CHAPTER 1

INTRODUCTION

1.1 Difference Equations

Difference equations are of interest both as approximations to differential equations and as methods for describing fundamentally discrete systems in biology, economics, population dynamics and other problems in science where the system is described by a discrete variable. In the past few decades the study of difference equations has already drawn a great deal of attention not only among mathematicians themselves, but also from various other disciplines as well. For the basic theory of difference equations and its applications, see the monographs by Agarwal [1], Kelley and Peterson [41] and Lakshmikanthan and Trigiante [47].

As in the case of differential equations, the corresponding difference equations can also be classified as ordinary, delay and neutral type difference equations. A difference equation of the form

$$\Delta^m x_n = f\left(n, x_n, \Delta x_n, \cdots, \Delta^{m-1} x_n\right), n \in \mathbb{N} \qquad (1.1.1)$$

where $\mathbb{N} = \{1, 2, 3, \cdots\}$ and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$,$\Delta^{j+1} x_n = \Delta^j (\Delta x_n), j = 1, 2, 3, \cdots, m$, is called an m^{th} order ordinary difference equation and the difference equation of the form

$$\Delta^m x_n = f\left(n, x_n, x_{n-\ell}\right), n \in \mathbb{N} \qquad (1.1.2)$$

is called a delay difference equation if ℓ is a positive integer and advanced type difference equation if ℓ is a negative integer. An equation of the form

$$\Delta^m \left(x_n + p_n x_{n-k}\right) = f\left(n, x_n, x_{n-\ell}\right), n \in \mathbb{N}$$

where k and ℓ are positive integers is called a neutral type difference equations.

Neutral type difference equations arise in population dynamics when maturation and gestation are included; in cobweb models in economics when demand depends on current price but supply depends on the price at an earlier time, and electric network containing lossless transmission lines.

1.2 Examples

Example 1.2.1. *One often looks toward physical systems to find chaos, but it also exhibits itself in biology. Biologists had been studying the variability in populations of various species and they found an equation that predicted animal populations reasonably well. This equation was a simple quadratic equation called the logistic difference equation. On the surface, one would not expect this equation to provide the fantastically complex and chaotic behavior that it exhibits. The logistic difference equation is given by*

$$x_{n+1} = rx_n(1 - x_n)$$

where r *is the so-called driving parameter. The equation is used in the following manner. Start with a fixed value of the driving parameter,* r*, and an initial value of* x_0*. One then runs the equation recursively, obtaining* $x_1, x_2, x_3, \cdots, x_n$*. For low values of* r, x_n *(as* n *goes to infinity) eventually converges to a single number. In biology, this number (*x_n *as* n *approaches infinity) represents the population of the species.*

Example 1.2.2. *Suppose a certain population of owls is growing at the rate of* 2% *per year. If we let* x_0 *represent the size of the initial population of owls and* x_n *the number of owls* n *years later, then*

$$x_{n+1} = x_n + 0.02x_n = 1.02x_n \tag{1.2.1}$$

for $n = 0, 1, 2, \cdots$*. That is, the number of owls in any given year is equal to the number of owls in the previous year plus* 2% *of the number of owls in the previous year. Equation (1.2.1) is an example of a first-order difference equation.*

Example 1.2.3. *Radium is a radioactive element which decays at a rate of* 1% *every 25 years.This means that the amount left at the* *beginning of any given 25 year period is equal to the amount at the beginning of the previous 25 year period minus* 1% *of that amount. That is, if* x_0 *is the initial amount of radium and* x_n *is the amount of radium still remaining after* $25n$ *years, then*

$$x_{n+1} = x_n - 0.01x_n = 0.99x_n \tag{1.2.2}$$

for $n = 0, 1, 2, \cdots$, *which is again a first order difference equation.*

Example 1.2.4. *Suppose a cup of tea, initially at a temperature of* 180^0F, *is placed in a room which is held at a constant temperature of* 80^0F. *Moreover, suppose that after one minute the tea has cooled to* 175^0F. *What will the temperature be after 20 minutes?*

Let T_n *be the temperature of the tea after* n *minutes and we let* S *be the temperature of the room, then we have* $T_0 = 180, T_1 = 175$, *and* $S = 80$. *Newton's law of cooling states that*

$$T_{n+1} - T_n = k(T_n - 80), \tag{1.2.3}$$

$n = 0, 1, 2, \cdots$, *where* k *is a constant which we will have to determine. To do so, we make use of the information given about the change in the temperature of the tea during the first minute.*

Namely, applying (1.2.3) with $n = 0$, *we must have*

$$T_1 - T_0 = k(T_0 - 80).$$

That is,

$$175 - 180 = k(180 - 80).$$

Hence

$$-5 = 100k,$$

and so

$$k = -5100 = -0.05.$$

Thus (1.2.3) becomes

$$T_{n+1} - T_n = -0.05(T_n - 80) = -0.05T_n + 4.$$

Hence

$$T_{n+1} = T_n - 0.05T_n + 4 = 0.95T_n + 4$$

for $n = 0, 1, 2, \cdots$. *which is the standard form of a first-order linear difference equation.*

1.3 Motivation

The origin of the modern theory of difference equations may be traced back to the fundamental work of Poincare at the end of nineteenth century, see [1] and the references cited therein. The landmark paper by Philip Hartman [32] has attracted many researchers and increased the interest in the study of qualitative theory of difference equations. In the qualitative theory of difference equations oscillatory behavior of solutions play an important role.

A nontrivial solution of a difference equation is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. For example $\{x_n\} = \{(-1)^n\}$, $\{x_n\} = \{1 + (-1)^n\}$ are oscillatory and $\{x_n\} = \{2^n\}, \{x_n\} = \{3 + (-1)^n\}$ are nonoscillatory. These types of solutions occurs in many physical phenomena such as, for example, vibrating mechanical systems, electrical circuits and in population dynamics.

The literature on the oscillation and asymptotic behavior of solutions of difference equations has grown to such an extent that it is quite impossible to mention all the authors who contributed on this topic. For example, Agarwal et al [1], Hooker and Patula [34 Graef et al [28], Gyori et al [29], Stavroulakis et al [77], Saker et al [75], Cheng et al [14], Szmanda et al [83], Dahiya et al [10], Philos [67], Thandapani et al [89], and Zhang et al [112] have done extensive work on this topic.

1.4 Contribution of the Author

Following this trend in this thesis, the author has obtained some significant results on the following topics:

1. **Oscillation of unstable type neutral difference equation with nonlinear neutral term.**

2. **Oscillatory behavior of neutral difference equation with nonlinear neutral term.**

3. **Oscillatory behavior of neutral difference equation with mixed type.**

4. **Oscillatory behavior of neutral difference equation with mixed nonlinearities.**

5. **Oscillatory behavior of neutral difference equation with positive and negative coefficients.**

The thesis contains six chapters. First chapter we are presenting necessary introduction and motivation for the present study.

Chapter 2 deals with the oscillatory behavior of the neutral difference equation of the form

$$\Delta\left(a_n(\Delta\left(x_n + px_{n-k}^{\alpha}\right))^{\lambda}\right) - q_n f\left(x_{n-\ell}\right) = 0, n \in \mathbb{N}, \qquad (1.4.1)$$

where $\{a_n\}$ is a positive real sequence, $\{q_n\}$ is a nonnegative real sequence, p is a real number, k and ℓ are nonnegative integers, α and λ are ratio of odd positive integers and $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

Section 2.1 presents necessary introduction and motivation. In Section 2.2, we establish sufficient conditions for the bounded/unbounded solutions of equation (1.4.1) to be oscillatory. Examples are provided in Section 2.3 to illustrate the main results. The results obtained in this chapter generalize those obtained by Thandapani, Arul and Raja [89] and Thandapani, Pandian and Balasubramanian [95].

In Chapter 3, we consider the neutral difference equation of the form

$$\Delta\left(a_n(\Delta\left(x_n + p_n x_{n-k}^{\alpha}\right))^{\lambda}\right) + q_n x_{n-\ell}^{\beta} = 0, n \in \mathbb{N}, \qquad (1.4.2)$$

where $\{a_n\}$ is a positive real sequences, $\{p_n\}$ and $\{q_n\}$ are real sequences, α, λ and β are ratio of odd positive integers and k and ℓ are nonnegative integers.

Section 3.1 provides necessary introduction and in Section 3.2 we establish conditions for the oscillation of all solutions of equation (1.4.2). Section 3.3 deals with the existence of nonoscillatory solutions of equation (1.4.2), and examples are provided in Section 3.4 to illustrate the results. The results obtained in this chapter generalize those obtained by Thandapani and Mahalingam [93] and Zhang [112].

Chapter 4 deals with the oscillatory behavior of the neutral difference equation of the form

$$\Delta\left(a_n \Delta\left(x_n + b x_{n-k} - c x_{n+\ell}\right)\right) = q_n f\left(x_{n-\sigma_1}\right) + p_n f\left(x_{n+\sigma_2}\right), \qquad (1.4.3)$$

where $n \in \mathbb{N}$ and b, c are positive constants, k, ℓ, σ_1 and σ_2 are positive integers, $\{a_n\}, \{q_n\}$ and $\{p_n\}$ are positive real sequences, and $f : \mathbb{R} \to \mathbb{R}$ is continuous function with $uf(u) > 0$ for $u \neq 0$.

Section 4.1 presents necessary introduction and in Section 4.2, we establish sufficient conditions for the oscillation of all solutions of equation (1.4.3). Examples are provided in Section 4.3 to illustrate the results. The results obtained in this chapter generalize some of the results in [6].

Chapter 5 deals with the neutral difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(x_n + c_n x_{n-k}\right)\right)^{\alpha}\right) + p_n x_{n-\ell}^{\beta} = e_n + q_n x_{n-\ell}^{\gamma},\ n \in \mathbb{N} \quad (1.4.4)$$

where $\{a_n\}$ is a positive real sequence, $\{c_n\}$ and $\{e_n\}$ are real sequences, $\{p_n\}$ and $\{q_n\}$ are positive real sequences, k and ℓ are positive integers and α, β and γ are ratio of odd positive integers.

In Section 5.1 we provide necessary introduction and in Section 5.2 we establish condition for the oscillation of all solutions of equation (1.4.4). Examples are provided in Section 5.3 to illustrate the results. The results obtained in this chapter generalize some of the results obtained in [3].

In the final Chapter 6, we consider the neutral type difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(x_n+c_nx_{n-k}\right)\right)\right)+p_nf(x_{n-m})-q_nx_{n-\ell}=0, n\in\mathbb{N}, \tag{1.4.5}$$

where $\{a_n\}$ is a positive sequence, $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are non-negative real sequences, k, ℓ, m are positive integers with $m \geq \ell$, and $f:\mathbb{R}\to\mathbb{R}$ is continuous, non decreasing with $uf(u)>0$ for $u \neq 0$.

Section 6.1 presents necessary introduction. In Section 6.2 we establish conditions for the oscillatory and asymptotic behavior of solutions of equation (1.4.5), and examples are presented in Section 6.3 to illustrate the results. The results obtained here complement to those of Jinfa [37] and Karpuz, Ocalan and Yildiz [40].

Thus, we have established new results and improved and generalized some of the existing results on the oscillatory and asymptotic behavior of neutral type difference equations. Moreover, examples are provided in the text to illustrate the results presented in the thesis.