CHAPTER 4

OSCILLATION OF MIXED TYPE NEUTRAL DIFFERENCE EQUATION

4.1 Introduction

In this chapter we study the oscillatory behavior of solutions of mixed type neutral difference equation of the form

$$\Delta\left(a_n\Delta\left(x_n+bx_{n-k}-cx_{n+\ell}\right)\right)=q_nf\left(x_{n-\sigma_1}\right)+p_nf\left(x_{n+\sigma_2}\right) \tag{4.1.1}$$

where $n\in\mathbb{N}, b, c$ are real constants, k, ℓ, σ_1 and σ_2 are positive integers, $\{a_n\}, \{q_n\}$ and $\{p_n\}$ are positive real sequences and f :

$\mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, \sigma_1\}$. By a solution of equation (4.1.1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\theta$ and satisfies equation (4.1.1) for all $n \in \mathbb{N}$.

The oscillation, nonoscillation and asymptotic behavior of equation (4.1.1) when $c = 0$ and $p_n \equiv 0$ or $b = 0$ and $p_n = 0$ or $c = 0$ and $q_n = 0$ have been considered by many authors, see for example [1, 2, 6] and the references cited therein.

The plan of the chapter is as follows. In Section 4.2, we present conditions for the oscillation of all solutions of equation (4.1.1) when b, c and f satisfy different set of conditions. Examples are provided in Section 4.3 to illustrate the results. The results established in this chapter generalize some of the results obtained in [4, 5, 24] and discrete analogue of the results given in [19].

4.2 Oscillation Results

In this section we obtain sufficient conditions for the oscillation of all solutions of equation (4.1.1). For our first result we assume that $f(u) \equiv u$ that is, consider the equation of the form

$$\Delta\left(a_n \Delta\left(x_n + b x_{n-k} - c x_{n+\ell}\right)\right) = q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2}. \qquad (4.2.1)$$

Theorem 4.2.1. *Let* $\sigma_1 > k$ *and* $\sigma_2 > \ell$. *Let* $\{q_n^*\}$ *and* $\{p_n^*\}$ *be two positive real sequences such that* $q_{n+k} \leq q_n^* \leq \min\{q_n, q_{n-k}\}$ *and* $p_{n+\ell} \leq p_n^* \leq \min\{p_n, p_{n-\ell}\}$. *Assume that* $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$, *and*

(i) the difference inequality

$$\Delta(a_n \Delta y_n) - \frac{p_n^*}{1+b} y_{n+\sigma_2} \geq 0 \qquad (4.2.2)$$

has no eventually positive increasing solution;

(ii) the difference inequality

$$\Delta(a_n \Delta y_n) - \frac{q_n^*}{1+b} y_{n-\sigma_1+k} \geq 0 \qquad (4.2.3)$$

has no eventually positive decreasing solution;

(iii) the difference inequality

$$\Delta(a_n \Delta u_n) + \frac{q_n^*}{c} u_{n-\sigma_1-\ell} + \frac{p_n^*}{c} u_{n+\sigma_2-\ell} \leq 0 \qquad (4.2.4)$$

has no eventually positive solution.

Then every solution of equation (4.2.1) is oscillatory.

Proof. Assume that equation (4.2.1) has an eventually positive solution $\{x_n\}$, say $x_{n-\theta} > 0$ for all $n \geq n_0 \in \mathbb{N}$. Set

$$z_n = x_n + bx_{n-k} - cx_{n+\ell}.$$

Then

$$\Delta\left(a_n \Delta z_n\right) = q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2} > 0, n \geq n_1 \geq n_0,$$

which implies that the sequences $\{z_n\}, \{\Delta z_n\}$ are of one sign for $n \geq n_2 \geq n_1$. We claim that $z_n > 0$ eventually. To prove it assume that $z_n < 0$. Then we let

$$0 < u_n = -z_n = cx_{n+\ell} - bx_{n-k} - x_n \leq cx_{n+\ell}.$$

Thus

$$x_n \geq \frac{1}{c} u_{n-\ell}, n \geq n_2.$$

From equation (4.2.1) one gets

$$\begin{aligned} 0 &= \Delta\left(a_n \Delta u_n\right) + q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2} \\ &\geq \Delta\left(a_n \Delta u_n\right) + \frac{q_n}{c} u_{n-\sigma_1-\ell} + \frac{p_n}{c} u_{n+\sigma_2-\ell} \end{aligned}$$

$$\geq \Delta\left(a_n \Delta u_n\right) + \frac{q_n^*}{c} u_{n-\sigma_1-\ell} + \frac{p_n^*}{c} u_{n+\sigma_2-\ell}.$$

Hence $\{u_n\}$ is a positive solution of (4.2.4), a contradiction. Therefore $z_n > 0$. We define

$$y_n = z_n + b z_{n-k} - c z_{n+\ell}. \tag{4.2.5}$$

Then

$$\begin{aligned} \Delta\left(a_n \Delta y_n\right) &= q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2} + b q_{n-k} x_{n-\sigma_1-k} \\ &\quad + b p_{n-k} x_{n+\sigma_2-k} - c q_{n+\ell} x_{n-\sigma_1+\ell} \\ &\quad - c p_{n+\ell} x_{n+\sigma_2+\ell}. \end{aligned}$$

Thus

$$\Delta\left(a_n \Delta y_n\right) \geq q_n^* z_{n-\sigma_1} + p_n^* z_{n+\sigma_2} > 0. \tag{4.2.6}$$

Hence $\{y_n\}, \{\Delta y_n\}$ are of one sign eventually. Now we shal prove that $y_n > 0$. If not, then we repeat the procedure used above. We let

$$0 < v_n = -y_n = c z_{n+\ell} - b z_{n-k} - z_n \leq c z_{n+\ell}.$$

Hence

$$z_n \geq \frac{1}{c} v_{n-\ell}$$

and (4.2.6) implies

$$\begin{aligned} 0 &\geq \Delta\left(a_n \Delta v_n\right) + q_n^* z_{n-\sigma_1} + p_n^* z_{n+\sigma_2} \\ &\geq \Delta\left(a_n \Delta v_n\right) + \frac{q_n^*}{c} v_{n-\sigma_1-\ell} + \frac{p_n^*}{c} v_{n+\sigma_2-\ell}. \end{aligned}$$

We get that $\{v_n\}$ is a positive solution of (4.2.4), a contradiction.

Next we consider the following two cases.

Case 1. Let $\Delta z_n < 0$ for $n \geq n_3$. We claim that $\Delta y_n < 0$ for $n \geq n_3$. If not then we have $y_n > 0, \Delta y_n > 0, \Delta\left(a_n \Delta y_n\right) > 0$ which implies that $\lim_{n\to\infty} y_n = \infty$. On the other hand $z_n > 0, \Delta z_n < 0$ implies that $\lim_{n\to\infty} z_n = k < \infty$. Then applying limits on both sides of (4.2.4) we have a contradiction. Thus $\Delta y_n < 0$ for $n \geq n_3$. Using the monotonicity of $\{z_n\}$ we now get

$$\begin{aligned} y_{n-\sigma_1} &= z_{n-\sigma_1} + b z_{n-\sigma_1-k} - c z_{n+\sigma_2+\ell} \\ &\leq z_{n-\sigma_1+} b z_{n-\sigma_1-k} \leq z_{n-\sigma_1-k}\left(1+b\right), \end{aligned}$$

that is $z_{n-\sigma_1} \geq \dfrac{y_{n-\sigma_1+k}}{1+b}$ which together with (4.2.6) provides

$$\Delta\left(a_n \Delta y_n\right) \geq q_n^* z_{n-\sigma_1} \geq \frac{q_n^*}{1+b} y_{n-\sigma_1+k}.$$

Thus $\{y_n\}$ is a positive decreasing solution of (4.2.3), a contradiction.

Case 2. Let $\Delta z_n > 0$ for $n \geq n_3$. Now we consider the following cases:

Case (i). Assume that $\Delta y_n < 0$. Proceeding similarly as above and using the monotonicity of z_n,we obtain

$$y_{n-\sigma_1} \leq (1+b)\, z_{n-\sigma_1}.$$

Then using (4.2.6) and monotonicity of $\{y_n\}$ we get

$$\Delta\left(a_n \Delta y_n\right) \geq q_n^* z_{n-\sigma_1} \geq \frac{q_n^*}{1+b} y_{n-\sigma_1} \geq \frac{q_n^*}{1+b} y_{n-\sigma_1+k}$$

and again $\{y_n\}$ is a positive, decreasing solution of (4.2.3), a contradiction.

Case (ii). Assume that $\Delta y_n > 0$. Then

$$y_{n+\sigma_2} \leq (1+b)\, z_{n+\sigma_2},$$

which in view of (4.2.6) implies

$$\Delta\left(a_n \Delta y_n\right) \geq p_n^* z_{n+\sigma_2} \geq \frac{p_n^*}{1+b} y_{n+\sigma_2}$$

that is (4.2.2) possesses a positive, increasing solution, a contradiction. The proof is now complete. □

Remark 4.2.1. *If the sequences $\{p_n\}$ and $\{q_n\}$ are decreasing then* *we can put* $p_n^* = p_n$ *and* $q_n^* = q_n$.

Remark 4.2.2. *In [24], the authors for the equation (4.2.1) with* $p_n = p$ *and* $q_n = q$ *the condition* $1 + b - c > 0$ *is imposed.*

Remark 4.2.3. *Applying existing conditions sufficient for the inequalities (4.2.2),(4.2.3),(4.2.4) to have no above mentioned solutions, we immediately obtain various oscillation criteria for equation (4.2.1).*

Theorem 4.2.2. *Let* $\sigma_1 > k$ *and* $\sigma_2 > 2$. *Assume that*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_2-2} \left(\sum_{t=s}^{n+\sigma_2-2} \frac{1}{a_t} \right) p_s^* > 1 + b \tag{4.2.7}$$

and

$$\limsup_{n\to\infty} \sum_{s=n-\sigma_1+k}^{n} \left(\sum_{t=s-\sigma_1+k}^{n} \frac{1}{a_t} \right) q_s^* > 1 + b \tag{4.2.8}$$

and that the difference inequality (4.2.4) has no eventually positive solution. Then every solution of equation (4.2.1) is oscillatory.

Proof. Conditions (4.2.7) and (4.2.8) are sufficient for (4.2.2) to have no increasing positive solution and for (4.2.3) to have no decreasing positive solution, respectively(see e.g. Lemma 7.6.15 [29]). The proof then follows from Theorem 4.2.1. □

Next we consider a difference equation slightly varient from equation (4.1.1)

$$\Delta\left(a_n\Delta\left(x_n + bx_{n-k} + cx_{n+k}\right)\right) = q_n f\left(x_{n-\sigma_1}\right) + p_n f\left(x_{n+\sigma_2}\right), \tag{4.2.9}$$

where $n \in \mathbb{N}$, with $\dfrac{f(u)}{u} \geq M > 0$ and $q_{n\pm k} = q_n$ and $p_{n\pm k} = p_n$.

Theorem 4.2.3. *Let $\sigma_2 - k > 2$ and $\sigma_1 > k$ be even integers. If*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_2-k-2} \left(\sum_{t=s}^{n+\sigma_2-k-2} \frac{1}{a_t} \right) p_s^* > \frac{1+b+c}{M} \tag{4.2.10}$$

and

$$\limsup_{n\to\infty} \sum_{s=n-\sigma_1+k}^{n} \left(\sum_{t=s-\sigma_1+k}^{n} \frac{1}{a_t} \right) q_s^* > \frac{1+b+c}{M} \tag{4.2.11}$$

then every solution of equation (4.2.9) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (4.2.9), say $x_{n-\theta} > 0$ for all $n \geq n_0 \in \mathbb{N}$. Set

$$z_n = x_n + bx_{n-k} + cx_{n+k}. \tag{4.2.12}$$

Then

$$\Delta\left(a_n \Delta z_n\right) = q_n f\left(x_{n-\sigma_1}\right) + p_n f\left(x_{n+\sigma_2}\right)$$

$$\geq Mq_n x_{n-\sigma_1} + Mp_n x_{n+\sigma_2} \geq 0,$$

eventually and hence we see that $z_n > 0$ and Δz_n is eventually of one sign. Next, we let

$$w_n = z_n + bz_{n-k} + cz_{n+k}. \tag{4.2.13}$$

Then

$$\Delta(a_n \Delta w_n) \geq Mq_n z_{n-\sigma_1} + Mp_n z_{n+\sigma_2} \tag{4.2.14}$$

and

$$\Delta(a_n \Delta(w_n + bw_{n-k} + cw_{n+k})) \geq Mq_n w_{n-\sigma_1} + Mp_n w_{n+\sigma_2}. \tag{4.2.15}$$

Clearly $w_n > 0$ eventually. Now we consider the two cases.

Case 1. Let $\Delta z_n > 0$ eventually, that is $\Delta z_n > 0$ for $n \geq n_1 \geq n_0$. Then $\Delta(a_n \Delta w_n) > 0$ for $n \geq n_1$ and hence we conclude that $w_n > 0$ and $\Delta w_n > 0$ for $n \geq n_1$. From the inequality (4.2.15) we have

$$\Delta(a_n \Delta w_n) \geq \frac{Mp_n}{1+b+c} w_{n+\sigma_2-k}, n \geq n_1$$

and by condition, we arrive at a contradiction.

Case 2. Let $\Delta z_n < 0$ for $n \geq n_1$. Then one can easily see that $\Delta(a_n \Delta w_n) > 0$ for $n \geq n_1$. We claim that $\Delta w_n < 0$ for $n \geq n_2 \geq n_1$. Otherwise, $\Delta w_n > 0$ for $n \geq n_2$ and hence we see that $w_n > 0, \Delta w_n > 0$ and $\Delta^2 w_n > 0$. Using these facts in (4.2.15), we obtain

$$\begin{aligned}\Delta(a_n \Delta w_n) &\geq \frac{M p_n}{(1+b+c)} w_{n+\sigma_2+k} \\ &\geq \frac{M p_n}{(1+b+c)} w_{n+\sigma_2} \\ &> \frac{M p_n}{(1+b+c)} w_{n+\sigma_2-k}\end{aligned}$$

and again we are led to a contradiction. Thus $\Delta w_n < 0$ for $n \geq n_2$ and hence we conclude that $w_n > 0, \Delta w_n < 0$ and $\Delta(a_n \Delta w_n) > 0$ for all $n \geq n_2$. From (4.2.15) we have

$$\Delta(a_n \Delta w_n) \geq \frac{M q_n}{(1+b+c)} w_{n-\sigma_1+k}$$

and again arrive at a contradiction. The proof is now complete.

□

Next we consider the following equation

$$\Delta(a_n \Delta(x_n - b x_{n-k} - c x_{n+k})) = q_n\, x_{n-\sigma_1} + p_n\, x_{n+\sigma_2}, \quad (4.2.16)$$

where $n \in \mathbb{N}, \{p_n\}$ and $\{q_n\}$ satisfy the conditions as in equation (4.2.9). We establish sufficient conditions for the oscillation of all solutions of equation (4.2.16).

Theorem 4.2.4. *If* $\sigma_2 \geq 2$,

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_2-2} \left(\sum_{t=s}^{n+\sigma_2-s-2} \frac{1}{a_t} \right) p_s > 1 \tag{4.2.17}$$

$$\limsup_{n\to\infty} \sum_{s=k-\sigma_1}^{n} \left(\sum_{t=s}^{n} \frac{1}{a_t} \right) q_s > 1 \tag{4.2.18}$$

then every solution of equation (4.2.16) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (4.2.16), say $x_n > 0$ for $n \geq n_0 \in \mathbb{N}$. Set

$$z_n = x_n - bx_{n-k} - cx_{n+k}.$$

Then

$$\Delta\left(a_n \Delta z_n\right) = q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2} \geq 0$$

eventually and hence $\{z_n\}, \{a_n \Delta z_n\}$ and $\{\Delta\left(a_n \Delta z_n\right)\}$ are eventually of one sign. There are two cases to consider.

Case 1. Let $z_n > 0$ eventually for some $n > n_1 \geq n_0$. Clearly $x_n \geq z_n, n \geq n_1$. We consider the following two sub cases.

If $\Delta z_n > 0$ for some $n \geq n_2$, then $z_n > 0, a_n \Delta z_n > 0$ and $\Delta(a_n \Delta z_n) > 0$ for $n \geq n_2$ and $\Delta(a_n \Delta z_n) \geq p_n z_{n+\sigma_2}$ for $n \geq n_2$. By condition (4.2.17), we arrive at a contradiction.

If $\Delta z_n < 0$ for $n \geq n_2$, then we have $z_n > 0, a_n \Delta z_n < 0$ and $\Delta(a_n \Delta z_n) > 0$ for $n \geq n_2$. Also we have $\Delta(a_n \Delta z_n) \geq q_n z_{n-\sigma_1}$ for $n \geq n_1$. Again by condition (4.2.18), we obtain the desired contradiction.

Case 2. Let $\Delta z_n < 0$ for $n \geq n_1$. Set

$$0 < u_n = -z_n = b x_{n-k} + c x_{n+k} - x_n.$$

Then

$$\Delta(a_n \Delta u_n) + q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2} = 0,$$

and hence we conclude that $\Delta u_n > 0$ for $n \geq n_2$. Define $w_n = b u_{n-k} + c u_{n+k} - u_n$. Then we have

$$\Delta(a_n \Delta w_n) + q_n u_{n-\sigma_1} + p_n u_{n+\sigma_2} = 0, \tag{4.2.19}$$

and

$$\Delta(a_n \Delta(b w_{n-k} + c w_{n+k} - w_n)) + q_n w_{n-\sigma_1} + p_n w_{n+\sigma_2} = 0. \tag{4.2.20}$$

Now it is easy to check that $w_n > 0$ and hence $\Delta w_n > 0$ for $n \geq n_3 \geq n_2$. There exists positive constants α_1 and α_2 and an integer $n_4 \geq n_3$ such that

$$u_{n-\sigma_1} \geq \alpha_1, u_{n+\sigma_2} \geq \alpha_2, n \geq n_4. \tag{4.2.21}$$

Using this in equation (4.2.19), we obtain

$$\Delta\left(a_n \Delta w_n\right) \leq -\alpha_1 q_n - \alpha_2 p_n, n \geq n_4.$$

Summing both sides of the last inequality from n_4 to $n-1$ we obtain

$$0 < a_n \Delta w_n \leq a_{n_4} \Delta w_{n_4} - \sum_{s=n_4}^{n+1} [\alpha_1 q_s + \alpha_2 p_s] \rightarrow -\infty$$

as $n \rightarrow \infty$, a contradiction and the proof is complete. □

Finally we consider the equation

$$\Delta\left(a_n \Delta\left(x_n - b x_{n-k} + c x_{n+k}\right)\right) = q_n x_{n-\sigma_1} + p_n x_{n+\sigma_2}, \tag{4.2.22}$$

where $n \in \mathbb{N}$, $\{p_n\}$ and $\{q_n\}$ satisfy the conditions as in equation (4.2.9).

Next, we give the following conditions for the oscillation of all solutions of equation (4.2.22). The proof of the result can be modelled as of previous theorems, and hence we omit the details.

Theorem 4.2.5. *Let* $\sigma_2 > k+2$. *If*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_2-k-2} \left(\sum_{t=s}^{n+\sigma_2-k-2} \frac{1}{a_t} \right) p_s > 1+c$$

and

$$\limsup_{n\to\infty} \sum_{s=n-\sigma_1}^{n} \left(\sum_{t=s}^{n} \frac{1}{a_t} \right) q_s > 1+c$$

then every solution of equation (4.2.22) is oscillatory.

Remark 4.2.4. *From the results presented in this section, we observe that when the coefficient* $p_n = 0$ *or the condition on* $\{p_n\}$ *is violated the conclusion of the theorem may be replaced by "every solution* $\{x_n\}$ *of each equation (4.2.1),(4.2.9),(4.2.16), (4.2.22) is oscillatory or* $x_n \to \infty$ *as* $n \to \infty$."

Remark 4.2.5. *Once again from the proofs of the theorems we see that if* $q_n = 0$ *or condition on* $\{q_n\}$ *is violated then the conclusion of the theorems may be replaced by " every solution* $\{x_n\}$ *of each equation (4.2.1), (4.2.9), (4.2.16), (4.2.22) is oscillatory or* $x_n \to 0$ *as* $n \to \infty$."

4.3 Examples

In this section we present some examples to illustrate the results given in the last section.

Example 4.3.1. *Consider the equation*

$$\Delta^2 \left(x_n + 2^k x_{n-k} - 2^{-\ell} x_{n+\ell}\right) = \frac{1}{2}\left(2^{\sigma_1} x_{n-\sigma_1} + 2^{-\sigma_2} x_{n+\sigma_2}\right). \tag{4.3.1}$$

The conditions (4.2.7) of Theorem 4.2.2 is not satisfied and hence the equation (4.3.1) has a nonoscillatory solution $\{x_n\} = \{2^n\} \to \infty$ *as* $n \to \infty$.

Example 4.3.2. *Consider the equation*

$$\Delta^2 \left(x_n + 2^{-k} x_{n-k} - 2^{\ell} x_{n+\ell}\right) = \frac{1}{8}\left(2^{-\sigma_1} x_{n-\sigma_1} + 2^{\sigma_2} x_{n+\sigma_2}\right). \tag{4.3.2}$$

The condition (4.2.8) of Theorem 4.2.2 is not satisfied and hence the equation (4.3.2) has a nonoscillatory solution $\{x_n\} = \{2^{-n}\} \to 0$ *as* $n \to \infty$.