CHAPTER - 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction.

The activity 'Operations Research' has become increasingly important in the face of fast moving technology and increasing complexities in business and industry. In recent years, Operations Research has an increasingly great impact on the management of organisations. Both the number and variety of its applications continue to grow rapidly. Many industries including aircraft missile, automobile, communication, computer, electronics, mining, paper, petroleum and transportation have made widespread use of Operations Research in determining their tactical and strategical decisions more scientifically.

Among the wide range of applications theory of queues, reliability and inventory have received considerable attention in recent years. Inventory problems are very important in business organisations and economics. The study of inventory theory is essentially a mathematical study of the underlying stochastic processes. This thesis makes an attempt on a probabilistic analysis of some inventory models, aiming at an improved understanding of the behaviour of such models which may wider their applications in real life situations. Various inventory systems with different demand and service patterns are considered. Renewal and Convolution techniques are applied to obtain the solution of inventory models with general distributions. For the Markovian models, the infinitesimal generators of continuous time Markov chains are written and steady state probability vectors of different inventory levels are obtained.
using Neuts' block partitioned matrix method. This chapter deals with the introduction of the thesis and a review of relevant topics.

1.2. Inventory Theory.

The single commodity inventory models are mainly characterized by the demand pattern and the policy of replenishing the stock. The demand process is of two types, unit demand and bulk demand (during calamity).

In both cases the interarrival time distributions may be general or exponential. The two basic reordering policies are a) orders are placed at regular intervals of time b) the \((s, S)\) policy under which an order is placed to fill up the inventory, only when level falls to a fixed level \(s\). The replenishments ordered as above are assumed to arrive after a time lag, which may be fixed or a random variable. This time lag is called lead time. During lead time, the inventory may fall to zero. The time duration for which the level of inventory continually remains at zero is called a dry period. The other parameter which has direct impact on the behavior of an inventory system is server's vacation. The server may be given vacation for a fixed or random length of time either during dry period or after serving fixed number of demands. These inventory models give rise to several other application oriented models such as queues, dams, insurance risks, industrial productions etc.

The quantitative analysis of the inventory started with the work of Harris [13] (1915). The theory of continuous time storage was also initiated by Moran, Gani and Prabu [27]. A stochastic inventory problem was analyzed for the first time by Masse [26] (1946) in which a formulation of the optimum storage was capacity was done by Hurst [14] (1951).
After that, several studies were made. The development of the theory up to 1952 had been summarized by Whitin [55]. Using the renewal theoretic arguments, a probabilistic treatment was given by Arrow, Karlin and Scarf [1]. For a systematic study of classical models one can refer to their work [1, 2]. Hadley and Whitin [12] (1963) dealt with application of such models to practical situations. Veinott [54] (1966) gave a detailed review of the work carried out in \((s, S)\) inventory systems up to 1966. Tijms [52, 53] (1972) gave a detailed analysis of the system under \((s, S)\) policy.

Gross and Harris [11] (1971) dealt with the system with state dependent lead times. The study of random lead times was provided by Ryshikov [41](1973). Sivazlian [47] (1974) investigated an \((s, S)\) inventory system with arbitrary inter-arrival time distribution between unit demands. Srinivasan [50] (1979) extended this result to the system in which lead times are *i.i.d.* random variables having a general distribution. Sahin [42] (1979) considered a system in which demand quantities are random but lead time is constant. Again in 1983 Sahin [43] discussed on inventory system in which the inter-arrival times between consecutive demands, quantities demanded and lead times are *i.i.d.* random variables with general distributions.

Multi-commodity inventory systems under \((s, S)\) policy has been studied extensively by Sivazlian [46, 47, 48, 49]. Thangaraj and Ramanarayanan [51] (1983) considered an inventory system with two ordering levels. Kalpakam and Arivaringnan [15] (1985) dealt with an inventory system having one exhibiting item subject to random failure. Daniel and Ramanarayanan [6, 7] (1988) considered inventory systems with vacation to the server. An \((s, S)\) inventory system with bulk demand and vacation
to the server has been discussed using renewal theory by Madhusoodanan and Jacob [22](1989). Perishable inventory has been studied by Manoharan and Krishnamoorthy [25] (1989). Other researchers like Jacob [16], Lakshmy [21] and Mahmut Parlar [23, 24] have also contributed a lot to the study of inventory theory.

Numerical Analysis of \((s, S)\) inventory systems with repeated attempts has been studied by Krishnamoorthy and Lopez-Herrero [17] (2006). \((s, S)\) inventory policy with service time, vocation to server and correlated lead time has been analyzed by Narayanan, Deepak, Krishnamoorthy and Krishnakumar [36] (2007). Effective utilization of idle time in an \((s, S)\) inventory with positive service time was discussed by Krishnamoorthy, Deepak, Narayanan and Vineetha [18] (2006). Control polices for Inventory with service time was discussed by Krishnamoorthy, Deepak, Narayanan and Vineetha [19] (2006). Setting the Clock Back to Zero \((SCBZ)\) property has been analyzed by Raja Rao [39] (1998).

1.3. Renewal Theory.

Let \(\{X_n : n = 1, 2, 3 \ldots \}\) be a sequence of non negative independent random variables with a common distribution function \(F(x)\). Let

\[
S_n = \sum_{i=1}^{n} X_i \text{ for } n \geq 1 \text{ and } S_0 = 0.
\]

Define the random variable \(N(t) = \{n : S_n \leq t\}\). The random variable \(N(t)\) gives the number of renewals occurring in \([0, t]\). The random variable \(X_n\) gives the inter arrival time(waiting time) between \((n - 1)^{th}\) and \(n^{th}\) renewals. The inter arrival times are independent identically distributed random variables.
The function $M(t) = E\{N(t)\}$ is called the renewal function of the
process with distribution $F$. Here $N(t) \geq n$ if and only if $S_n \leq t$. The
distribution of $N(t)$ is given by

$$P_n(t) = P\{N(t) = n\} = F^{*n}(t) - F^{*(n+1)}(t)$$

and the expected number of renewals by

$$M(t) = \sum_{n=1}^{\infty} F^{*n}(t).$$

The renewal function $M$ satisfies the equation

$$M(t) = F(t) + \int_{0}^{t} M(t-x)dF(x).$$

Let $m(t) = M'(t)$; $m(t)$ is called the renewal density function and

$$m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$$

provided $F''(x) = f(x)$ exists. For more details of the renewal theory,
references may be made to Samuel Karlin and Howord M.Taylor [44, 45].

1.4. Markov Chains.

Let

$$P_n(t) = P\{N(t) = n\} \text{ and } \sum_{n=0}^{\infty} p_n(t) = 1.$$  

The family of random variables $\{N(t); t \geq 0\}$ is stochastic process. The
stochastic process $\{X_n: n = 0, 1, 2, \ldots\}$ is called a Markov chain, if for
$j, k, j_1, j_2 \ldots j_{n-1} \in N.$

$$P\{X_n = k|X_{n-1} = j, X_{n-2} = j_1 \ldots X_0 = j_{n-1}\} = P\{X_n = k|X_{n-1} = j\} = P_{jk} \ (say)$$
The transition probabilities $P_{jk}$ satisfy $P_{jk} \geq 0$ and $\sum_k P_{jk} = 1$ for all $j$. These probabilities may be written in the form $\mathbf{P} = (P_{ij})$. This matrix is called the transition probability matrix of the Markov chain. $\mathbf{P}$ is a stochastic or Markov matrix which is a square matrix with non-negative elements and unit row sums to one. If every state can be reached from every other state the chain is said to be irreducible.

1.5. Solutions of Inventory Models.

The objective of the solution of an inventory model is to find the probability distribution of the number of units in the system at a given system. One way of obtaining probabilities is to formulate a system of different differential equations and solve them using the method of generating functions and Laplace-Stieltjes transforms. The solution thus obtained is known as transient solution. This transient solution depends on the choice of initial conditions and the computations itself are laborious. When $t$ tends to infinity, $P_n(t)$ tends to $P_n$ which is independent of $t$. This solution obtained under certain conditions, is called steady-state solution. For the analysis of such solution, initial conditions are not required. The distribution $\{P_n\}$ is obtained by using theory of semi Markov processes.

1.6. Block Partitioned Matrix Method.

The steady-state solution is supposed to throw more light on many behavioural features of a model. For the first time, Neuts M.F [37] discussed an algorithmic approach for obtaining computational probabilities on a class of structured markov chains. He has investigated a new approach which has efficient and stable algorithms involving only real
arithmetic. Neuts shows that a class of infinite, block partitioned stochastic matrices has an invariant probability vector of a matrix-geometric form. He considers a Markov chain with state space \(((i, j), i \geq 0, j \geq 0)\) and a transition probability matrix \(P\) of the form:

\[
P = \begin{pmatrix}
B_0 & A_0 & 0 & 0 & 0 & \ldots \\
B_1 & A_1 & A_0 & 0 & 0 & \ldots \\
& & \vdots & & & \\
B_k & A_k & A_{k-1} & \ldots 
\end{pmatrix}
\]

where the matrices \(A_k\) and \(B_k\), \(k \geq 0\) are \(m \times m\) non negative matrices satisfying

\[
\sum_{r=0}^{k} A_r e + B_k e = e, \quad \text{for} \quad k \geq 0
\]

where

\[
e = (1, 1, 1, \ldots)^t.
\]

The stochastic matrix \(\sum_{r=0}^{\infty} A_r\) will be denoted by \(A\) and it is assumed irreducible. Let \(\Pi\) be an invariant probability vector such that \(\Pi \geq 0\) and \(\Pi P = \Pi\) with \(\Pi e = 1\). Partitioning the vector \(\Pi\) into \(m\)-vectors \(\Pi_0\Pi_1\ldots\) the above equations may be written as

\[
\Pi_k = \sum_{r=0}^{\infty} \Pi_{k+r-1} A_r, \quad \text{for} \quad k \geq 1
\]

\[
\Pi_0 = \sum_{r=0}^{\infty} \Pi_r B_r
\]

\[
\sum_{k=0}^{\infty} \Pi_k e = 1
\]

In [37], the matrix-geometric solutions are available for the above system of equations.
1.7. SCBZ Property.

Let $X$ be a random variable such that the pdf of $X$ is

\[
h(x) = \begin{cases} 
\theta_1 e^{-\theta_1 x} & \text{if } x \leq x_0, \\
\theta_2 e^{-\theta_2 x} e^{x_0(\theta_2 - \theta_1)} & \text{if } x > x_0.
\end{cases}
\]

Here $x_0$ is called truncation point and parameter $\theta_1$ changes to $\theta_2$ at truncation point. It can be shown that this random variables satisfies SCBZ \((\text{Setting Clock Back to Zero})\) property. Here $x_0$ is constant or random variable. If $x_0$ is taken to be a random variable with p.d.f. $\lambda e^{\lambda x_0}$, then

\[
h(x) = \theta_1 e^{-\theta_1 x} p(x < x_0) + \theta_2 e^{-\theta_2 x} e^{x_0(\theta_2 - \theta_1)} p(x_0 < x)
\]

\[
= \theta_1 e^{-\theta_1 x} e^{-\lambda x} + \theta_2 e^{-\theta_2 x} \int_0^x e^{x_0(\theta_2 - \theta_1)} \lambda e^{-\lambda x_0} dx_0
\]

\[
= \theta_1 e^{-x(\theta_1 + \lambda)} + \theta_2 e^{-\theta_2 x} \int_0^x e^{-x_0(\theta_1 - \theta_2 + \lambda)} dx_0
\]

\[
= \theta_1 e^{-x(\theta_1 + \lambda)} + \frac{\lambda \theta_2 e^{-\theta_2 x}}{\theta_1 + \lambda - \theta_2} - \frac{\lambda \theta_2 e^{-x(\theta_1 + \lambda)}}{\theta_1 + \lambda - \theta_2}
\]

\[
= e^{-x(\theta_1 + \lambda)} \left[ \theta_1 - \frac{\lambda \theta_2}{\theta_1 + \lambda - \theta_2} \right] + \frac{\lambda \theta_2 e^{-\theta_2 x}}{\theta_1 + \lambda - \theta_2}
\]

\[
= e^{-x(\theta_1 + \lambda)} \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{\lambda + \theta_1 - \theta_2} + \frac{\lambda \theta_2 e^{-\theta_2 x}}{\theta_1 + \lambda - \theta_2}
\]

Next distribution function is given by

\[
H(x) = \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{\lambda + \theta_1 - \theta_2} \int_0^x e^{-t(\lambda + \theta_1)} dt + \frac{\theta_2 \lambda}{\lambda + \theta_1 - \theta_2} \int_0^x e^{-\theta_2 t} dt
\]

\[
H(x) = 1 - \left[ \frac{\theta_1 - \theta_2}{\lambda + \theta_1 - \theta_2} \right] e^{-(\theta_1 + \lambda)x} - \left[ \frac{\lambda}{\lambda + \theta_1 - \theta_2} \right] e^{-\theta_2 x}.
\]

Here we take

\[
p = \frac{\theta_1 - \theta_2}{\lambda + \theta_1 - \theta_2}
\]

and

\[
q = \frac{\lambda}{\lambda + \theta_1 - \theta_2}.
\]
clearly $p + q = 1$.


The first chapter deals with the introduction to the thesis, with a review of relevant research articles.

Chapter II deals with the $(s, S)$ inventory system with defective supplies. In this model, we discuss both general case and Markovian model. The demands are assumed to occur for one unit at a time and the time intervals between the arrivals of consecutive demands form a family of independent and identically distributed random variables. The number of defective items in a lot is random variable. The supply is accepted only when at least $s + 1$ units are good. In the first part, assuming that all the random variables follow general distributions, inventory level probabilities are presented. In the second part, the model is treated under Markovian environment and the steady state probability vector is obtained. Numerical examples are also presented.

In chapter III, two $(s, S)$ inventory systems A and B with perishable units in A are discussed. Inter occurrence times of demands and the lead times for order are random variables. When the inventory level falls to $s$ from $S$, order is made for $S - s$ units to fill up inventories A or B or both as the case may be. Inventory level probabilities are obtained for exponential distribution. Numerical examples are also presented.

Chapter IV deals with the inventory system exposed to calamities with SCBZ arrival property. The demand process is assumed to be a combination of single and bulk demand for entire inventory where the rates of demand have the SCBZ property. The lead times and intervals
of time between successive demands are independent. We discuss the exponential case of two models. In model 1, unit demand rates are varying and in model 2, bulk demand rates are varying. Steady state probability vector of inventory levels is obtained using Neuts matrix method. Numerical examples are also presented.

Chapter V deals with the two (s, S) inventory systems with binary choice of demand and optional accessories with SCBZ arrival property. We discuss two models under Markovian environment. In model 1, there are two types of units. The demand occurs for any one of them with known probability. In model 2, there are two inventories namely primary and secondary. When a demand occurs primary unit is sold and secondary unit is demanded only with known probability. Steady state probability vector of inventory levels is obtained using Neuts matrix method. Numerical examples are also presented.

In chapter VI, we discuss two ordering level inventory systems with perishable units and exposed to calamity. We discuss two models. In model 1, inventory system contains perishable units and perishable time is exponential. In model 2, we discuss the inventory which is exposed to calamity. Calamity time is exponential. The inter occurrence times between successive demands, the lead times at two different levels are assumed to follow exponential distributions which are mutually independent. Inventory level probabilities are obtained and numerical examples are also presented.

Chapter VII deals with one ordering and two ordering levels inventory systems with SCBZ lead times. We discuss two models. In model 1, we discuss the one ordering level inventory system. In model 2, we discuss the two ordering level inventory system. In both the models
we consider that lead times have SCBZ property. The inter occurrence times between successive demands and the lead time are assumed to follow exponential distribution which are mutually independent. Inventory level probabilities are obtained using Neuts matrix method. Numerical examples are also presented.

Chapter VIII deals with two ordering levels inventory with different lead times and rest time to the server. The lead times at the different ordering levels s and 0 are distinct and server is allowed taking rest. In model 1, service time and rest time follow exponential distributions. Demands occur in accordance with a general renewal process and inter occurrence times follow general distributions. In model 2, we discuss Markovian model using Neuts matrix method. Inter arrival times, service times and lead times are all independent. Numerical examples are also presented.

Chapter IX deals with two ordering levels inventory system with different lead times and machine service. In the previous Chapter VIII, we have considered server to serve the arriving demands. In this chapter we consider a machine to serve the arriving demands. Machine's service time and repair time distribution are exponentials. After repair time machine is switched on and its service time starts immediately. In model 1, demands occur in accordance with a general renewal process and inter occurrence times follow general distribution. In model 2, we discuss Markovian model using Neuts matrix method. Inter arrival times, service times and lead times are all independent. Numerical examples are also presented.