CHAPTER 5

NEUTRAL DIFFERENCE EQUATION WITH MIXED NONLINEARITIES

5.1 Introduction

In this chapter we consider a second order neutral difference equation with mixed nonlinearities of the type

$$\Delta\left(a_n\left(\Delta\left(x_n+c_n x_{n-k}\right)\right)^{\alpha}\right)+p_n x_{n-\ell}^{\beta}=e_n+q_n x_{n-\ell}^{\gamma}, n\in\mathbb{N} \quad (5.1.1)$$

where $\{c_n\}$ is a nonnegative real sequence, $\{e_n\}$ is a real sequence, $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are positive real sequences, k and ℓ are positive integers and α, β and γ are ratio of odd positive integers.

Let $\theta = \max\{k, \ell\}$. By a solution of equation (5.1.1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\theta$ and satisfies equation (5.1.1) for all $n \in \mathbb{N}$. It is easy to see that under the initial conditions

$$x_n = \phi_n, n = -\theta, -\theta+1, \cdots, 1 \tag{5.1.2}$$

equation (5.1.1) has a unique solution satisfying (5.1.2).

The problem of determining oscillation and nonoscillation of solutions of difference equation with mixed nonlinearities received less attention, see for example [2, 6]. However, to study the oscillatory behavior of forced difference equation (5.1.1) with mixed nonlinearities, the known techniques either do not work or impose severe restrictions on the forcing term $\{e_n\}$. Thus in this chapter we shall provide some easily verifiable sufficient conditions for the oscillation of all solutions of equation (5.1.1).

In Section 5.2 we present oscillation theorems for the equation (5.1.1) and in Section 5.3 we provide some examples to illustrate the results. The results established in this chapter generalize some of the results obtained in [3, 7, 38, 112].

5.2 Oscillation Results

In order to discuss our results we shall need the following lemma from [31].

Lemma 5.2.1. *If X and Y are nonnegative, then*

(I) $X^{\lambda} - \lambda XY^{\lambda-1} + (\lambda - 1) Y^{\lambda} \geq 0$ *for all* $\lambda \geq 1$;

(II) $X^{\mu} - \mu XY^{\mu-1} - (1 - \mu) Y^{\mu} \leq 0$ *for all* $0 < \mu < 1$.

In the above inequalities, equality holds if and only if $X = Y$.

Theorem 5.2.2. *Assume that* $\gamma = 1$ *and* $\beta > 1$. *If*

$$\liminf_{n\to\infty} \sum_{j=N}^{n-1} \left[\frac{c}{a_j} + \frac{1}{a_j} \sum_{s=N}^{j-1} (e_s + A_s) \right]^{\frac{1}{\alpha}} = -\infty \tag{5.2.1}$$

and

$$\limsup_{n\to\infty} \sum_{j=N}^{n-1} \left[\frac{c}{a_j} + \frac{1}{a_j} \sum_{s=N}^{j-1} (e_s - A_s) \right]^{\frac{1}{\alpha}} = \infty$$

hold for all $n \geq N \in \mathbb{N}$ *and all* $c \in \mathbb{R}$, *where*

$$A_n = (\beta - 1)\, \beta^{\frac{\beta}{1-\beta}} q_n^{\frac{\beta}{\beta-1}} p_n^{\frac{1}{1-\beta}}, \tag{5.2.2}$$

then all solutions of equation (5.1.1) are oscillatory.

oof. Let $\{x_n\}$ be an eventually positive solution of equation (5.1.1). Set $z_n = x_n + c_n x_{n-k}$ then $z_n > 0$ for all $n \geq n_0 \in \mathbb{N}$, and from equation (5.1.1), we have

$$\Delta\left(a_n\left(\Delta z_n\right)^{\alpha}\right) = e_n + \left[q_n x_{n-\ell} - p_n x_{n-\ell}^{\beta}\right] \tag{5.2.3}$$

Let $X = p_n^{\frac{1}{\beta}} x_{n-l}$ and $Y = \left(\frac{1}{\beta} q_n p_n^{-\frac{1}{\beta}}\right)^{\frac{1}{\beta-1}}$ apply Lemma 5.2.1(I) in equation (5.2.3) to obtain

$$\Delta\left(a_n\left(\Delta z_n\right)^{\alpha}\right) \leq e_n + (\beta - 1)\,\beta^{\frac{\beta}{1-\beta}} q_n^{\frac{\beta}{\beta-1}} p_n^{\frac{1}{1-\beta}}, n \geq n_0 \in \mathbb{N}. \tag{5.2.4}$$

Summing both sides of equation (5.1.1) from n_0 to $n-1$, we obtain

$$a_n\left(\Delta z_n\right)^{\alpha} \leq a_{n_1}\left(\Delta z_{n_1}\right)^{\alpha} + \sum_{s=n_0}^{n-1}\left[e_s + A_s\right]$$

or

$$\Delta z_n \leq \left(\frac{a_{n_1}\left(\Delta z_{n_1}\right)^{\alpha}}{a_n} + \frac{1}{a_n}\sum_{s=n_0}^{n-1}\left[e_s + A_s\right]\right)^{\frac{1}{\alpha}}.$$

Again summing we obtain

$$z_n \leq z_{n_0} + \sum_{j=n_0}^{n-1}\left(\frac{a_{n_0}\left(\Delta z_{n_0}\right)^{\alpha}}{a_j} + \frac{1}{a_j}\sum_{j=n_0}^{s-1}\left[e_j + A_j\right]\right)^{\frac{1}{\alpha}}$$

a contradiction with (5.2.1). The proof when $\{x_n\}$ is eventually negative is similar and the proof is complete. □

Theorem 5.2.3. *Assume that* $\beta = 1$ *and* $0 < \gamma < 1$. *If*

$$\liminf_{n\to\infty} \sum_{j=N}^{n-1} \left[\frac{c}{a_j} + \frac{1}{a_j}\sum_{s=N}^{j-1}(e_s + B_s)\right]^{\frac{1}{\alpha}} = -\infty$$
$$\text{and } \limsup_{n\to\infty} \sum_{j=N}^{n-1} \left[\frac{c}{a_j} + \frac{1}{a_j}\sum_{s=N}^{j-1}(e_s - B_s)\right]^{\frac{1}{\alpha}} = \infty \tag{5.2.5}$$

hold for all $n \geq N \in \mathbb{N}$ *and all* $c \in \mathbb{R}$, *where*

$$B_n = (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}} p_n^{\frac{\gamma}{\gamma-1}} q_n^{\frac{1}{1-\gamma}}, \tag{5.2.6}$$

then all solutions of equation (5.1.1) are oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (5.1.1) and let $z_n = x_n + c_n x_{n-k}$ then $z_n > 0$ for all $n \geq n_0 \in \mathbb{N}$, and from equation (5.1.1), we have

$$\Delta\left(a_n (\Delta z_n)^{\alpha}\right) = e_n + \left[q_n x_{n-\ell}^{\gamma} - p_n x_{n-\ell}\right]. \tag{5.2.7}$$

Set $X = q_n^{\frac{1}{\gamma}} x_{n-\ell}$ and $Y = \left(\frac{1}{\gamma} p_n q_n^{-\frac{1}{\gamma}}\right)^{\frac{1}{\gamma-1}}$, $n \geq n_0 \in \mathbb{N}$ and apply Lemma 5.2.1(II) to equation (5.2.7) to obtain

$$\Delta\left(a_n (\Delta z_n)^{\alpha}\right) \leq e_n + (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}} p_n^{\frac{\gamma}{\gamma-1}} q_n^{\frac{1}{1-\gamma}}, n \geq n_0 \in \mathbb{N}.$$

The rest of the proof is similar to Theorem 5.2.2 and hence is omitted. □

From Theorems 5.2.2 and 5.2.3 we obtain the following corollaries.

Corollary 5.2.4. *Assume that* $\gamma = 1$ *and* $\beta > 1$. *Further assume*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} e_j = -\infty$$

and

$$\limsup_{n\to\infty} \sum_{j=n_0}^{n} e_j = \infty$$

and

$$\sum_{n=n_0}^{\infty} q_n^{\frac{\beta}{\beta-1}} p_n^{\frac{1}{1-\beta}} < \infty$$

hold. If

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} \left[\frac{1}{a_j} \sum_{s=n_0}^{j-1} e_s\right]^{\frac{1}{\alpha}} = -\infty \tag{5.2.8}$$

and

$$\limsup_{n\to\infty} \sum_{j=n_0}^{n} \left[\frac{1}{a_j} \sum_{s=n_0}^{j-1} e_s\right]^{\frac{1}{\alpha}} = +\infty,$$

then every solution of equation (5.1.1) is oscillatory.

Corollary 5.2.5. *Assume that* $\beta = 1$ *and* $0 < \gamma < 1$. *Further assume*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} e_j = -\infty$$

and

$$\lim_{n\to\infty}\sup\sum_{j=n_0}^{n} e_j = \infty$$

and

$$\sum_{n=n_0}^{\infty} p_n^{\frac{\gamma}{\gamma-1}} q_n^{\frac{1}{1-\gamma}} < \infty$$

hold. If (5.2.8) holds then every solution of equation (5.1.1) is oscillatory.

Theorem 5.2.6. *Assume that* $0 < \gamma < 1$ *and* $\beta > 1$ *hold. If there exists a positive sequence* $\{b_n\}$ *such that*

$$\lim_{n\to\infty}\inf\sum_{j=n_0}^{n}\left[\frac{c}{a_j}+\frac{1}{a_j}\sum_{s=n_0}^{j-1}(e_s+D_s)\right]^{\frac{1}{\alpha}} = -\infty \qquad (5.2.9)$$

and

$$\lim_{n\to\infty}\sup\sum_{j=n_0}^{n}\left[\frac{c}{a_j}+\frac{1}{a_j}\sum_{s=n_0}^{j-1}(e_s-D_s)\right]^{\frac{1}{\alpha}} = \infty$$

holds for all $n \ge n_0 \in \mathbb{N}$ *and all* $c \in R$, *we have*

$$D_n = (\beta-1)\,\beta^{\frac{\beta}{1-\beta}} b_n^{\frac{\beta}{\beta-1}} p_n^{\frac{1}{1-\beta}} + (1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}} b_n^{\frac{\gamma}{\gamma-1}} q_n^{\frac{1}{1-\gamma}} \qquad (5.2.10)$$

then every solution of equation (5.1.1) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (5.1.1). Set $z_n = x_n + c_n x_{n-k}$ then $z_n > 0$ for all $n \ge n_0 \in \mathbb{N}$, and from equation (5.1.1), we have

$$\Delta\left(a_n\left(\Delta z_n\right)^{\alpha}\right)=e_n+\left[b_n x_{n-\ell}-p_n x_{n-\ell}^{\beta}\right]+\left[q_n x_{n-\ell}^{\gamma}-b_n x_{n-\ell}\right]. \tag{5.2.11}$$

The rest of the proof is similar to the proofs of Theorems 5.2.2 and 5.2.3 and hence is omitted. □

From the above proofs, one can easily establish criteria for the boundedness of all solutions of the equation (5.1.1).

Theorem 5.2.7. *Assume that* $\gamma=1$ *and* $\beta>1$ *hold. If*

$$\sum_{n=n_0}^{\infty}\left(\frac{c}{a_n}+\frac{1}{a_n}\sum_{s=n_0}^{n-1}\left(|e_s|+|A_s|\right)\right)^{\frac{1}{\alpha}}<\infty,\, n_0\in\mathbb{N}, \tag{5.2.12}$$

with A_n *defined in (5.2.2), then all nonoscillatory solutions of equation (5.1.1) are bounded.*

Proof. As in the proof of Theorem 5.2.2, we obtain

$$|x_n|\le|z_n|\le|z_{n_0}|+\sum_{j=n_0}^{n-1}\left(\frac{a_{n_0}\left(\Delta z_{n_0}\right)^{\alpha}}{a_j}+\frac{1}{a_j}\sum_{s=n_0}^{j-1}\left(|e_s|+|A_s|\right)\right)^{\frac{1}{\alpha}}.$$

The conclusion now follows from the condition (5.2.12). □

Theorem 5.2.8. *Assume that* $\beta=1$ *and* $0<\gamma<1$ *hold. If*

$$\sum_{n=n_0}^{\infty}\left(\frac{c}{a_n}+\frac{1}{a_n}\sum_{s=n_0}^{n-1}\left(|e_s|+|B_s|\right)\right)^{\frac{1}{\alpha}}<\infty,\, n_0\in\mathbb{N},$$

where B_n is as defined in (5.2.6), then all nonoscillatory solutions of equation (5.1.1) are bounded.

Theorem 5.2.9. *Assume that $\beta > 1$ and $0 < \gamma < 1$ hold. If there exists a positive sequence $\{b_n\}$ so that for all $n \geq n_0 \in \mathbb{N}$, and $c \in \mathbb{R}$*

$$\sum_{n=n_0}^{\infty}\left(\frac{c}{a_n}+\frac{1}{a_n}\sum_{s=n_0}^{n-1}\left(|e_s|+|D_s|\right)\right)^{\frac{1}{\alpha}} < \infty, n_0 \in \mathbb{N},$$

where D_n is as defined in (5.2.10), then all nonoscillatory solutions of equation (5.1.1) are bounded.

Next we consider the following difference equations of the form

$$\Delta^2 z_{n-1} + q_n x^{\gamma}_{n-\ell} + b_n x_{n+1} = p_n x^{\beta}_{n+1-\ell} + e_n, n \in \mathbb{N} \quad (5.2.13)$$

where $z_n = x_n + c_n x_{n-\ell}$, $\{c_n\}$ is a nonnegative real sequence, $\{q_n\}$ and $\{p_n\}$ are positive real sequences, $\{e_n\}$, $\{b_n\}$ are sequences of real numbers and β, γ and ℓ are as in equation (5.1.1).

Theorem 5.2.10. *Let $\{H(m,n) : m, n \in \mathbb{N}, m \geq n \geq 0\}$ be a double sequence satisfying*

$$H(m,m) = 0 \textit{ for } m \geq 0, H(m,n) > 0 \textit{ for } m > n \geq 0 \quad (5.2.14)$$

$$h(m,n) = H(m,n) - H(m,n+1) > 0 \, for \, m > n \geq 0 \tag{5.2.15}$$

$$-P(m,n) = h(m,n) + b_n H(m,n) < 0 \, for \, m > n \geq 0. \tag{5.2.16}$$

If $0 < \gamma < 1, \beta > 1$ *and for* $N \geq n_0 \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup \frac{1}{H(m,N)} \sum_{n=N}^{m-1} [H(m,n) e_n - Q(m,n)] = +\infty \tag{5.2.17}$$

$$\lim_{n\to\infty} \inf \frac{1}{H(m,N)} \sum_{n=N}^{m-1} [H(m,n) e_n - Q(m,n)] = -\infty \tag{5.2.18}$$

where

$$\begin{aligned} Q(m,n) &= (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} (h(m,n) c_n)^{\frac{\gamma}{\gamma-1}} (H(m,n) q_n)^{\frac{1}{1-\gamma}} \\ &+ (\beta-1)\beta^{\frac{\beta}{1-\beta}} (h(m,n) c_{n+1})^{\frac{\beta}{\beta-1}} (H(m,n) p_n)^{\frac{\beta}{1-\beta}} \end{aligned}$$

then every solution of equation (5.2.13) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of (5.2.13) and so $\{z_n\}$ be eventually positive. Multiply (5.2.13) by $H(m,n)$ for $m > n \geq N \in \mathbb{N}$. Summing from N to $m-1$ and using (5.2.14), we obtain

$$\frac{1}{H(m,N)}\sum_{n=N}^{m-1} H(m,n)\, e_n$$
$$\le -\Delta z_{N-1} + \frac{1}{H(m,N)}\sum_{n=N}^{m-1}\left(H(m,n)\, q_n x_{n-\ell}^{\gamma} - h(m,n)\, c_n x_{n-\ell}\right)$$
$$+\frac{1}{H(m,N)}\sum_{n=N}^{m-1}\left(h(m,n)\, c_{n+1}x_{n+1-\ell} - H(m,n)\, p_n x_{n+1-\ell}^{\beta}\right).$$

Now using Lemma 5.2.1, we obtain

$$\frac{1}{H(m,N)}\sum_{n=N}^{m-1} H(m,n)\, e_n$$
$$\le -\Delta z_{N-1} + \frac{1}{H(m,N)}\sum_{n=N}^{m-1}(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\left(h(m,n)\, c_n\right)^{\frac{\gamma}{\gamma-1}}$$
$$\times\left(H(m,n)\, q_n\right)^{\frac{1}{1-\gamma}}$$
$$+\frac{1}{H(m,N)}\sum_{n=N}^{m-1}(\beta-1)\,\beta^{\frac{\beta}{1-\beta}}\left(h(m,n)\, c_{n+1}\right)^{\frac{\beta}{\beta-1}}$$
$$\times\left(H(m,n)\, p_n\right)^{\frac{1}{1-\beta}}$$

or

$$\frac{1}{H(m,N)}\sum_{n=N}^{m-1}\left[H(m,n)\, e_n - Q(m,n)\right] \le -\Delta z_{N-1}. \qquad (5.2.19)$$

Taking $\limsup$ as $m \to \infty$ in the inequality (5.2.17) we obtain a contradiction to condition (5.2.17). If $\{x_n\}$ is eventually negative, then reasoning as above leads to a contradiction with the condition (5.2.18). This completes the proof. □

Similarly, we present the following result for the equation

$$\Delta^2 z_{n-1} + b_n x_{n+1} + q_n x_{n-\ell}^{\gamma} = e_n + p_n x_{n+1-\ell}, n \in \mathbb{N}, \qquad (5.2.20)$$

where $z_n, c_n, b_n, q_n, p_n, e_n, \gamma_n$ and ℓ are as in equation (5.2.13).

Theorem 5.2.11. *Let the sequence* $\{H(m,n)\}$ *be as in Theorem 5.2.10 such that conditions (5.2.14) - (5.2.16) hold and*

$$h(m,n)\, c_{n+1} - p_n H(m,n) \leq 0 \, \textit{for} \; m > n \geq N. \qquad (5.2.21)$$

If $0 < \gamma < 1$ *and conditions (5.2.17) and (5.2.18) hold with*

$$Q(m,n) = (1-\gamma)\, \gamma^{\frac{\gamma}{1-\gamma}} (h(m,n)\, c_n)^{\frac{\mu}{\mu-1}} (H(m,n))^{\frac{\mu}{1-\mu}} q_n^{\frac{1}{1-\mu}} \qquad (5.2.22)$$

then every solution of equation (5.2.20) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (5.2.20). As in the proof of Theorem 5.2.10, we have

$$\sum_{n=N}^{m-1} H(m,n)\, e_n \;\leq\; -H(m,N)\, \Delta z_{N-1}$$

$$+\sum_{n=N}^{m-1} (h(m,n) + b_n H(m,n))\, x_{n+1}$$
$$+\sum_{n=N}^{m-1} [h(m,n)\, c_{n+1} - p_n H(m,n)]\, x_{n+1-\ell}$$
$$+\sum_{n=N}^{m-1} \left[H(m,n)\, q_n x_{n-\ell}^{\gamma} - h(m,n)\, c_n x_{n-\ell}\right]$$

$$\frac{1}{H(m,N)} \sum_{n=N}^{m-1} H(m,n)\, e_n \le -\Delta z_{N-1}$$
$$+\frac{1}{H(m,N)} \sum_{n=N}^{m-1} \left(H(m,n)\, q_n x_{n-\ell}^{\gamma} - h(m,n)\, c_n x_{n-\ell}\right)$$

where we have used (5.2.16) and (5.2.21). Set

$$X = [H(m,n)\, q_n]^{\frac{1}{\mu}}\, x_{n-\ell}; Y = \left(\frac{1}{\gamma} h(m,n) \left(H(m,n)\, q_n\right)^{-\frac{1}{\gamma}}\right)^{\frac{1}{1-\gamma}}$$

and the rest of the proof is similar to that of Theorem 5.2.3 and hence omitted. □

Also we discuss the oscillatory behavior of the following difference equation

$$\Delta^2 z_{n-1} + b_n x_{n+1} = e_n + p_n x_{n+1-\ell}^{\beta}, n \in \mathbb{N}, \qquad (5.2.23)$$

where $z_n, b_n, e_n, p_n, \beta$ and ℓ are as in equation (5.2.13).

Theorem 5.2.12. *Let the sequence* $\{H(m,n)\}$ *be as in Theorem 5.2.10 such that conditions (5.2.14) - (5.2.16) hold. If* $\beta > 1$ *and (5.2.17) and (5.2.18) hold with*

$$Q(m,n) = (\beta - 1)\beta^{\frac{\beta}{1-\beta}} (h(m,n)c_{n+1})^{\frac{\beta}{\beta-1}} (H(m,n)p_n)^{\frac{1}{1-\beta}},$$

then every solution of equation (5.2.23) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (5.2.23). As in the proof of Theorem 5.2.10, we have

$$\begin{aligned}
&\sum_{n=N}^{m-1} H(m,n)e_n \\
&= -H(m,N)\Delta z_{N-1} + \sum_{n=N}^{m-1} h(m,n)\Delta z_n \\
&\quad + \sum_{n=N}^{m-1} H(m,n)b_n x_{n+1} - \sum_{n=N}^{m-1} H(m,n)p_n x^{\beta}_{n+1-\ell} \\
&\le -H(m,N)\Delta z_{N-1} + \sum_{n=N}^{m-1} (h(m,n) + H(m,n)b_n)x_{n+1} \\
&\quad + \sum_{n=N}^{m-1} \left[h(m,n)c_{n+1}x_{n+1-\ell} - H(m,n)p_n x^{\beta}_{n+1-\ell}\right]
\end{aligned}$$

$$\frac{1}{H(m,N)} \sum_{n=N}^{m-1} H(m,n)e_n \le -\Delta z_{N-1}$$

$$+\frac{1}{H(m,N)}\sum_{n=N}^{m-1}\left(h(m,n)\,c_{n+1}x_{n+1-\ell}-H(m,n)\,p_n x_{n+1-\ell}^{\beta}\right)$$

where we have used (5.2.16). The rest of the proof is similar to that of Theorem 5.2.2 and hence omitted. □

Next we consider the equation

$$\Delta^2 z_{n-1}+p_n x_n^{\gamma}+q_n x_{n-\ell}^{\beta}=e_n, n\in\mathbb{N} \tag{5.2.24}$$

where $z_n = x_n + cx_{n-\ell}$, c is a positive constant, $\{p_n\}, \{q_n\}$ and $\{e_n\}$ are as in equation (5.2.13) , and $\beta < 1$ and $\gamma < 1$.

Theorem 5.2.13. *If*

$$\limsup_{n\to\infty}\left[e_n-(1-\gamma)\left(\frac{\gamma}{2}\right)^{\frac{\gamma}{1-\gamma}}p_n^{\frac{1}{1-\gamma}}\right.$$
$$\left.-(1-\beta)\left(\frac{\beta}{2c}\right)^{\frac{\beta}{1-\beta}}q_n^{\frac{1}{1-\beta}}\right]=+\infty \tag{5.2.25}$$

and

$$\liminf_{n\to\infty}\left[e_n-(1-\gamma)\left(\frac{\gamma}{2}\right)^{\frac{\gamma}{1-\gamma}}p_n^{\frac{1}{1-\gamma}}\right.$$
$$\left.-(1-\beta)\left(\frac{\beta}{2c}\right)^{\frac{\beta}{1-\beta}}q_n^{\frac{1}{1-\beta}}\right]=-\infty \tag{5.2.26}$$

then all bounded solutions of equation (5.2.24) are oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive and bounded solution of equation (5.2.24). Then $\{z_n\}$ is eventually positive and bounded. Now

$$e_n = z_{n+1} + z_{n-1} + (p_n x_n^\gamma - 2x_n) + (q_n x_{n-\ell}^\beta - 2cx_{n-\ell}).$$

As in the proof of Theorem 5.2.3, we have

$$e_n - (1-\gamma)\left(\frac{\gamma}{2}\right)^{\frac{\gamma}{1-\gamma}} p_n^{\frac{1}{1-\gamma}} - (1-\beta)\left(\frac{\beta}{2c}\right)^{\frac{\beta}{1-\beta}} q_n^{\frac{1}{1-\beta}}$$
$$\leq z_{n+1} + z_{n-1} < \infty.$$

Taking $\limsup$ as $n \to \infty$, we obtain a contradiction to (5.2.25). This completes the proof. □

Finally we consider the equation

$$\Delta^2 z_{n-1} - p_n x_{n+1} - q_n x_{n+1-\ell} = e_n, n \in \mathbb{N} \tag{5.2.27}$$

where z_n, p_n, q_n and e_n are as in equation (5.2.13).

Theorem 5.2.14. *Let the sequence $\{H(m,n)\}$ be as in Theorem 5.2.10 such that conditions (5.2.14) and (5.2.15) hold, and*

$$p_n H(m,n) - h(m,n) \geq 0 \tag{5.2.28}$$

and

$$q_n H(m,n) - c_{n+1} h(m,n) \geq 0 \tag{5.2.29}$$

for $m \geq n \geq N$*. If*

$$\lim_{m\to\infty} \sup \frac{1}{H(m,N)} \sum_{n=N}^{m-1} H(m,n)\, e_n = +\infty,$$

and

$$\lim_{m\to\infty} \inf \frac{1}{H(m,N)} \sum_{n=N}^{m-1} H(m,n)\, e_n = +\infty,$$

then every solution of equation (5.2.27) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (5.2.27). As in the proof of Theorem 5.2.10, we have

$$\sum_{n=N}^{m-1} H(m,n)\, e_n \leq -H(m,N)\, \Delta z_{N-1}$$
$$+ \sum_{n=N}^{m-1} \left(h(m,n) - p_n H(m,n)\right) x_{n+1}$$
$$+ \sum_{n=N}^{m-1} \left(h(m,n)\, c_{n+1} - q_n H(m,n)\right) x_{n+1-\ell}.$$

Using (5.2.28) and (5.2.29) in the last inequality, we obtain

$$\frac{1}{H(m,N)} \sum_{n=N}^{m-1} H(m,n)\, e_n \leq -\Delta z_{N-1}.$$

The rest of the proof is similar to that of Theorem 5.2.13 and hence omitted. □

5.3 Examples

In this section we present some examples to illustrate the results established in the previous section.

Example 5.3.1. *Consider the difference equation*

$$\Delta\left(n\Delta\left(x_n+\frac{1}{2}x_{n-1}\right)\right)+\frac{1}{(n-1)^5}x_{n-1}^3$$
$$=\frac{(-1)^n}{2}\left(4n^2+10n+5\right)+\frac{1}{(n-1)^3}x_{n-1}, \qquad (5.3.1)$$

where $n \geq 2$.*Here* $\beta = 3, c_n = \frac{1}{2}, a_n = n, p_n = \frac{1}{(n-1)^5}, q_n = \frac{1}{(n-1)^3}, e_n = \frac{(-1)^n}{2}\left(4n^2+10n+5\right)$. *It is easy to see that all conditions of Theorem 5.2.2 are hold and hence every solution of equation (5.3.1) is oscillatory. Infact* $\{x_n\} = \{n(-1)^n\}$ *is one such solution of equation (5.3.1).*

Example 5.3.2. *Consider the difference equation*

$$\Delta\left(n\left(\Delta\left(x_n+2x_{n-2}\right)\right)\right)+\frac{1}{(n-1)^3}x_{n-1}$$
$$=(-1)^n\left(12n^2+2n+1\right)+\frac{1}{(n-1)^{\frac{7}{3}}}x_{n-1}^{\frac{1}{3}} \qquad (5.3.2)$$

where $n \geq 2$. *Here* $\gamma = \frac{1}{3}, c_n = 2, a_n = n, p_n = \frac{1}{(n-1)^3}, q_n = \frac{1}{(n-1)^{\frac{7}{3}}}$ *and* $e_n = (-1)^n\left(12n^2+2n+1\right)$. *It is easy to see that all* *conditions of Theorem 5.2.3 are hold and hence every solution of equation (5.3.2) is oscillatory. Infact* $\{x_n\} = \{n(-1)^n\}$ *is one such solution of equation (5.3.2).*

Example 5.3.3. *Consider the difference equation*

$$\Delta\left(n\left(\Delta\left(x_n+\frac{1}{2}x_{n-1}\right)\right)^3\right)+\frac{1}{(n-1)^2}x_{n-1}^3$$
$$=(-1)^n(2n+1)+\frac{1}{(n-1)^2}x_{n-1}^{\frac{1}{3}} \qquad (5.3.3)$$

where $n \geq 2$. *Here* $a_n = n, \alpha = \beta = 3, \gamma = \frac{1}{3}, c_n = \frac{1}{2}, p_n = q_n = \frac{1}{(n-1)^2}$ *and* $e_n = (-1)^n(2n+1)$. *By taking* $\{b_n\} = \left\{\frac{1}{(n-1)^2}\right\}$, *it is easy to see that all the conditions of Theorem 5.2.6 are satisfied. Hence all solutions of equation (5.3.3) are oscillatory. Infact* $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation (5.3.3).*

Example 5.3.4. *Consider the following difference equation*

$$\Delta^2\left(x_n+\frac{1}{2}x_{n-1}\right)+\frac{1}{n^{\frac{1}{5}}}x_n^{\frac{1}{5}}+\frac{1}{(n-1)^{\frac{1}{3}}}x_{n-1}^{\frac{1}{3}}=2, n \geq 2 \quad (5.3.4)$$

which has an unbounded solution $\{x_n\} = \{n\}$. *All conditions of Theorem 5.2.13 are satisfied except those on* $\{e_n\}$, *that is, conditions (5.2.25) and (5.2.26).*

Remark 5.3.1. *We note that the results of this chapter are not applicable to unforced equations, that is , when* $e_n \equiv 0$.