CHAPTER 3

OSCILLATION OF QUASILINEAR SECOND ORDER DIFFERENCE EQUATION-I

3.1 Introduction

In this chapter we study the oscillatory and nonoscillatory behavior of solutions of second order quasilinear difference equations of the type

\[ \Delta\left(a_n \Psi(\Delta x_n)\right) + f(n, x_{n+1}) = 0, \quad n \in N(n_0), \]  

(3.1.1)
where \( N(n_0) = \{n_0, n_0 + 1, \ldots \} \), \( n_0 \in \mathbb{N} = \{0, 1, 2, \ldots \} \), \( \{a_n\} \) is a real sequence with \( a_n > 0 \) for all \( n \in N(n_0) \), \( \lim_{n \to \infty} a_n = \infty \), \( \Psi : \mathbb{R} \to \mathbb{R} \) is continuous and strictly increasing with \( \Psi(-s) = -\Psi(s) \), and \( f : N(n_0) \times \mathbb{R} \to \mathbb{R} \) is continuous, \( uf(n, u) > 0 \) for \( u \neq 0 \), and \( f(n, u) \) is nondecreasing in \( u \) for each fixed \( n \in N(n_0) \). By a solution of equation (3.1.1) we mean a nontrivial sequence \( \{x_n\} \) satisfying equation (3.1.1) for all \( n \in N(n_0) \).

While equation (3.1.1) can be considered as a discrete analogue of the differential equation

\[
(a(t)\Psi(x'))' + f(t, x) = 0,
\]

(3.1.2)

it is known [32] that some properties of equations of the type (3.1.2) (for example, when \( \Psi(u) = u \), \( f(t, u) = p(t)u' \), and \( \gamma \) is a quotient of odd positive integers) do not carry over directly to the corresponding difference equation. Furthermore, it is known [37] that oscillation results obtained for equation (3.1.2) can be applied to derive similar properties for solutions of certain
partial differential equations. Hence, the study of oscillatory and asymptotic behavior of solutions of (3.1.1) extends beyond the obvious self interest.

Our objective is to establish criteria for equation (3.1.1) to have various types of nonoscillatory solutions, as well as criteria for all solutions to be oscillatory when \( \sum_{n=n_0}^{\infty} \frac{1}{\alpha_n} = \infty \). The desired results will be developed in Sections 3.2 and 3.3. Example illustrating the results are provided in Section 3.4.

### 3.2 Nonoscillation Theorems

In this section we observe that \( \psi \) has an inverse defined on \( \psi(\mathbb{R}) \), and we will denote this inverse function by \( \phi \). It is clear that \( \phi(-s) = -\phi(s) \) and \( \phi(s) \) is strictly increasing for all \( s \in \text{dom} \phi = \psi(\mathbb{R}) \). Moreover, for any positive \( \alpha \in \text{dom} \phi \) there exist positive constants \( b(\alpha) \) and \( c(\alpha) \) such that

\[
b(\alpha)s \leq \phi(s) \leq c(\alpha)s \quad \text{for } 0 \leq s \leq \alpha.
\] (3.2.1)

Define
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\[ R(n,s) = \sum_{i=s}^{n-1} \frac{1}{a_i} \] for \( n \geq s \in N(n_0) \) and \( R(n) = R(n,n_0) \). (3.2.2)

Clearly, \( R(n,s) \to \infty \) as \( n \to \infty \) for any fixed \( s \in N(n_0) \).

**Lemma 3.2.1.** Each nonoscillatory solution \( \{x_n\} \) of equation (3.1.1) must belong to one of the following three types:

(I) \( \lim_{n \to \infty} x_n / R(n) = \text{constant} \neq 0 \);

(II) \( \lim_{n \to \infty} x_n / R(n) = 0 \) and \( \lim_{n \to \infty} |x_n| = \infty \);

(III) \( \lim_{n \to \infty} x_n = \text{constant} \neq 0 \).

**Proof:** Let \( \{x_n\} \) be a nonoscillatory solution of equation (3.1.1). With no loss in generality, we may assume that \( x_n > 0 \) for \( n \geq n_1 \in N(n_0) \). From equation (3.1.1), it follows that \( \Delta(a_n \Psi(\Delta x_n)) < 0 \) for \( n \in N(n_1) \), and therefore \( \{a_n \Psi(\Delta x_n)\} \) is a decreasing sequence for \( n \in N(n_1) \). We claim that \( a_n \Psi(\Delta x_n) > 0 \) for \( n \in N(n_1) \). If \( a_{n_2} \Psi(\Delta x_{n_2}) = -k < 0 \) for some \( n_2 \geq n_1 \) and \( k > 0 \), then \( a_n \Psi(\Delta x_n) \leq -k \) for \( n \in N(n_2) \), so

\[ \Psi(\Delta x_n) \leq -k / a_n \] for \( n \in N(n_2) \). (3.2.3)
Since \( k / a_n \to 0 \) as \( n \to \infty \), for any fixed positive \( \alpha \in \text{dom}\phi \), there exists \( n_3 \geq n_2 \) such that \( k | a_n | \leq \alpha \) for \( n \in N(n_j) \). From (3.2.3) and (3.2.1) we have

\[
\Delta x_n \leq -\phi(k/a_n) \leq \frac{-b(\alpha)k}{a_n}
\]

for \( n \in N(n_j) \). Summing the last inequality from \( n_j \) to \( n-1 \) and letting \( n \to \infty \), we have \( x_n \to -\infty \), which is a contradiction. Hence,

\[
a_n \Psi(\Delta x_n) > 0 \text{ for } n \in N(n_j) \text{ and } \lim_{n \to \infty} a_n \Psi(\Delta x_n) = \text{constant} \geq 0.
\]

This implies that \( \Psi(\Delta x_n) \to 0 \) and \( \Delta x_n \to 0 \) as \( n \to \infty \) since \( a_n \to \infty \) as \( n \to \infty \).

Suppose that \( \lim_{n \to \infty} a_n \Psi(\Delta x_n) = c > 0 \): then by Stolz’s theorem [10] we have

\[
\lim_{n \to \infty} \frac{x_n}{R(n)} = \lim_{n \to \infty} a_n \Delta x_n = \lim_{n \to \infty} a_n \Psi(\Delta x_n) \frac{\Delta x_n}{\Psi(\Delta x_n)}
\]

\[
= c \lim_{t \to 0} \frac{t}{\Psi(t)} = c \Delta \phi(0) > 0,
\]

which implies that \( \{x_n\} \) is a type (I) solution.
If \( \lim_{n \to \infty} a_n \Psi(\Delta x_n) = 0 \), then, as above, we have \( \lim_{n \to \infty} \frac{x_n}{R(n)} = 0 \).

Since \( \Delta x_n > 0 \) for \( n \in N(n_0) \), \( \{x_n\} \) is increasing and either tends to \( \infty \) or to a finite limit as \( n \to \infty \). In the former case \( \{x_n\} \) is of type (II), and in the latter case \( \{x_n\} \) is of type (III). This completes the proof of the lemma.

**Theorem 3.2.2.** A necessary and sufficient condition for equation (3.1.1) to have a nonoscillatory solution \( \{x_n\} \) that belongs to type (I) is that

\[
\sum_{n=n_0}^{\infty} \left| f \left( n, c R (n+1) \right) \right| < \infty \tag{3.2.4}
\]

for some nonzero constant \( c \).

**Proof:** (Necessity) Let \( \{x_n\} \) be a nonoscillatory type (I) solution of equation (3.1.1). We may suppose that \( x_n > 0 \) for \( n \geq n_1 \in N(n_0) \) since a similar argument holds if \( \{x_n\} \) is eventually negative. Summing equation (3.1.1), we obtain

\[
\sum_{s=n_1}^{\infty} f(s, x_{s+1}) < a_{n_1} \Psi(\Delta x_{n_1}).
\]
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Now \( x_n \geq c_i R(n) \) for \( n \in N(n_i) \) and some constant \( c_i > 0 \) we have

\[
\sum_{s=n_i}^{\infty} f(s, c_i R(s + 1)) < \infty.
\]

(Sufficiency): Assume that (3.2.4) holds with \( c > 0 \); a similar argument can be used if \( c < 0 \). Let \( \alpha \) be a fixed constant in \( \text{dom}\phi \) and let \( b(\alpha) \) and \( c(\alpha) \) be the constants appearing in (3.2.1). Choose constants \( k \geq 1 \) and \( \ell > 0 \) such that

\[
(k + 1) \ell \leq c
\]  
(3.2.5)

and choose \( n_i \in N(n_0) \) so large that

\[
\frac{1}{a_n} \left( \frac{\ell}{b(\alpha)} + \sum_{s=n_i}^{\infty} f(s, (k + 1) \ell R(s + 1, n_i)) \right) \leq \alpha
\]  
(3.2.6)

for \( n \in N(n_i) \) and

\[
c(\alpha) \left( \frac{\ell}{b(\alpha)} + \sum_{s=n_i}^{\infty} f(s, (k + 1) \ell R(s + 1, n_i)) \right) \leq (k + 1) \ell.
\]  
(3.2.7)

Let \( B(n_i) \) be the Banach space of all real sequences \( X = \{x_n\}, n \in N(n_i) \), with norm \( \|X\| = \sup_{n \in N(n_i)} |x_n| \). We define a
partial ordering on $B(n_1)$ as follows: for given $Y, X \in B(n_1)$, $Y \leq X$ means that $y_n \leq x_n$ for $n \in N(n_1)$. Let

$$S = \{ X \in B(n_1) : \ell R(n, n_1) \leq x_n \leq (k+1) \ell R(n, n_1), n \in N(n_1) \}$$

and define $T : S \to B(n_1)$ by

$$(TX)_n = \sum_{s=n_1}^{n-1} \left( \frac{1}{a_s} \left[ \frac{\ell}{b(\alpha)} + \sum_{t=s}^{\infty} f(t, x_{t+1}) \right] \right), n \in N(n_1).$$

From (3.2.5)-(3.2.7), it is clear that $TS \subseteq S$ and that $T$ is an increasing mapping. Hence, by the Knaster-Tarski fixed point theorem [47], there exists $X \in S$ such that $TX = X$, i.e.,

$$(X)_n = x_n = \sum_{s=n_1}^{n-1} \phi \left( \frac{1}{a_s} \left[ \frac{\ell}{b(\alpha)} + \sum_{t=s}^{\infty} f(t, x_{t+1}) \right] \right), n \in N(n_1).$$

It follows that $\{x_n\}$ is a nonoscillatory solution of (3.1.1) and satisfies $x_n / R(n)$ approaches a positive constant as $n \to \infty$. This completes the proof of the theorem.

**Theorem 3.2.3.** A necessary and sufficient condition for equation (3.1.1) to have a nonoscillatory solution $\{x_n\}$ belonging to type (III) is that
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\[
\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} |f(n, c)| < \infty \tag{3.2.8}
\]

for some nonzero constant \(c\).

**Proof:** (Necessity) Suppose that equation (3.1.1) has a positive solution \(\{x_n\}\) of type (III). Then \(\Delta x_n > 0\) eventually and \(\lim_{n \to \infty} \Delta x_n = 0\). Let \(\alpha > 0\) be in \(\text{dom} \phi\) and let \(b(\alpha)\) satisfy the left half of the inequality in (3.2.1). For \(C_3 = 1/b(\alpha)\) there exists \(n_1 \in N(n_0)\) such that \(0 < C_3 \Delta x_n \leq \alpha\) for \(n \geq n_1\). Letting \(s = C_3 \Delta x_n\) in the left half of (3.2.1), we have \(b(\alpha) C_3 \Delta x_n = \Delta x_n \leq \phi(C_3 \Delta x_n)\).

Hence, there exist positive constants \(C_1, C_2\) and \(C_3\) such that

\[
C_1 \leq x_n \leq C_2 \quad \text{and} \quad \Psi(\Delta x_n) \leq C_3 \Delta x_n \quad \text{for} \quad n \in N(n_1). \tag{3.2.9}
\]

Summing equation (3.1.1), using (3.2.9) and the fact that \(a_n \Psi(\Delta x_n) \to 0\) as \(n \to \infty\), we have

\[
\frac{1}{a_n} \sum_{s=n}^{\infty} f(s, C_1) \leq \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, x_{s+1}) = \Psi(\Delta x_n) \leq C_3 \Delta x_n
\]

for \(n \in N(n_1)\). Summing once again, we obtain
\[
\sum_{n=\text{n}_1}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, C_i) \leq C_2 C_3,
\]

which implies (3.2.8). The proof when \( \{x_n\} \) is eventually negative is similar.

(Sufficiency) Assume that (3.2.8) holds with \( c > 0 \); a similar argument holds if \( c < 0 \). Let \( \alpha \in \text{dom} \phi \), with \( \alpha > 0 \), be fixed.

Choose \( n_1 \in N(n_0) \) so large that

\[
\frac{1}{a_n} \sum_{s=n}^{\infty} f(s, c) \leq \alpha
\]

for \( n \in N(n_1) \) and

\[
c(\alpha) \sum_{n=n_1}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, c) \leq \frac{c}{2}.
\]

Let \( B(n_1) \) be the same as in the proof of Theorem 3.2.2 and let

\[S = \left\{ X \in B(n_1): \frac{c}{2} \leq x_n \leq c, n \in N(n_1) \right\}\]

Define \( T: S \to B(n_1) \) by

\[(TX)_n = \frac{c}{2} + \sum_{s=n_1}^{n-1} \phi \left( \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, x_{t+1}) \right), \quad n \in N(n_1).\]
As in the proof of Theorem 3.2.2, we can show that the mapping $T$ satisfies the assumptions of the Knaster-Tarski fixed point theorem. Therefore, there exists $X \in S$ such that $TX = X$, i.e., $\{x_n\}$ is a nonoscillatory solution of equation (3.1.1) satisfying $\lim_{n \to \infty} x_n = \frac{c}{2} > 0$. The proof of the theorem is now complete.

**Corollary 3.2.4.** If condition (3.2.8) is replaced by

$$\sum_{n=n_0}^{\infty} R(n) |f(n,c)| < \infty,$$

for some nonzero constant $c$, then Theorem 3.2.3 remains true.

Next, we give sufficient conditions for the existence of a type (II) solution of equation (3.1.1).

**Theorem 3.2.5.** Equation (3.1.1) has a nonoscillatory solution of type (II) if

$$\sum_{n=n_0}^{\infty} |f(n,cR(n+1))| < \infty \quad (3.2.10)$$

for some nonzero constant $c$, and
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\[ \sum_{n=n_0}^{\infty} R(n) |f(n, d)| = \infty \quad (3.2.11) \]

for all nonzero constants \( d \) with \( cd > 0 \).

**Proof:** Assume that \( c > 0 \) in (3.2.10) and let \( \alpha \in \text{dom} \phi \) and \( \ell \in (0, c) \) be fixed. Choose \( n_1 \in N(n_0) \) such that

\[ \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, \ell + \ell R(s + 1, n_1)) \leq \alpha \]

for \( n \in N(n_1) \) and

\[ c(\alpha) \sum_{n=n_1}^{\infty} f(n, \ell + \ell R(n + 1, n_1)) \leq \ell. \]

Let the Banach space \( B(n_1) \) be the same as in the proof of Theorem 3.2.2 and define \( S \subseteq B(n_1) \) and an operator \( T : S \to B(n_1) \) by

\[ S = \{ X \in B(n_1) : \ell \leq x_n \leq \ell + \ell R(n, n_1), n \in N(n_1) \} \]

\[ (TX)_n = \ell + \sum_{s=n_1}^{n-1} \phi \left( \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, x_{i+1}) \right), \quad n \in N(n_1). \]

Similar to the proof of Theorem 3.2.2, we can show that the mapping \( T \) satisfies the hypotheses of the Knaster-Tarski fixed point theorem. Hence, there exists \( X \in S \) such that
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\[ x_n = \ell + \sum_{s=n_1}^{n-\ell} \phi \left( \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, x_{t+1}) \right). \]

This is clearly a nonoscillatory solution of equation (3.1.1).

Furthermore,

\[ \Psi(\Delta x_n) = \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, x_{s+1}) \leq \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, \ell + \ell R(s+1)) \]  
(3.2.12)

for \( n \in N(n_1) \) and

\[ x_n \geq \ell + b(\alpha) \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, x_{t+1}), \]  
(3.2.13)

for \( n \in N(n_1) \). From (3.2.12), we see that \( \lim_{n \to \infty} x_n = 0 \), and from (3.2.13) we conclude that \( \lim_{n \to \infty} x_n = \infty \). Thus, \( \{x_n\} \) is a nonoscillatory type (II) solution of equation (3.1.1).

**Remark 3.2.1.** In a recent paper, He [33] considered the problem of finding necessary and sufficient conditions for solutions of the special case of (3.1.1) where \( \Psi(u)=u \) to have nonoscillatory solutions with a prescribed asymptotic behavior different from those discussed here. Due to the fact that He
requires more restrictive conditions on $f$ but less restrictive conditions on $\{a_n\}$, direct comparisons are not easily made.

### 3.3 Oscillation Theorems

In this section we study the oscillatory behavior of solution of equation (3.1.1). In view of the results of Hooker and Patula [32] and Zhang [96], it is reasonable to expect that a characterization of oscillation for equation (3.1.1) can be obtained under suitable additional conditions on the nonlinear functions $f$ and $\Psi$.

**Definition 3.3.1.** (i) The function $f(n,u)$ is said to be strongly superlinear if there is a constant $\gamma > 1$ such that $|u|^\gamma f(n,u)$ is nondecreasing in $u$ for each fixed $n \in N(n_0)$.

(ii) The function $f(n,u)$ is said to be strongly sublinear if there is a constant $\delta, 0 < \delta < 1$, such that $|u|^{-\delta} f(n,u)$ is nonincreasing in $u$ for each fixed $n \in N(n_0)$. 
Theorem 3.3.2. Let \( f(n,u) \) be strongly superlinear. All solutions of equations (3.1.1) are oscillatory if and only if

\[
\sum_{n=n_0}^{\infty} R(n)|f(n,c)| = \infty
\]  

(3.3.1)

for every nonzero constant \( c \).

Proof: The necessary part follows from Theorem 3.2.3. To prove sufficiency, suppose that equation (3.1.1) has a nonoscillatory solution \( \{x_n\} \), say \( x_n > 0 \) for \( n \geq n_1 \in N(n_0) \). As in the proof of Lemma 3.2.1, \( \Delta x_n > 0 \) for \( n \in N(n_1) \) and \( \Delta x_n \rightarrow 0 \) as \( n \rightarrow \infty \). Again, there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
x_n \geq C_1 \text{ and } \Psi(\Delta x_n) \leq C_2 \Delta x_n \text{ for } n \in N(n_1).
\]

(3.3.2)

A summation of equation (3.1.1) yields

\[
a_n \Psi(\Delta x_n) = \sum_{s=n}^{\infty} f(s, x_{s+1})
\]

for \( n \in N(n_1) \), and in view of (3.3.2), this implies

\[
C_2 \Delta x_n \geq \frac{1}{a_n} \sum_{s=n}^{\infty} f(s, x_{s+1})
\]

(3.3.3)

for \( n \in N(n_1) \). Now, from the strong superlinearity of \( f \) and the
first inequality in (3.3.2), it follows that

\[ f(n, x_{n+1}) - \frac{x_{n+1}^{-r}}{a_n} \leq \frac{x_{n+1}^{r}}{a_n} \geq C_1^{-r} f(n, C_1) \]

(3.3.4)

Using (3.3.4) in (3.3.3) and noting that \( \{x_n\} \) is increasing, we have

\[ C_2 \Delta x_n \geq \frac{C_1^{-r}}{a_n} \sum_{s=n}^{\infty} f(s, C_1) x_{s+1}^{r} \geq \frac{x_{n+1}^{r} C_1^{-r}}{a_n} \sum_{s=n}^{\infty} f(s, C_1) \]

for \( n \in \mathbb{N}(n_1) \). Dividing the last inequality by \( x_{n+1}^{r} \) and summing from \( n_1 \) to \( n-1 \), we obtain

\[ C_1^{-r} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, C_1) = C_1^{-r} \sum_{s=n_1}^{n-1} R(s) f(s, C_1) \leq \sum_{s=n_1}^{n-1} \frac{C_2 \Delta x_s}{x_{s+1}^{r}} \]

(3.3.5)

Since \( 1/x_n^{r} \geq x_{n+1}^{r} \) for \( x_n \leq y \leq x_{n+1} \), we have

\[ \Delta \frac{x_n}{x_{n+1}^{r}} \leq \int_{x_n}^{x_{n+1}} \frac{dy}{y^{r}}. \]

(3.3.6)

Substituting (3.3.6) into (3.3.5), we obtain
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\[ C_1^{-\gamma} \sum_{s=n_1}^{n-1} R(s) f(s,C_1) \leq C_2 \int_{x_n}^{x} \frac{dy}{y^{\gamma}} \leq \frac{C_2 x_1^{1-\gamma}}{\gamma - 1} < \infty, \]

which is a contradiction. This completes the proof of the theorem.

**Theorem 3.3.3.** Let \( f(n,u) \) be strongly sublinear. All solutions of equation (3.1.1) are oscillatory if and only if

\[ \sum_{n=n_0}^{\infty} \left| f(n,cR(n+1)) \right| = \infty \quad (3.3.7) \]

for every nonzero constant \( c \).

**Proof:** First, note that the necessity of condition (3.3.7) follows from Theorem 3.2.2. Next, let \( \{x_n\} \) be a nonoscillatory solution of equation (3.1.1), say \( x_n > 0 \) for \( n \geq n_1 \in N(n_0) \). As in the proof of Theorem 3.3.2, inequality (3.3.3) holds for some constant \( C_2 > 0 \). Letting \( n_2 > n_1 \) be fixed and summing (3.3.3) from \( n_1 \) to \( n \in N(n_2) \), we have

\[ C_3 x_{n+1} \geq R(n+1) \sum_{s=n}^{\infty} f(s,x_{s+1}) \quad (3.3.8) \]

for some constant \( C_3 > 0 \). Since there is a constant \( C_4 > 0 \) such
that \( x_n \leq C_4 R(n) \) for \( n \in N(n_2) \), the strong sublinearity of \( f \) implies that

\[
 f(n, x_{n+1}) = x_{n+1}^{-\delta} f(n, x_{n+1}) x_{n+1}^{\delta} \\
\geq C_4^{-\delta} f(n, C_4 R(n+1)) \left( \frac{x_{n+1}}{R(n+1)} \right)^{\delta}, n \in N(n_2). \tag{3.3.9}
\]

Combining (3.3.8) with (3.3.9), we have

\[
 C_3 \frac{x_{n+1}}{R(n+1)} \geq C_4^{-\delta} \sum_{s=n}^{\infty} f(s, C_4 R(s+1)) \left( \frac{x_{s+1}}{R(s+1)} \right)^{\delta}, n \in N(n_2).
\]

Denoting the right hand side of last inequality by \( z_n \), we see that

\[
 (C_4 C_3)^{-\delta} f(n, C_4 R(n+1)) \leq \frac{z_n - z_{n+1}}{z_{n+1}^{\delta}} \leq (1 - \delta) \int_{z_{n+1}}^{z_n} \frac{dy}{y^{\delta}}, n \in N(n_2).
\]

A summation of the above inequality shows that

\[
 \sum_{n=n_2}^{\infty} f(n, C_1 R(n+1)) < \infty,
\]

which contradicts (3.3.7). This completes the proof of the theorem.
3.4 Example

In this section we provide an example to illustrate the results obtained in this chapter.

Example 3.4.1. Consider the difference equation

\[ \Delta(n \sinh \Delta x_n) + q_n |x_{n+1}|^\gamma \sgn x_{n+1} = 0, \quad n \in N(1), \]  

(3.4.1)

where \( \{q_n\} \) is a positive real sequence and \( \gamma > 0 \) is a constant.

The sequence \( R(n) \) defined in Section 3.2 is

\[ R(n) = \sum_{s=1}^{n-1} \frac{1}{n}. \]

Applying the results in Section 3.2 to equation (3.4.1), we have the following results:

(i) There exists a nonoscillatory solution \( \{x_n\} \) of (3.4.1)

such that \( \lim_{n \to \infty} x_n / R(n) = \text{constant} \neq 0 \) if and only if

\[ \sum_{n=1}^{\infty} q_n R^n (n + 1) < \infty. \]  

(3.4.2)

(ii) There exists a nonoscillatory solution \( \{x_n\} \) of (3.4.1)

such that \( \lim_{n \to \infty} x_n = \text{constant} \neq 0 \) if and only if
\[
\sum_{n=1}^{\infty} q_n R(n) < \infty. \quad (3.4.3)
\]

(iii) There exists a nonoscillatory solution \( \{x_n\} \) of (3.4.1) such that \( \lim_{n \to \infty} x_n / R(n) = 0 \) and \( \lim_{n \to \infty} x_n = \infty (\pm \infty) \) if (3.4.2) holds and

\[
\sum_{n=1}^{\infty} q_n R(n) = \infty. \quad (3.4.4)
\]

(iv) All solutions of (3.4.1) with \( \gamma > 1 \) are oscillatory if and only if (3.4.4) holds.

(v) All solutions of (3.4.1) with \( 0 < \gamma < 1 \) are oscillatory if and only if

\[
\sum_{n=1}^{\infty} q_n R^\gamma(n + 1) = \infty. \quad (3.4.5)
\]