Chapter 0

Introduction

0.1 Fuzzy Sets

Among the various paradigmatic changes in science and mathematics, one such change concerns the concept of uncertainty. We deal problems, in general, in terms of systems that are constructed as models of some aspects of reality. In constructing a model we always attempt to maximize its usefulness and this aim is closely connected with the relationship among the three key characteristics of every systems model such as complexity, credibility and uncertainty. In general, allowing more uncertainty tends to reduce complexity and increase credibility of the resulting model. Identification of this important role of uncertainty by some researchers began the stage of transition from the traditional view to the modern view of uncertainty and such transition is characterized by the emergence of several new theories of
uncertainty distinct from probability theory.

An important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh [79]. In his paper Zadeh introduced a theory whose objects - **fuzzy sets** - are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of a degree. The significance of his concept was that it challenged not only probability theory but also the very foundations upon which probability theory is based: Aristotelian two-valued logic. When $\lambda$ is a fuzzy set and $x$ is relevant object, the proposition “$x$ is a member of $\lambda$” is not necessarily either true or false, as required by two-valued logic, but it may be true only to some degree, the degree to which $x$ is actually a member of $\lambda$. It is most common, but not required, to express degrees of membership in fuzzy set as well as degrees of truth of the associated propositions by numbers in the closed unit interval $[0, 1]$. The extreme values in this interval 0 and 1, then represent, respectively, the total denial and affirmation of the membership in given fuzzy set, as well as the falsity and truth of the associated proposition.

The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and non-members (those that certainly do not belong in the set).
A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. The basic concept of fuzzy set is nothing but a generalization of the crisp set.

0.2 Review of Literature

The fundamental concept of fuzzy sets introduced by Zadeh [79] provided a natural foundation for building new branches. Fuzzy sets have application in applied fields such as information [65], control [70, 78] and pattern recognition [30, 40]. This concept has been applied by many authors to several branches of mathematics particularly in the fields such as fuzzy numbers [14], fuzzy groups [58], fuzzy topological groups [18], L-fuzzy sets [21], fuzzy linear spaces [22], fuzzy algebra [29], fuzzy vector spaces [33] and fuzzy proximity spaces [34]. One of the applications was the study of fuzzy topological spaces [11] introduced and studied by Chang in 1968. In his paper various notions such as fuzzy topological spaces, fuzzy open (closed) set, fuzzy neighbourhood, interior (closure) of a fuzzy set, fuzzy continuity and fuzzy compactness were defined and studied. After the introduction of fuzzy sets and fuzzy topological spaces many fuzzy topologists have contributed to this theory like in [2, 4, 5, 6, 7, 8, 10, 12, 24, 25, 32, 37, 42, 43, 51, 52, 53, 54, 55, 56, 57, 75, 81] by means of generalization of concepts of
general topology into fuzzy setting. However, generalization of certain general topological structures failed to carry their properties in fuzzy setting. For example,

1. In point-set topology the closure of the product is the product of the closures (which leads to the general fact that every topological space is product related to any other topological space). But this is not so in fuzzy setting ([2] cf. Example 3.5). In order to declare this as true in fuzzy setting, the concept of "a fuzzy topological space product related to any other topological space" was introduced ([2] cf. Definition 3.7).

2. In ordinary topological setting the intersection of a semiopen set with an open set is a semiopen set, but the intersection (union) of a fuzzy semiopen (semiclosed) set with a fuzzy open (closed) set need not be a fuzzy semiopen (semiclosed) set ([2] cf. Remark 4.4).

3. Consider the following theorem ([2] cf. Theorem 6.8): "Let $f : X \to Y$ be a mapping from a fuzzy topological space $X$ to another fuzzy topological space $Y$. Then if the graph $g : X \to X \times Y$ of $f$ is fuzzy semicontinuous, $f$ is also fuzzy semicontinuous". The converse of this theorem is false. However in ordinary topological setting the converse is also true ([2] cf. Remark 6.9).

4. The intersection of fuzzy open set and fuzzy $\beta$-open set need not be a
fuzzy $\beta$-open set [44]: a contrast to the situation in general topology [1].

Ever since the introduction of fuzzy topological spaces various notions like fuzzy semi (respectively almost, weakly) continuity [2], fuzzy strong semi-continuity [9], fuzzy connectedness [17], stronger forms of fuzzy disconnectedness [5], fuzzy semiconnectedness [64], and fuzzy compactness [42] were extended from general topological structures. We may note that at the earlier stage some of the extensions were not only defined new concepts but also acted as a tool to examine the preservation of certain fuzzy topological properties (invariants) like fuzzy connectedness and fuzzy disconnectedness. For example,

1. Fuzzy continuous (respectively almost continuous, weakly continuous, strongly semicontinuous) image of a fuzzy connected space is fuzzy connected ([10] cf. Theorem 2.3 to Theorem 2.6).

2. Fuzzy continuous image of a fuzzy super connected (fuzzy strongly connected) space is fuzzy super (fuzzy strongly) connected ([17] cf. Theorem 6.5 and Theorem 8.4).

3. Fuzzy almost continuous image of a fuzzy super connected space is fuzzy super connected ([10] cf. Theorem 3.3).

On the other hand,
1. Fuzzy connectedness is preserved neither by semicontinuous mappings nor by fuzzy precontinuous mappings ([10] cf. Remark 2.8).

2. Fuzzy super connectedness is preserved neither by fuzzy weakly continuous mappings nor by fuzzy precontinuous mappings ([10] cf. Remark 3.6).

3. Fuzzy strong connectedness is not preserved by any one of the following weaker forms of fuzzy continuity: fuzzy almost continuity, fuzzy weakly continuity, fuzzy strong continuity, fuzzy semicontinuity and fuzzy precontinuity ([10] cf. Remark 3.4).

Also, in the study of extensions of fuzzy topologies, even the question, like "If a fuzzy topological space \((X, T)\) has a property \(P\) under what conditions will \((X, T^*)\) where \(T^*\) is a simple extension of \(T\) also have the same property?" was also answered ([4] cf. Theorems 2,3,4,5 and 6). On its further development and motivated by the fact that there are some non-symmetric fuzzy topological structures, Kubiak [38] in 1983 introduced the bitopological aspects in the theory of fuzzy topological spaces, particularly concerned with fuzzy bitopological spaces related to fuzzy quasi-proximity spaces. He also established that such fuzzy bitopological spaces admits a fuzzy quasi-proximity \(iff\) it is pathwise fuzzy completely regular. On the other hand,
in 1989 Kandil [31] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces [11] and an extension of bitopological spaces [35] in fuzzy setting. Thereafter many fuzzy topologists have introduced and studied several concepts like fuzzy quasi-proximities [38], connectedness and semiconnectedness [62], pairwise fuzzy continuity [73], pairwise fuzzy connectedness between fuzzy sets [74] in fuzzy bitopological spaces using quasi-coincident for fuzzy sets. Also notions like, fuzzy pairwise α-continuity (precontinuity) using fuzzy α-open sets (preopen sets) [60], pairwise fuzzy irresolute mappings [73] and semiopen sets, semicontinuity and semiopen mappings in fuzzy bitopological spaces [59] were defined and their properties were also investigated. Based on the concepts like fuzzy semiopen sets [9], fuzzy β-open sets [7], fuzzy regular open sets [2], fuzzy connectedness [17] (respectively super connectedness, strongly connectedness) fuzzy disconnectedness [5] (respectively total disconnectedness, extremally disconnectedness) generalized fuzzy open [6] (closed) sets and the results related to them, we generalize and extend some of them in both fuzzy topological spaces and fuzzy bitopological spaces.
0.3 Basic Assumptions

In this thesis we use the notion of fuzzy topological space in the original sense of Chang [11] and not in the modified sense of Lowen [42] and the notion of fuzzy bitopological space in the original sense of Kandil [31] and not in the modified sense of Kubiak [38].

0.4 Outline of the Thesis

The concept of totally continuous functions was introduced in [26] and as a consequence of this totally semicontinuous function was defined and studied in [48]. In Chapter 1 we have two parts. In the first part we have introduced and studied the concept of totally fuzzy semicontinuous function as a generalization of perfectly fuzzy continuous function defined and studied in [6]. In the second part a new class of mapping called almost fuzzy semicontinuous function is defined as a generalization of the concept introduced by B.M. Munshi and D.S. Bassan [45] in fuzzy setting and some of its properties are also studied.

The concept of $\alpha$-open set was introduced and studied by Njastad [47] and further this concept in fuzzy setting was defined by Bin Shahna [10] with the introduction of fuzzy $\alpha$-open sets. In Chapter 2, using the concept of fuzzy $\alpha$-open set we defined various notions of fuzzy $\alpha$-connectedness
and fuzzy $\alpha$-disconnectedness in fuzzy topological spaces and employing the same means, we studied questions similar to those of [17] and [5].

Chapter 3 deals with the extension of the notions of connectedness and disconnectedness [49], total disconnectedness [71] and extremally disconnectedness [3] in fuzzy setting. In the same spirit two weaker forms of connectedness namely, pairwise fuzzy super connectedness and pairwise fuzzy strong connectedness are introduced and studied in fuzzy bitopological spaces. Also two stronger forms of disconnectedness namely, pairwise fuzzy total disconnectedness and pairwise fuzzy extremally disconnectedness in fuzzy bitopological spaces are defined and studied. Some characterizations and properties of these spaces are also established.

In the study of bitopological spaces extremally disconnected spaces were defined and studied in [23] where it is pointed out that the concept arose earlier in a paper of Stone [67]. The importance of these spaces lies in their connection with the completeness of $C(X)$ as a lattice. Basically disconnected spaces introduced in [20] also arose in this connection [46, 67]. Based on Gillman and Jerison’s [20] concept of basic disconnectedness for topological spaces, the corresponding notion for bitopological spaces was defined and studied in [63] using $F_\sigma$-set. In Chapter 4 an exclusive and an important space called pairwise fuzzy basically disconnected space is introduced and some of its properties and characterizations are also derived. Some of the
results of [63] to fuzzy bitopological structures have been extended.

The concept of $\beta$-open sets was introduced in [1] and has been generalized to fuzzy setting in [7]. In Chapter 5 we generalized the concepts of [61] in fuzzy setting with the introduction of pairwise fuzzy $\beta$-open set in fuzzy bitopological spaces. Also pairwise fuzzy $\beta$-open (closed) mappings, pairwise fuzzy $\beta$-continuous, pairwise fuzzy $\beta$-irresolute mappings are defined and their relationship with various types of pairwise fuzzy continuities are investigated. Also some results of [7, 44] are extended to fuzzy bitopological situation.

Based on the definition of generalized closed sets [41] in topological spaces, the concept of generalized fuzzy (in short $gf$) closed set was introduced in fuzzy topological spaces. The concepts of $gf$-continuity, $gf$-connectedness, $gf$-extremally disconnectedness and $gf$-compactness were introduced and studied in [6]. Chapter 6 is devoted to define and study pairwise generalized fuzzy open (closed) sets. Several properties are established and some results in [6] are extended to fuzzy bitopological spaces using generalized fuzzy open sets.

As a generalization of the concept of quasi-open sets in bitopological spaces in fuzzy setting, new concepts namely, fuzzy quasi-open set and fuzzy quasi pre-open set are introduced and their properties are discussed in Chapter 7. The notion of fuzzy quasi-connectedness between fuzzy sets
is defined and studied. Also using fuzzy quasi-preopen sets a concept of fuzzy quasi-pre-separated sets is defined and their properties are discussed. Finally, we have also introduced a concept of fuzzy quasi-preconnectedness in fuzzy bitopological spaces as an extension of the concept of [72] in fuzzy setting.

0.5 Basic Definitions

The aim of this section is to present some basic definitions, results and important remarks.

Notations: Throughout the thesis I will denote the unit interval [0, 1] of the real line \( \mathbb{R} \). \( X, Y, Z \) will denote non-empty sets and \( \Gamma, \Lambda \) will be index sets. The symbols \( \lambda, \mu, \nu, \eta, \cdots \) are used to denote fuzzy sets and all other symbols have their usual meaning unless explicitly stated.

Definition 0.5.1. A fuzzy set [79] in \( X \) is an element of the set \( I^X \) of all functions from \( X \) to \( I \).

Definition 0.5.2. A fuzzy topology \( T \) [11] on \( X \) is a collection of subsets of \( I^X \) such that

(i) \( 0, 1 \in T \), (or \( 0_X, 1_X \in T \))

(ii) If \( \lambda, \mu \in T \), then \( \lambda \wedge \mu \in T \),
(iii) If \( \lambda_i \in T \) for each \( i \in \Gamma \) then \( \vee \lambda_i \in T \).

The ordered pair \((X, T)\) is called a fuzzy topological space (in short \(fts\)) and members of \(T\) are called \(T\)-fuzzy open sets or simply fuzzy open sets.

**Definition 0.5.3.** A fuzzy set \( \lambda \) in a fuzzy topological space is called a fuzzy closed set, if its complement \( \lambda' \) or \((1 - \lambda)\) is fuzzy open.

**Definition 0.5.4.** A fuzzy set \( \lambda \) which is both fuzzy open and fuzzy closed is called fuzzy clopen set.

**Definition 0.5.5.** For \( t \in (0, 1] \), \( x_t \) stands for the function from \((X, T) \to [0, 1]\) defined by

\[
x_t(y) = \begin{cases} 
0 & \text{if } y \neq x \\
t & \text{if } y = x, \quad y \in X 
\end{cases}
\]

and is referred to as the fuzzy point with support \( x \) and value \( t \). The fuzzy point \( x_1 \) is called the crisp point.

**Definition 0.5.6.** Let \( \lambda \) be fuzzy set in \( X \) and \( x_t \) a fuzzy point in \( X \). We say that \( x_t \in \lambda \iff t \leq \lambda(x), \, x \in X \).

A fuzzy set \( \lambda \) is the union of all fuzzy points which belong to \( \lambda \).

We also use the following notation [76] for fuzzy points:

A fuzzy point \( p \) in \( X \) is defined by

\[
p(x) = \begin{cases} 
t & \text{for } x = x_0 \\
0 & \text{otherwise}
\end{cases}
\]
where $0 < t < 1$. $p$ is said to have support $x_0$ and value $t$.

**Definition 0.5.7.** A fuzzy set on $(X, T)$ is called a fuzzy singleton [36] if and only if it takes the value zero (0) for all points in $X$ except one, say $x \in X$. The point at which a fuzzy singleton takes the nonzero value is called the support of the singleton and the corresponding element of $[0, 1]$, its value. A fuzzy singleton with value 1 is called a crisp singleton.

**Definition 0.5.8.** Let $\lambda$ and $\mu$ be any two fuzzy sets in $\mathcal{F} (X, T)$. Then we define $\lambda \vee \mu : X \rightarrow [0, 1]$ as follows:

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}, \forall x \in X.$$ 

Also we define $\lambda \wedge \mu : X \rightarrow [0, 1]$ as follows:

$$(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}, \forall x \in X.$$ 

**Definition 0.5.9.** The closure and the interior of a fuzzy set $\lambda$ are respectively denoted by $\text{Cl}(\lambda)$ and $\text{Int}(\lambda)$ and are defined as follows:

1. $\text{Cl}(\lambda) = \{\mu / \mu \text{ is a fuzzy closed and } \mu \geq \lambda\}$ and

2. $\text{Int}(\lambda) = \{\mu / \mu \text{ is a fuzzy open and } \mu \leq \lambda\}$

respectively.

For any fuzzy set $\lambda$ in a $\mathcal{F} (X, T)$, we have

(i) $1 - \text{Cl}(\lambda) = \text{Int}(1 - \lambda)$ and
(ii) \( 1 - Int(\lambda) = Cl(1 - \lambda) \).

**Definition 0.5.10.** The *characteristic function* of a subset \( A \) of \( X \) is denoted by \( \chi_A \) and is defined as

\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases}
\]

**Definition 0.5.11.** A \( fts \ (X, T) \) is said to be fuzzy connected [24] if it has no proper fuzzy clopen set. Otherwise it is called fuzzy disconnected.

**Definition 0.5.12.** A fuzzy set \( \lambda \) in \( fts \ (X, T) \) is proper if \( \lambda \neq 0 \) and \( \lambda \neq 1 \).

**Definition 0.5.13.** A \( fts \ (X, T) \) is called fuzzy super connected [17] if it has no proper fuzzy regular open set.

**Definition 0.5.14.** A \( fts \ (X, T) \) is called fuzzy strongly connected [17] if it has no non-zero fuzzy closed sets \( \lambda \) and \( \mu \) such that \( \lambda + \mu \leq 1 \).

**Definition 0.5.15.** A \( fts \ (X, T) \) is said to be fuzzy extremally disconnected [5] if \( \lambda \in T \) implies \( Cl(\lambda) \in T \).

**Definition 0.5.16.** A \( fts \ (X, T) \) is said to be fuzzy totally disconnected [5] if and only if for every pair of fuzzy points \( p, q \) with \( p \neq q \) in \( (X, T) \) there exists non-zero fuzzy open sets \( \lambda, \mu \) such that \( \lambda + \mu = 1 \), \( \lambda \) contains \( p \) and \( \mu \) contains \( q \). Suppose \( A \subseteq X \). \( A \) is said to be a fuzzy totally disconnected subset of \( X \) if \( A \) as a fuzzy subspace of \( (X, T) \) is fuzzy totally disconnected.
Definition 0.5.17. Let \((X, T)\) be fts and \(Y\) be an ordinary subset of \(X\). Then \(T/Y = \{\lambda/Y : \lambda \in T\}\) is a fuzzy topology on \(Y\) and is called the induced or relative fuzzy topology. The pair \((Y, T/Y)\) is called a fuzzy subspace \([19]\) of \((X, T)\). \((Y, T/Y)\) is called fuzzy open (fuzzy closed) subspace if the characteristic function of \(Y\) viz., \(\chi_Y\) is fuzzy open (fuzzy closed).

Definition 0.5.18. Let \((X, T)\) be a fts. A subfamily \(\mathcal{M}\) of \(T\) is called a fuzzy base for \(T\) if each member of \(T\) is a union of some members of \(\mathcal{M}\) and a subfamily \(\mathcal{N}\) of \(T\) is called fuzzy subbase for \(T\) if the collection of all finite intersections of members of \(\mathcal{N}\) forms a fuzzy base for \(T\). The members of \(\mathcal{M}\) are called fuzzy basic open sets.

Definition 0.5.19. A fuzzy set \(\lambda\) in a fts \((X, T)\) is called

(i) fuzzy semiopen \([2]\) if \(\lambda \leq \text{ClInt}(\lambda)\);

(ii) fuzzy regularopen \([2]\) if \(\lambda = \text{IntCl}(\lambda)\);

(iii) fuzzy preopen \([9]\) if \(\lambda \leq \text{IntCl}(\lambda)\);

(iv) fuzzy \(\beta\)-open \([7]\) if \(\lambda \leq \text{ClIntCl}(\lambda)\).

Remark 0.5.1. From Theorem 4.2 of \([2]\) the following are equivalent in a fuzzy topological space \((X, T)\).

(a) \(\lambda\) is a fuzzy semiclosed set.

(b) \(1 - \lambda\) is a fuzzy semiopen set.
(c) \( \text{IntCl}\lambda \leq \lambda \).

(d) \( \text{ClInt}(1 - \lambda) \geq 1 - \lambda \).

**Definition 0.5.20.** Let \( X \) and \( Y \) be any two non-empty sets and \( f : X \to Y \) be any function. Let \( \lambda \) be any fuzzy set in \( Y \). Then we define inverse image of \( \lambda \), denoted by \( f^{-1}(\lambda) \), as follows:-

\[
f^{-1}(\lambda) : X \to I, \quad f^{-1}(\lambda)(x) = \lambda[f(x)].
\]

**Definition 0.5.21.** If \( \lambda \) is a fuzzy set in \( X \) and \( \delta \) is a fuzzy set in \( Y \) then the cartesian product of \( \lambda \) and \( \delta \) denoted by \( \lambda \times \delta \) [2] is defined as a fuzzy set in \( X \times Y \) as follows:-

\[
(\lambda \times \delta)(x, y) = \min\{\lambda(x), \delta(y)\} \quad \text{for each} \ (x, y) \in X \times Y.
\]

**Definition 0.5.22.** Let \( \{(X_k, T_k) : k \in \Gamma\} \) be any family of fuzzy topological spaces.

Let \( X = \prod_{k \in \Gamma} X_k \). Let \( P_k : X \to X_k \) be the projection map from \( X \) onto \( X_k \).

Put \( \mathcal{A} = \{P_k^{-1}(\lambda) : \lambda \in T_k, k \in \Gamma\} \). Let \( \mathcal{B} \) the family of all finite intersections of members of \( \mathcal{A} \). Let \( T \) be the family of all unions of members of \( \mathcal{B} \). Then \( T \) is a fuzzy topology called the product fuzzy topology [50] for \( X \) which we shall denote by \( T = \prod_{k \in \Gamma} T_k \) and \( (X, T) \) is called the product fuzzy topological space. In particular, if \( (X, T) \) and \( (Y, S) \) are fuzzy topological spaces we shall denote the product of the fuzzy topological spaces
(X, T) and (Y, S) by (X × Y, T × S).

**Definition 0.5.23.** By a fuzzy T₁-space [5] we mean fits (X, T) such that given fuzzy points p, q in X such that p ≠ q, there exists a fuzzy open sets δ₁, δ₂ such that p ∈ δ₁, q ∉ δ₁ and q ∈ δ₂, p ∉ δ₂.

**Definition 0.5.24.** A fuzzy topological space (X, T) is said to be product related [2] to a fuzzy topological space (Y, S) if for any fuzzy set γ in X and τ in Y whenever (1 − λ) ∉ γ and (1 − δ) ∉ τ imply [(1 − λ) × 1] ∨ [1 × (1 − δ)] ≥ γ × τ where λ is a fuzzy open set in X and δ is a fuzzy open set in Y, there exist a fuzzy open set λ₁ in X and a fuzzy open set δ₁ in Y such that

\[(1 − λ₁) ≥ γ or (1 − δ₁) ≥ τ and\]

\[([(1 − λ₁) × 1] ∨ [1 × (1 − δ₁)]) = [(1 − λ) × 1] ∨ [1 × (1 − δ)].\]

**Definition 0.5.25.** A fuzzy set λ in a fits(X, T) is called generalized fuzzy closed [6] iff Cl(λ) ≤ μ whenever λ ≤ μ and μ is fuzzy open.

**Definition 0.5.26.** A fuzzy bitopological space [31] (in short fbts) is an ordered triple (X, T₁, T₂) where T₁ and T₂ are fuzzy topologies on X and members of Tᵢ are called Tᵢ-fuzzy open sets for i = 1, 2. A fuzzy set λ in fbts (X, T₁, T₂) is called Tᵢ-fuzzy closed set if its complement λ' or 1 − λ is Tᵢ-fuzzy open. In a fbts (X, T₁, T₂) a family of Tᵢ-fuzzy closed sets is denoted by Tᵢ for i = 1, 2.
Definition 0.5.27. Let $\lambda$ be a fuzzy set in the fbts $(X, T_1, T_2)$. Then we define $T_i$-fuzzy interior and $T_i$-fuzzy closure of $\lambda$, denoted by $Int_{T_i}(\lambda)$ and $Cl_{T_i}(\lambda)$ respectively, for $i = 1, 2$, as follows:

(i) $Int_{T_i}(\lambda) = \vee\{\mu/\mu \text{ is } T_i\text{-fuzzy open set and } \mu \leq \lambda\}$

(ii) $Cl_{T_i}(\lambda) = \wedge\{\mu/\mu \text{ is } T_i\text{-fuzzy closed set and } \mu \geq \lambda\}$

For any fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ we have the following results. (i) $1 - Cl_{T_i}(\lambda) = Int_{T_i}(1 - \lambda)$ and (ii) $1 - Int_{T_i}(\lambda) = Cl_{T_i}(1 - \lambda)$ for $i = 1, 2$.

Definition 0.5.28. A fuzzy set $\lambda$ in fbts $(X, T_1, T_2)$ is said to be $(T_1, T_2)$-fuzzy clopen if it is $T_1$-fuzzy closed as well as $T_2$-fuzzy open.

A fuzzy set $\lambda$ in a fbts $(X, T_1, T_2)$ is $T_i$-fuzzy open iff $\lambda = Int_{T_i}(\lambda)$ and $T_i$-fuzzy closed iff $\lambda = Cl_{T_i}(\lambda)$ for $i = 1, 2$.

Definition 0.5.29. The constant fuzzy topology $C_X$ in $X$ is defined as follows:

$$C_X = \{\lambda \in I^X/\lambda \text{ is a constant}\}.$$ 

Definition 0.5.30. A fbts $(X, T_1, T_2)$ is said to be pairwise fuzzy weakly Hausdorff, if for each two distinct fuzzy points $p, q$ in $X$, there exists a $T_1$-distance.

Definition 0.5.31. A fbts $(X, T_1, T_2)$ is said to be pairwise fuzzy weakly Hausdorff, if for each two distinct fuzzy points $p, q$ in $X$, there exists a
$T_1$-fuzzy neighborhood $\lambda$ for $p$ which does not contain $q$ and a $T_2$-fuzzy neighborhood $\mu$ for $q$ which does not contain $p$. If the roles of the points are interchangeable then $(X, T_1, T_2)$ is pairwise fuzzy Hausdorff.

**Definition 0.5.32.** Let $(X, T_1, T_2)$ be a fbts. For any non-empty subset $A$ of $X$ we shall write $T_1/A = \{\lambda/A|\lambda \in T_1\}$ and $T_2/A = \{\mu/A|\mu \in T_2\}$. Clearly $T_1/A$ and $T_2/A$ are fuzzy topologies on $A$ and the fuzzy bitopological space $(A, T_1/A, T_2/A)$ is called pairwise fuzzy subspace of $(X, T_1, T_2)$.

**Definition 0.5.33.** A fuzzy set $\lambda$ in a fbts $(X, T_1, T_2)$ is said to be $(i, j)$-fuzzy semiopen ($(i, j)$-fuzzy semiclosed) set if

$$\lambda \leq C1_{T_j}(Int_{T_i}(\lambda))(\lambda \geq Int_{T_j}(Cl_{T_i}(\lambda))).$$

**Definition 0.5.34.** A mapping $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ is said to be pairwise fuzzy semicontinuous [73] (pairwise fuzzy continuous) if the inverse image of each $S_i$-fuzzy open ($S_i$-fuzzy open) set in $Y$ is an $(i, j)$-fuzzy semiopen ($T_i$-fuzzy open) set in $X$ for $i \neq j$ and $i, j = 1, 2$.

**Definition 0.5.35.** A mapping $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ is said to be pairwise fuzzy semiopen (pairwise fuzzy open) if the image of each $T_i$-fuzzy open ($T_i$-fuzzy open) set in $X$ is an $(i, j)$-fuzzy semiopen, ($S_i$-fuzzy open) set in $Y$ for $i \neq j$ and $i, j = 1, 2$.

In a similar manner one can define pairwise fuzzy semiclosed, pairwise fuzzy closed mappings.
**Definition 0.5.36.** In a bitopological space \((X, P, Q)\), a set \(A \subseteq X\) is said to be quasi-open [13] if \(E = U \cup V\) for some \(U \in P\) and \(V \in Q\).