4.1 Introduction

In this chapter we continue the study of oscillatory and nonoscillatory behavior of solutions of second order quasilinear difference equations of the type considered in Chapter 3, that is,

\[ \Delta(a_n \Psi \Delta x_n) + f(n, x_{n+1}) = 0, \quad n \in N(n_0). \quad (4.1.1) \]
Our purpose is to establish criteria for equation (4.1.1) to have various types of nonoscillatory solutions, as well as criteria for all solutions to be oscillatory when \( \sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty \). The desired results will be developed in Sections 4.2 and 4.3. Example illustrating the results are provided in Section 4.4.

4.2 Nonoscillation Theorems

In this section we discuss when the sequence \( \{a_n\} \) in equation (4.1.1) satisfies

\[
\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{4.2.1}
\]

We define \( \rho(n) \) by

\[
\rho(n) = \sum_{s=n}^{\infty} \frac{1}{a_s} \tag{4.2.2}
\]

for \( n \in N(n_0) \). In this section, we make the additional assumption that there is a constant \( K > 0 \) such that the inverse function \( \phi \) of \( \Psi \) in equation (4.1.1) satisfies

\[
\phi(s) \geq Ks \quad \text{for } s \in \text{dom}\phi \quad \text{with } s > 0. \tag{4.2.3}
\]
Lemma 4.2.1. If \( \{x_n\} \) is a positive solution of equation (4.1.1) such that \( \{\Delta x_n\} \) is eventually negative, then

\[
x_n \geq -Ka_n \Psi(\Delta x_n) \rho(n)
\]  

(4.2.4)

for all large \( n \), where \( K \) is the constant defined in (4.2.3).

Proof: Suppose that \( x_n > 0 \) and \( \Delta x_n < 0 \) for all \( n \in N(n_1) \). Since \( \{a_n \Psi(\Delta x_n)\} \) is decreasing, \( a_n \Psi(\Delta x_n) \geq a_s \Psi(\Delta x_s) \) for \( s \geq n \), that is,

\[
\Psi(-\Delta x_s) \geq \frac{a_n \Psi(-\Delta x_n)}{a_s}
\]

for \( s \geq n \in N(n_1) \). Using the inverse function \( \phi \) and (4.2.3), we have

\[
-\Delta x_s \geq \frac{Ka_n \Psi(-\Delta x_n)}{a_s}
\]

for \( s \geq n \in N(n_1) \). A summation of this inequality yields

\[
x_n \geq Ka_n \Psi(-\Delta x_n) \rho(n), \quad n \in N(n_1).
\]

Lemma 4.2.2. If \( \{x_n\} \) is a nonoscillatory solution of equation (4.1.1), then there exist positive constants \( C_1, C_2 \) and \( n_1 \in N(n_0) \) such that
C_1 \rho(n) \leq |x_n| \leq C_2 \quad \text{for } n \in N(n_1).

(4.2.5)

**Proof:** Let \{x_n\} be a nonoscillatory solution of equation (4.1.1), and without loss of generality, assume that \{x_n\} is eventually positive. Then, \{a_n \Psi(\Delta x_n)\} is eventually decreasing, so \{\Delta x_n\} is eventually of constant sign. If \Delta x_n > 0 \text{ for } n \in N(n_1) \text{ for some } n_1 \geq n_0, \text{ then } a_n \Psi(\Delta x_n) \leq a_n \Psi(\Delta x_{n_1}) = K_1 \text{ for } n \in N(n_1). \text{ So } 

\Psi(\Delta x_n) \leq K_1 / a_n \text{ for } n \in N(n_1). \text{ Since } K_1 / a_n \to 0 \text{ as } n \to \infty, \text{ from } b(\alpha)s \leq \phi(s) \leq c(\alpha)s \text{ for } 0 \leq s \leq \alpha, \text{ we see that there is a constant } K_2 \text{ and an integer } n_2 \geq n_1 \text{ such that } \Delta x_n \leq K_2 / a_n \text{ for } n \in N(n_2). \text{ This implies } x_n \leq K_2 \rho(n_2) + x_{n_2} \text{ for } n \in N(n_2). \text{ This proves the right half of the inequality in (4.2.5). If } \Delta x_n < 0 \text{ for } n \in N(n_1), \text{ then (4.2.4) holds for } n \geq n_2 \text{ for some } n_2 \in N(n_1). \text{ Since } a_n \Psi(\Delta x_n) \leq a_{n_2} \Psi(\Delta x_{n_2}) \text{ for } n \in N(n_2), \text{ we have }

x_n \geq -Ka_{n_2} \Psi(\Delta x_{n_2}) \rho(n).

This proves the left half of the inequality in (4.2.5). A similar argument holds if \{x_n\} is eventually negative.
Lemma 4.2.2 shows that under condition (4.2.1) the following three types of asymptotic behavior are possible for nonoscillatory solutions \( \{x_n\} \) of equation (4.1.1):

(A) \( \lim_{n \to \infty} x_n = \text{constant} \neq 0; \)

(B) \( \lim_{n \to \infty} x_n = 0 \) and \( \lim_{n \to \infty} \frac{|x_n|}{\rho(n)} = \infty; \)

(C) \( 0 < \liminf_{n \to \infty} \frac{|x_n|}{\rho(n)}, \limsup_{n \to \infty} \frac{|x_n|}{\rho(n)} < \infty. \)

First, we characterize the type (C) solutions of equation (4.1.1).

**Theorem 4.2.3.** A necessary and sufficient condition for equation (4.1.1) to have a type (C) nonoscillatory solution \( \{x_n\} \) is that

\[
\sum_{n=n_0}^{\infty} |f(n, c \rho(n+1))| < \infty \tag{4.2.6}
\]

for some nonzero constant \( c \).

**Proof:** (Necessity). Let \( \{x_n\} \) be a type (C) nonoscillatory solution of equation (4.1.1). We may assume that \( \{x_n\} \) is
eventually positive, in which case \( \{\Delta x_n\} \) is eventually negative. Hence, there is an integer \( n_i \in N(n_0) \) such that \( x_n > 0 \) and \( \Delta x_n < 0 \) for \( n \in N(n_i) \). Summing equation (4.1.1) and using (4.2.4) we obtain

\[
\frac{x_n}{\rho(n)} \geq K \sum_{s=n_i}^{n-1} f(s, x_{s+1}), \quad n \in N(n_i).
\] (4.2.7)

Since \( C_1 \rho(n) \leq x_n \leq C_3 \rho(n) \) for \( n \in N(n_i) \) and some positive constants \( C_1 \) and \( C_3 \), it follows from (4.2.7) that

\[
K \sum_{n=n_i}^{\infty} f(n, C_1 \rho(n+1)) \leq C_3.
\]

(Sufficiency) We may assume that the constant \( c \) in (4.2.6) is positive. Let \( \alpha \in \text{dom}\phi, \alpha > 0 \), be fixed, and let \( k \geq 1 \). Choose \( \ell > 0 \) and \( n_i > n_0 \) so that

\[
\frac{\ell}{a_n} \left( \frac{\ell}{b(\alpha)} + \sum_{s=n_i}^{n-1} f(s, (k+1) \ell \rho(s+1)) \right) \leq \alpha
\]

for \( n \in N(n_i) \) and

\[
c(\alpha) \left( \frac{\ell}{b(\alpha)} + \sum_{s=n_i}^{\infty} f(s, (k+1) \ell \rho(s+1)) \right) \leq (k+1) \ell.
\]
Let $B(n_1)$ be the same Banach space as in the proof of Theorem 3.2.2. Define a subset $S$ and an operator $T : S \to B(n_1)$ by

$$S = \{ X \in B(n_1) : \ell \rho(n) \leq x_n \leq (k+1)\ell \rho(n), n \in N(n_1) \}$$

$$\left( TX \right)_n = \sum_{s=n}^{\infty} \phi \left( \frac{1}{a_s} \left[ \ell \frac{1 \rho(n)}{b(\alpha)} + \sum_{i=n_1}^{s-1} f(t,x_{i+1}) \right] \right), \quad n \in N(n_1).$$

As in the proof of Theorem 3.2.2, we can show that $T$ satisfies the assumptions of the Knaster-Tarski fixed point theorem. Hence, there exists $X \in S$ such that $TX = X$, and this gives rise to a positive type (C) solution of equation (4.1.1).

**Theorem 4.2.4.** A necessary and sufficient condition for equation (4.1.1) to have a nonoscillatory solution of the type (C) is that

$$1 \sum_{s=n_1}^{n-1} \left| f(s,c) \right| \in dom \phi \quad (4.2.8)$$

for $n \in N(n_1)$ and

$$\sum_{n=n_1}^{\infty} \phi \left( \frac{1}{a_n} \sum_{s=n_1}^{n-1} \left| f(s,c) \right| \right) < \infty \quad (4.2.9)$$

for some nonzero constant $c$. 
Proof: (Necessity) Let \( \{x_n\} \) be a type (C) solution of equation (4.1.1) which is positive for \( n \in N(n_i) \). Then, there is a constant \( c > 0 \) such that \( x_n > c \) for \( n \in N(n_i) \). If \( \Delta x_n > 0 \) for \( n \in N(n_i) \), a summation of equation (4.1.1) shows that

\[
\sum_{n=n_i}^{\infty} f(n, c) \leq \sum_{n=n_i}^{\infty} f(n, x_{n+1}) \leq a_{n_i} \Psi(\Delta x_{n_i}) < \infty
\]

from which both (4.2.8) and (4.2.9) follow easily. If \( \Delta x_n < 0 \) for \( n \in N(n_i) \), then

\[
\Psi(-\Delta x_n) \geq \frac{1}{a_n} \sum_{s=n_i}^{n-1} f(s, x_{s+1}) \geq \frac{1}{a_n} \sum_{s=n_i}^{n-1} f(s, c) \quad (4.2.10)
\]

for \( n \in N(n_i) \), so \( \frac{1}{a_n} \sum_{s=n_i}^{n-1} f(s, c) \in \Psi(\mathbb{R}_+) \subset \text{dom} \phi \) for \( n \in N(n_i) \).

Summing the inequality

\[
-\Delta x_n \geq \phi \left( \frac{1}{a_n} \sum_{s=n_i}^{n-1} f(s, c) \right),
\]

we have

\[
\sum_{n=n_i}^{\infty} \phi \left( \frac{1}{a_n} \sum_{s=n_i}^{n-1} f(s, c) \right) \leq x_{n_i} < \infty.
\]
(Sufficiency) Suppose that the constant $c$ in (4.2.8) and (4.2.9) is positive, and let $n_2 \geq n_1$ be large enough so that

$$\frac{1}{a_n} \sum_{s=n_2}^{n-1} f(s, c) \in \text{dom} \phi$$

for $n \in N(n_2)$ and

$$\sum_{n=n_2}^{\infty} \phi \left( \frac{1}{a_n} \sum_{s=n_2}^{n-1} f(s, c) \right) \leq \frac{c}{2}.$$

The desired positive type (C) solution of equation (4.1.1) will be obtained as a fixed point of the mapping

$$(TX)_n = \frac{c}{2} + \sum_{n=n_2}^{\infty} \phi \left( \frac{1}{a_n} \sum_{s=n_2}^{n-1} f(s, x_{s+1}) \right), \ n \in N(n_2),$$

in the set $S = \left\{ X \in B(n_2): \frac{c}{2} \leq x_n \leq c, n \in N(n_2) \right\}$. The remaining details are omitted since they are similar to those in previous theorems.

We next give sufficient conditions for the existence of a type (B) solution of (4.1.1).
Theorem 4.2.5. Suppose that there are constants \( L_0 > 0, c \neq 0 \)

and \( n_i \in N(n_0) \) such that

\[
\frac{1}{a_n} \sum_{s=n_i}^{n-1} |f(s,c)| \leq L_0 < \sup (\text{dom} \phi), \tag{4.2.11}
\]

for \( n \in N(n_i) \) and

\[
\sum_{n=n_i}^{\infty} \frac{1}{a_n} \sum_{s=n_i}^{n-1} |f(s,c)| < \infty. \tag{4.2.12}
\]

If

\[
\sum_{n=n_i}^{\infty} |f(n,d\rho(n+1))| = \infty \tag{4.2.13}
\]

for every \( d \neq 0 \) with \( cd > 0 \), then equation (4.1.1) has a nonoscillatory solution type (B).

Proof: Suppose that the \( c \) in (4.2.11) and (4.2.12) is positive.

Choose \( L \) such that \( L_0 < L < \sup (\text{dom} \phi) \) and let \( \ell > 0 \) be fixed.

Choose \( n_2 \geq n_i \in N(n_0) \) so that

\[
\frac{1}{a_n} \left( \frac{\ell}{b(L)} + \sum_{s=n_2}^{\infty} f(s,c) \right) \leq L
\]

for \( n \in N(n_2) \) and

\[
c(L) \left( \frac{\ell}{b(L)} \rho(n_2) + \sum_{n=n_2}^{\infty} \frac{1}{a_n} \sum_{s=n_2}^{n-1} f(s,c) \right) \leq c,
\]
where \( b(L) \) and \( c(L) \) are the constants appearing in
\[ b(\alpha)s \leq \phi(s) \leq c(\alpha)s \]
for \( 0 \leq s \leq \alpha \) with \( \alpha = L \). Let \( B(n_2) \) be the same as in the proof of Theorem 3.2.2 and define a subset \( S \) and an operator \( T : S \to B(n_2) \) by
\[
S : \{ X \in B(n_2) : \ell \rho(n) \leq x_n \leq c, n \in N(n_2) \},
\]
\[
(TX)_n = \sum_{s=n}^{\infty} \phi \left( \frac{1}{a_s} \left[ \frac{\ell}{b(L)} + \sum_{t=n_2}^{s-1} f(t, x_{t+1}) \right] \right), \quad n \in N(n_2).
\]
Similar to the proofs of some of our previous theorems, we can show that the mapping satisfies the assumptions of Knaster-Tarski fixed point theorem. Hence, there exists \( X \in S \) such that \( TX = X \); that is, \( \{ x_n \} \) is a nonoscillatory solution of equation (4.1.1) satisfying \( \lim x_n = 0 \). To see that \( \frac{x_n}{\rho(n)} \to \infty \) as \( n \to \infty \), we use Stolz's theorem [10] and (4.2.13) to obtain
\[
\lim_{n \to \infty} \frac{x_n}{\rho(n)} = \lim_{n \to \infty} \frac{\Delta x_n}{\Delta \rho(n)} = \lim_{n \to \infty} a_n \phi \left( \frac{1}{a_n} \left[ \frac{\ell}{b(L)} + \sum_{s=n_2}^{n-1} f(s, x_{s+1}) \right] \right)
\]
\[
\geq \lim_{n \to \infty} \left( \ell + b(L) \sum_{s=n_2}^{n-1} f(s, \ell \rho(s+1)) \right) = \infty.
\]
4.3 Oscillation Theorems

The purpose of this section is to present criteria for the oscillation of all solutions of the equation (4.1.1) when $f$ is either strongly superlinear or strongly sublinear in the sense of definition of 3.3.1.

**Theorem 4.3.1.** Let $f$ be strongly superlinear. All solutions of equation (4.1.1) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty} |f(n, c \rho(n+1))| = \infty$$

(4.3.1)

for every nonzero constant $c$.

**Proof:** The necessity part follows from Theorem 4.2.3. Next, suppose that (4.3.1) holds and that equation (4.1.1) has nonoscillatory solution $\{x_n\}$. It suffices to consider the two cases: (i) $\{x_n > 0, \Delta x_n > 0 \text{ for } n \geq n_1 \in N(n_0)\}$ and

(ii) $\{x_n > 0, \Delta x_n < 0 \text{ for } n \geq n_1 \in N(n_0)\}$.

In the first case, it is easy to see that

$$\sum_{n=n_1}^{\infty} f(n, c_1) < \infty$$
for some $c_i > 0$. Since $\rho(n) \to 0$ as $n \to \infty$, there exists an integer $n_2 \in N(n_1)$ such that $x_n \geq c_i \rho(n)$ for $n \in N(n_2)$. Hence,

$$\sum_{n=n_2}^{\infty} f(n, c_i \rho(n + 1)) < \infty,$$

which contradicts (4.3.1). In case (ii), we first note that (4.2.7) holds. By Lemma 4.2.2 there is a constant $C > 0$ such that $\frac{x_n}{\rho(n)} \geq C$ for $n \in N(n_1)$, so

$$f(n, x_{n+1}) \geq C_2^{-y} f(n, C_2 \rho(n + 1)) \left( \frac{x_{n+1}}{\rho(n+1)} \right)^y, \quad n \in N(n_1) \quad (4.3.2)$$

by the strong superlinearity of $f$. From (4.3.2) and (4.2.7), we have

$$\frac{x_n}{\rho(n)} \geq K C_2^{-y} \sum_{s=n_1}^{n-1} f(s, C_2 \rho(s + 1)) \left( \frac{x_{s+1}}{\rho(s+1)} \right)^y, \quad n \in N(n_1).$$

If we denote the right hand side of the above inequality by $z_n$, we obtain

$$\frac{\Delta z_n}{z_{n+1}^y} \geq K C_2^{-y} f(n, C_2 \rho(n + 1)), \quad n \in N(n_1). \quad (4.3.3)$$

Observe that for $z_n \leq y \leq z_{n+1}$, $\frac{1}{y^y} \geq \frac{1}{z_{n+1}^y}$ and we have
4.3 Oscillation Theorems

\[
\int_{z_n}^{z_{n+1}} \frac{dy}{y^\gamma} \geq \frac{\Delta z_n}{z_{n+1}^\gamma}, \quad n \in N(n_1).
\]

Using the last inequality in (4.3.3) and summing the resulting inequality, we obtain

\[
K C_2^{-\gamma} \sum_{n=n_1}^\infty f(n, C_2 \rho(n+1)) \leq \frac{z_1^{1-\gamma}}{\gamma - 1} < \infty.
\]

This contradicts (4.3.1) and completes the proof of the theorem.

**Theorem 4.3.2.** Let \( f \) be strongly sublinear. All solutions of equation (4.1.1) are oscillatory if

\[
\sum_{n=n_0}^\infty \frac{1}{a_n} \sum_{s=n_0}^{n-1} \left| f(s, c) \right| = \infty \quad (4.3.4)
\]

for every nonzero constant \( c \).

**Proof:** Let \( \{x_n\} \) be a positive solution of equation (4.1.1) for \( n \in N(n_0) \). If \( \Delta x_n > 0 \) for \( n \geq n_1 \in N(n_0) \), then there is a constant \( C_1 > 0 \) such that \( \sum_{n=n_1}^\infty f(s, C_1) < \infty \), which contradicts (4.3.4). If \( \Delta x_n < 0 \) for \( n \in N(n_1) \), then from (4.2.10) we have

\[
-\Delta x_n \geq \frac{K}{a_n} \sum_{s=n_1}^{n-1} f(s, x_{s+1}), \quad n \in N(n_1). \quad (4.3.5)
\]
Now, \( x_n \leq C_2 \) for \( n \in N(n_1) \), for some constant \( C_2 > 0 \), and using the strong sublinearity of \( f \), (4.3.5) implies

\[
-\Delta x_n \geq \frac{K}{a_n} C_2^{-\delta} x_n^{\delta} \sum_{s=n_1}^{n-1} f(s, C_2), \quad n \in N(n_1).
\]

Dividing the above inequality by \( x_n^{\delta} \), we have

\[
\frac{K C_2^{-\delta}}{a_n} \sum_{s=n_1}^{n-1} f(s, C_2) \leq \frac{-\Delta x_n}{x_n^{\delta}} \leq \int \frac{dy}{y^{\delta}}.
\]

Summing the last inequality, we obtain

\[
KC_2^{-\delta} \sum_{n=n_1}^{\infty} \frac{1}{a_n} \sum_{s=n_1}^{n-1} f(s, C_2) \leq \frac{x_{n_1}^{1-\delta}}{1-\delta} < \infty,
\]

which contradicts (4.3.4) and completes the proof of the theorem.

Our final result in this chapter combines Theorems 4.2.5 and 4.3.2 to obtain a necessary and sufficient condition for all solutions of (4.1.1) to be oscillatory in the strongly sublinear case. The proof is immediate.
4.4 Example

**Theorem 4.3.3.** Suppose $f$ is strongly sublinear and conditions (4.2.11) and (4.2.13) hold. Then all solutions of equation (4.1.1) are oscillatory if and only if (4.3.4) holds.

**Remark 4.3.1.** Zhang [90] proved some oscillation theorems for special cases of equation (4.1.1) with (4.2.1) holding but under assumptions somewhat different from those imposed here.

4.4 Example

In this section we provide as example to illustrate the results obtained in this chapter.

**Example 4.4.1.** We consider the difference equation

$$\Delta \left[ 2^n \log \left( \Delta x_n + \sqrt{1 + (\Delta x_n)^2} \right) \right] + q_n |x_{n+1}|^\gamma \text{sgn} x_{n+1} = 0, \ n \in \mathbb{N}(1) \quad (4.4.1)$$

where $\gamma > 0$ and $\{q_n\}$ is a positive real sequence. Since the inverse function of $\Psi(s) = \log \left( s + \sqrt{1 + s^2} \right)$ is $\phi(s) = \sinh s$ and this clearly satisfies (4.2.3), the theorems in Section 4.2 can be applied to equation (4.4.1). Note that the sequence $\{\rho(n)\}$
4.4 Example

defined by (4.2.2) is $\rho(n) = 2^{-n}$ and we have the following results.

(i) Equation (4.4.1) has a nonoscillatory solution $\{x_n\}$ satisfying

$$0 < \liminf_{n \to \infty} 2^n |x_n|, \quad \limsup_{n \to \infty} 2^n |x_n| < \infty$$

if and only if

$$\sum_{n=1}^{\infty} 2^{-\gamma(n+1)} q_n < \infty. \quad (4.4.2)$$

(ii) Suppose that

$$\sum_{s=1}^{n-1} q_s = O(2^n) \text{ as } n \to \infty \quad (4.4.3)$$

and

$$\sum_{n=1}^{\infty} \sinh \left(2^{-n} \sum_{s=1}^{n-1} q_s \right) < \infty. \quad (4.4.4)$$

Then equation (4.4.1) has a nonoscillatory solution $\{x_n\}$ such that

$$\lim_{n \to \infty} x_n = \text{constant} \neq 0.$$ 

(iii) Suppose, in addition (4.4.2), that
Then (4.4.1) has a nonoscillatory solution \( \{x_n\} \) such that

\[
\lim_{n \to \infty} x_n = 0 \quad \text{and} \quad \lim_{n \to \infty} 2^n |x_n| = \infty.
\]

(iv) All solutions of (4.4.1) with \( \gamma > 1 \) are oscillatory if and only if (4.4.6) holds.

(v) Suppose (4.4.3) and (4.4.6) hold. All solutions of (4.4.1) with \( 0 < \gamma < 1 \) are oscillatory if

\[
\sum_{n=1}^{\infty} 2^{-n} \sum_{s=1}^{n-1} q_s = \infty.
\]