 are satisfied. Hence all bounded solutions of equation (2.4.1) are oscillatory.*

Example 2.4.2. *Consider the difference equation*

$$\Delta\left(n\left(\Delta\left(x_n - 2x_{n-k}^{\frac{1}{3}}\right)\right)^3\right) - \frac{128}{n(n+1)}x_{n-\ell}^3 = 0, n \geq 1. \quad (2.4.2)$$

It is easy to see that all assumptions of Theorem 2.2.3 are satisfied. Therefore every bounded solutions of equation (2.4.2) are oscillatory.

Example 2.4.3. *Consider the difference equation*

$$\Delta\left(\left(\Delta\left(x_n + px_{n-2}^3\right)\right)^3\right) - 8\left(1+p\right)^3 x_{n-4}^3\left(1+\left|x_{n-4}\right|\right) = 0, \quad (2.4.3)$$

where $n \geq 1$. All conditions of Theorem 2.2.4 are satisfied and hence all bounded solutions of equation (2.4.3) are oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is such a solution of equation (2.4.3).