CHAPTER 6

NEUTRAL DIFFERENCE EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

6.1 Introduction

In the last chapter of the thesis, we study the oscillatory behavior of second order neutral delay difference equations of the form

$$\Delta\left(a_n\left(\Delta\left(x_n + c_n x_{n-k}\right)\right)\right) + p_n f\left(x_{n-m}\right) - q_n x_{n-\ell} = 0, \quad (6.1.1)$$

and

$$\Delta\left(a_n\left(\Delta\left(x_n - c_n x_{n-k}\right)\right)\right) + p_n f\left(x_{n-m}\right) - q_n x_{n-\ell} = 0, \quad (6.1.2)$$

where $n \in \mathbb{N}$, $\{a_n\}$ is a positive sequence, $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, k, ℓ, m are positive integers with $m \geq \ell$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, non decreasing with $uf(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, \ell, m\}$. By a solution of equations (6.1.1), (6.1.2) we mean a real sequence $\{x_n\}$ which is defined for all $n \geq -\theta$, and satisfies equations (6.1.1), (6.1.2) for all $n \in \mathbb{N}$. We restrict our attention only to the nontrivial solution $\{x_n\}$ that is, to the solution $\{x_n\}$ such that $\sup\{|x_n| : n \geq N \in \mathbb{N}\} > 0$ for all $N \in \mathbb{N}$.

Sufficient conditions for oscillation of solutions of first order delay and neutral difference equations with positive and negative coefficients have investigated by many authors [1, 2, 6]. On the other hand very few results are available on the oscillatory and asymptotic behavior of solutions of second order neutral delay difference equations with positive and negative coefficients [12, 21, 39, 40, 51, 58, 92, 96]. Motivated by this observation in this chapter, we establish sufficient conditions for the oscillation of all solutions of equations (6.1.1) and (6.1.2).

In Section 6.2, we establish sufficient conditions for the oscillation of all solutions of equations (6.1.1) and (6.1.2). Oscillation results for equations (6.1.1) and (6.1.2) with forcing terms are also given in this section. Examples are provided in Section 6.3 to illustrate the results.

6.2 Oscillation Results

Here after we always assume without mentioning further that

(C_1) $\dfrac{f(u)}{u} \geq M > 0$ for $u \neq 0$;

(C_2) $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} = \infty$;

(C_3) $q_n \leq q_{n-\ell}$ and $Mp_n - q_{n-m} \geq b > 0$.

Theorem 6.2.1. *Assume that*

$$m \geq \ell \textit{ and } 0 \leq c_n \leq c, \textit{ for } n \in \mathbb{N} \tag{6.2.1}$$

hold. If

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell} q_t \leq 1 \tag{6.2.2}$$

then all solutions of equation (6.1.1) are oscillatory.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of equation (6.1.1). Without loss of generality, we assume that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_0 \in \mathbb{N}$. If we define

$$z_n = x_n + c_n x_{n-k} - \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t. \tag{6.2.3}$$

From equation (6.1.1) and conditions (C_1) and (C_3), it follows

$$\begin{aligned}
\Delta\left(a_n \Delta z_n\right) &= \Delta\left(a_n \Delta\left(x_n + c_n x_{n-k}\right)\right) - q_{n-\ell} x_{n-\ell} + q_{n-m} x_{n-m} \\
&= \left(-p_n f\left(x_{n-m}\right) + q_n x_{n-\ell}\right) - q_{n-\ell} x_{n-\ell} + q_{n-m} x_{n-m} \\
&\leq -\left(q_{n-\ell} - q_n\right) x_{n-\ell} - M p_n x_{n-m} + q_{n-m} x_{n-m} \\
&\leq -\left(M p_n - q_{n-m}\right) x_{n-m} \\
&\leq -b x_{n-m} \leq 0, n \geq n_0 + \theta. \qquad (6.2.4)
\end{aligned}$$

Hence $\{a_n \Delta z_n\}$ is eventually nonincreasing. Then, we have that $a_n \Delta z_n \geq 0$ or $a_n \Delta z_n < 0$ for all $n \geq n_1 \geq n_0 + \theta$.

If $a_n \Delta z_n < 0$ for all $n \geq n_1$, then the inequality

$$a_n \Delta z_n \leq a_{n_1} \Delta z_{n_1} < 0$$

implies that

$$z_n \leq z_{n_1} + a_{n_1} \Delta z_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{a_s},$$

which inview of (C_2) we have $\lim_{n\to\infty} z_n = -\infty$.

We claim that $\{x_n\}$ is bounded from above. If this is not the case, there is an integer, $n_2 \geq n_1$ satisfying

$$z_{n_2} < 0 \text{ and } \max_{n_1 \leq n \leq n_2} x_n = x_{n_2}.$$

Then we obtain

$$\begin{aligned} 0 &> z_{n_2} = x_{n_2} + c_{n_2} x_{n_2-k} - \sum_{s=n_0}^{n_2-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t \\ &\geq \left[1 - \sum_{s=n_0}^{n_2-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t \right] x_{n_2} \\ &\geq \left[1 - \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell} q_t \right] x_{n_2} \geq 0. \end{aligned}$$

This contradiction shows that $\{x_n\}$ must be bounded from above. Then there exists a constant $L > 0$ such that $x_n \leq L$ for all $n \geq n_1$. Accordingly, we have

$$\begin{aligned} z_n &\geq -L \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t \\ &\geq -L \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell} q_t \\ &\geq -L > -\infty \end{aligned}$$

which contradicts the fact that $\lim_{n\to\infty} z_n = -\infty$.We therefore have that $a_n\Delta z_n \geq 0$ for all $n \geq n_1$.Now summation of the inequality (6.2.4) from n_1 to $n-1$ yields

$$\infty > a_{n_1}\Delta z_{n_1} \geq -a_n\Delta z_n + a_{n_1}\Delta z_{n_1}$$
$$\geq b\sum_{s=n_1}^{n-1} x_{s-m} \geq b\sum_{n=n_1}^{\infty} x_{n-m} \qquad (6.2.5)$$

and therefore $\{x_n\}$ is summable. Thus from the condition (6.2.1), we have

$$y_n = x_n + c_n x_{n-k} \qquad (6.2.6)$$

also summable. On the other hand from (6.2.3), we obtain

$$\Delta y_n = \Delta z_n + \frac{1}{a_n}\sum_{t=n-m}^{n-\ell-1} q_t x_t \geq 0, n \geq n_1$$

so that $\{y_n\}$ is a nondecreasing sequence. But, then $y_n \geq y_{n_1}, n \geq n_1$ implies that $\{y_n\}$ is not summable, which is a contradiction with (6.2.6). Hence every solution of equation (6.1.1) is oscillatory. This completes the proof.

Next we obtain sufficient conditions for the oscillatory behavior of solutions of equation (6.1.2). □

Theorem 6.2.2. *Assume that* $m \geq \ell$ *and*

$$0 \leq c_n \leq c < 1 \textit{ for } n \in \mathbb{N} \tag{6.2.7}$$

hold. If

$$c + \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-m}^{n-\ell} q_s \leq 1 \tag{6.2.8}$$

then every solution of equation (6.1.2) either oscillates or satisfies

$\lim_{n\to\infty} x_n = 0.$

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (6.1.2). Without loss of generality, we assume that $x_n > 0$ for $n \geq n_0 \in \mathbb{N}$. If we define

$$\omega_n = x_n - c_n x_{n-k} - \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t \tag{6.2.9}$$

then as in the proof Theorem 6.2.1 from equation (6.1.2),we obtain

$$\Delta\left(a_n \Delta\omega_n\right) \leq -b x_{n-m} \leq 0, n \geq n_0 + \theta \tag{6.2.10}$$

and conclude that $\{a_n \Delta\omega_n\}$ is eventually nonincreasing.Therefore

$$a_n \Delta\omega_n < 0 \text{ or } a_n \Delta\omega_n \geq 0 \text{ for all } n \geq n_1 \geq n_0 + \theta.$$

Let us suppose that $a_n \Delta\omega_n < 0$ for $n \geq n_1$. Then as in the proof of Theorem 6.2.1, we have $\lim_{n\to\infty} \omega_n = -\infty$. We claim that $\{x_n\}$

is bounded from above. If it is not the case, there exists an integer $n_2 \geq n_1$ such that

$$\omega_{n_2} < 0 \text{ and } \max_{n_1 \leq n \leq n_2} x_n = x_{n_2}$$

and

$$\begin{aligned} 0 &> \omega_{n_2} \\ &= x_{n_2} - c_{n_2} x_{n_2 - k} - \sum_{s=n_0}^{n_2-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t \\ &\geq \left[1 - c - \sum_{s=n_0}^{n_2-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell} q_t \right] x_{n_2} \\ &\geq \left[1 - c - \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell} q_t \right] x_{n_2} \geq 0. \end{aligned}$$

This is a contradiction and it shows that $\{x_n\}$ must be bounded from above. Accordingly, there exists a constant $L > 0$ such that $x_n \leq L$ for all $n \geq n_1$. It follows from (6.2.8) that

$$\begin{aligned} \omega_n &\geq -L \left\{ 1 + c_n + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell} q_t \right\} \\ &\geq -L \left\{ 1 + c + \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell} q_t \right\} \\ &\geq -L > -\infty, n \geq n_1 \end{aligned}$$

which contradicts the fact that $\lim_{n \to \infty} \omega_n = -\infty$.Hence $\{a_n \Delta \omega_n\}$

be eventually positive sequence. Summation of the inequality (6.2.10) from n_1 to $n-1$ yields

$$\infty > a_{n_1}\Delta\omega_{n_1} \geq -a_n\Delta\omega_n + a_{n_1}\omega_{n_1} \geq b\sum_{s=n_1}^{\infty} x_{s-m}$$

and therefore $\{x_n\}$ is summable and thus $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Next we consider equations (6.1.1) and (6.1.2) with forcing terms of the form

$$\Delta\left(a_n\Delta\left(x_n + c_n x_{n-k}\right)\right) + p_n f\left(x_{n-m}\right) - q_n x_{n-\ell} = e_n, \quad (6.2.11)$$

and

$$\Delta\left(a_n\Delta\left(x_n - c_n x_{n-k}\right)\right) + p_n f\left(x_{n-m}\right) - q_n x_{n-\ell} = e_n, \quad (6.2.12)$$

where $n \in \mathbb{N}$, $\{e_n\}$ is a real sequence.

Theorem 6.2.3. *Assume that $m \geq \ell$ and (6.2.1) and (6.2.2) hold. If there exists a sequence $\{E_n\}$ such that*

$$\lim_{n\to\infty} E_n \textit{ is finite and } \Delta\left(a_n\Delta E_n\right) = e_n \textit{ for all } n \in \mathbb{N} \quad (6.2.13)$$

then every solution of equation (6.2.11) is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of equation (6.2.11) such that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_0 \in \mathbb{N}$.

If we denote

$$B_n = x_n + c_n x_{n-k} - \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t - E_n + A + 1 \quad (6.2.14)$$

where $\lim_{n\to\infty} E_n = A$, then from equation (6.2.11) we obtain

$$\Delta(a_n \Delta B_n) \leq -b x_{n-m} \leq 0, n \geq n_0 + \theta. \quad (6.2.15)$$

By (6.2.11), there exists an integer $n_1 \geq n_0 + \theta$ such that $\Delta B_n \geq 0$ or $\Delta B_n < 0$ for $n \geq n_1$. By hypothesis there exists sufficiently large integer n_2 such that $-E_n + A + 1 > 0$ for all $n \geq n_2$. Let $N = \max\{n_1, n_2\}$.

Let $\Delta B_n < 0$ for $n \geq N$. Then from (C_2) and (6.2.15), we have $\lim_{n\to\infty} B_n = -\infty$. First we show that $\{x_n\}$ is bounded. If this is not the case, there exists an integer $N_1 > N$ satisfying $B_{N_1} < 0$ and $\max_{N \leq n \leq N_1} x_n = x_{N_1}$. Then we have

$$0 > B_{N_1} = x_{N_1} + c_{N_1} x_{N_1-k} - \sum_{s=n_0}^{N_1-1} \frac{1}{a_s} \sum_{t=s-m}^{s-\ell-1} q_t x_t - E_{N_1} + A + 1$$
$$\geq \left[1 - \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-m}^{n-\ell-1} q_t\right] x_{N_1} \geq 0.$$

This contradiction shows that $\{x_n\}$ must be bounded . Then, there exists a constant $L > 0$ such that $x_n \leq L$ for all $n \geq N$. It follows from (6.2.1) and (6.2.14) that $\{B_n\}$ is bounded,which contradicts the fact that $\lim_{n\to\infty} B_n = -\infty$.

Let $\Delta B_n \geq 0$ for $n \geq N$. Summing (6.2.15), we have

$$\infty > a_N \Delta B_N \geq a_N \Delta B_N - a_n \Delta B_n \geq b \sum_{n=N}^{\infty} x_{n-m}$$

which implies that $\{x_n\}$ is summable and thus $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Theorem 6.2.4. *Assume that $m \geq \ell$ and (6.2.6) and (6.2.8) hold. If (6.2.13) holds,then every solution of equation (6.2.12) is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.*

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of equation (6.2.12) with $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_0 \in \mathbb{N}$. Let us denote with

$$W_n = \omega_n - E_n + A + 1 \tag{6.2.16}$$

where ω_n is defined by (6.2.9). Then we have

$$\Delta\left(a_n \Delta W_n\right) \leq -b x_{n-m} \leq 0, n \geq n_0 + \theta. \tag{6.2.17}$$

Therefore, we have the following two cases : $\Delta W_n < 0$ for $n \geq N \geq n_0 + \theta$ which implies that $\lim_{n\to\infty} W_n = -\infty$. It is not hard to prove that $\Delta W_n < 0$ is not possible by following the arguments as in the proof of Theorem 6.2.3.

Therefore, $\Delta W_n \geq 0$ for $n \geq N$. From (6.2.17), we obtain $\{x_n\}$ is summable and thus $\lim_{n\to\infty} x_n = 0$. The proof is now complete. □

6.3 Examples

In this section we present some examples to illustrate the results obtained in the previous section.

Example 6.3.1. *Consider the difference equation*

$$\Delta\left(n\left(\Delta\left(x_n + 2x_{n-1}\right)\right)\right) + \left(2n + 1 + \frac{1}{4^{n+5}}\right) x_{n-4}\left(1 + x_{n-4}^2\right) - \frac{2}{4^{n+5}} x_{n-2} = 0, n \geq 1. \tag{6.3.1}$$

Here $a_n = n, c_n = 2, m = 4, \ell = 2, p_n = 2n + 1 + \frac{1}{4^{n+5}}, q_n = \frac{2}{4^{n+5}}$ *and* $f(u) = u\left(1 + u^2\right)$. *With* $M = 1$, *all assumptions* (C_1) *and* (C_3) *hold.*

Further, we see that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, *and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{s=n-4}^{n-2} 2\left(\frac{1}{4^{s+5}}\right) = 2\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{4^{n+1}} + \frac{1}{4^{n+2}} + \frac{1}{4^{n+3}}\right)$$

$$< \frac{42}{64}\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{14}{64} < 1.$$

Hence by Theorem 6.2.1, all solutions of equation (6.3.1) are oscillatory. Infact $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation (6.3.1).*

Example 6.3.2. *Consider the difference equation*

$$\Delta\left(n\Delta\left(x_n - \frac{1}{2}x_{n-2}\right)\right) + \left(n + \frac{1}{2} + \frac{1}{3^{n+6}}\right)x_{n-3}\left(1 + x_{n-3}^2\right)$$

$$- \frac{2}{3^{n+6}}x_{n-1} = 0, n \geq 1. \qquad (6.3.2)$$

Here $a_n = n, c_n = \frac{1}{2}, m = 3, \ell = 1, p_n = n + \frac{1}{2} + \frac{1}{3^{n+6}}, q_n = \frac{2}{3^{n+6}}$ *and* $f(u) = u\left(1 + u^2\right)$. *With* $M = 1$, *it is easy to check that conditions* (C_1) *and* (C_3) *hold. Further, we see that*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and

$$c + \sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{s=n-m}^{n-\ell} q_s = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{s=n-3}^{n-1} \frac{2}{3^{s+6}}$$

$$< \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{3^n}\left(\frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5}\right)$$
$$< \frac{1}{2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} < 1.$$

Hence by Theorem 6.2.2, all solutions of equation (6.3.2) are oscillatory. Infact $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation (6.3.2).*

Example 6.3.3. *Consider the difference equation*

$$\Delta^2 (x_n + 2x_{n-2}) + \frac{n}{n+1} x_{n-2} (1 + |x_{n-2}|) - \frac{1}{2^{n+3}} x_{n-1} = \frac{2}{(n+1)(n+2)(n+3)}, \tag{6.3.3}$$

where $n \geq 1$. *For this equation, we see that* $a_n = 1, c_n = 2, m = 2, \ell = 1, p_n = \frac{n}{n+1}, q_n = \frac{1}{2^{n+3}}, e_n = \frac{2}{(n+1)(n+2)(n+3)}$ *and* $f(u) = u(1 + |u|)$. *We may set* $M = 1$, *and we see that* $(C_1) - (C_3)$ *hold. Further,*

$$E_n = \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$

and it is not hard to see that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{s=n-2}^{n-1} q_s = \sum_{n=1}^{\infty} \sum_{s=n-2}^{n-1} \frac{1}{2^{s+3}} = \frac{3}{4} < 1.$$

Therefore,all conditions of Theorem 6.2.3 are satisfied, and hence every solution of equation (6.3.3) is either oscillatory or tends to zero at infinity.

Example 6.3.4. *Consider the difference equation*

$$\Delta\left(n\Delta\left(x_n-\frac{1}{4}x_{n-2}\right)\right)+\left(\frac{n^2}{n^2+1}\right)x_{n-2}\left(1+|x_{n-2}|\right)$$
$$-\frac{1}{4^{n+2}}x_{n-1}=\frac{n-1}{2^{n+2}},\ n\geq 1. \quad (6.3.4)$$

For this equation $a_n=n, c_n=\frac{1}{4}, m=2, \ell=1, p_n=\dfrac{n^2}{n^2+1}, q_n=\dfrac{1}{4^{n+2}}, e_n=\dfrac{n-1}{2^{n+2}}$ *and* $f(u)=u(1+|u|)$. *With* $M=1$, *we see that conditions* $(C_1)-(C_3)$ *hold. Further,* $E_n=\dfrac{1}{2^n}\to 0$ *as* $n\to\infty$ *and*

$$\begin{aligned} c+\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{s=n-2}^{n-1}q_s &= \frac{1}{4}+\sum_{n=1}^{\infty}\frac{1}{n}\sum_{s=n-2}^{n-1}\frac{1}{4^{s+2}} \\ &= \frac{1}{4}+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{4^n}+\frac{1}{4^{n+1}}\right) \\ &< \frac{2}{3}<1. \end{aligned}$$

Therefore all conditions of Theorem 6.2.4 are satisfied, and hence every solution of equation (6.3.4) is either oscillatory or tends to zero at infinity

We conclude this chapter with the following remark,

Remark 6.3.1. *Note that the results in [21, 39, 40, 96] cannot be applicable to equations (6.3.1) to (6.3.4).*