Chapter 5

Pairwise Fuzzy $\beta$-Open Sets, Pairwise Fuzzy $\beta$-Continuity and Pairwise Fuzzy $\beta$-Extremally Disconnected Spaces

5.1 Introduction

The concept of fuzzy $\beta$-open sets was defined and studied in [7, 8] also fuzzy $\beta$-extremally disconnectedness [7] in fuzzy topological spaces was introduced and studied. In this chapter we first define fuzzy $(i, j)fuzzy \alpha$-open, $(i, j)$ fuzzy preopen sets in fuzzy bitopological spaces on the basis of concepts in [28]. Also we define pairwise fuzzy $\alpha$-continuous (precontinuous) mappings between two fuzzy bitopological spaces. On the basis of the concepts in [61] various forms of pairwise fuzzy $\beta$-continuity for fuzzy
bitopological spaces are introduced as an extension of [7,8] and generalization of [61] and their relations with pairwise fuzzy $\alpha$-continuous (semicontinuous [73], precontinuous) mappings are also investigated. Besides giving several properties, coincidence and non-coincidence (with examples) of some continuous mappings are also investigated. Finally, we introduce pairwise fuzzy $\beta$-extremally disconnected spaces and several characterizations of such spaces are also obtained.

**Definition 5.1.1.** A fuzzy set $\lambda$ in a $fbts$ $(X, T_1, T_2)$ is said to be

(i) $(i, j)$-fuzzy $\alpha$-open ($\alpha$-closed) set if

$$\lambda \subseteq \text{Int}_{T_i}(\text{Cl}_{T_j}(\text{Int}_{T_i}(\lambda)))(\lambda \text{Cl}_{T_i} \geq (\text{Int}_{T_i}(\text{Cl}_{T_j}(\lambda))).$$

(ii) $(i, j)$-fuzzy preopen (preclosed) set if

$$\lambda \subseteq \text{Int}_{T_i}(\text{Cl}_{T_j}(\lambda))\ (\lambda \geq \text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda))).$$

**Remark 5.1.1.** It is clear that every $T_i$-fuzzy open ($T_i$-fuzzy closed) set is an $(i, j)$ fuzzy $\alpha$-open ($\alpha$-closed) set. But the converse is not true in general, as shown in the following example 5.1.1.

**Example 5.1.1.** Let $X = \{a, b, c\}$, $T_1 = \{0, 1, \lambda_1\}$, $T_2 = \{0, 1, \lambda_2\}$, where $\lambda_1, \lambda_2 : X \to [0, 1]$ and are defined as $\lambda_1(x) = 0.5, x = a, \lambda_1(x) = 0, x = b, c; \lambda_2(x) = 0.7, x = a, \lambda_2(x) = 0, x = b, c$. Then $(X, T_1, T_2)$ is a $fbts$. Let $\lambda(x) = 0.9, x = a, \lambda_1(x) = 0, x = b, c$. Then $\lambda$ is $(i, j)$ fuzzy $\alpha$-open but not $T_i$-fuzzy open.
Also it is easy to see that a fuzzy set $\lambda$ of a $f$ $b$ $t$ $s$ $(X, T_1, T_2)$ is $(i, j)$-fuzzy $\alpha$-open if and only if it is $(i, j)$-fuzzy semiopen and $(i, j)$-fuzzy preopen for $i \neq j$ and $i, j = 1, 2$.

**Definition 5.1.2.** A mapping $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is said to be pairwise fuzzy $\alpha$-continuous (pairwise fuzzy precontinuous) if the inverse image of each $S_i$-fuzzy open, $(S_i$-fuzzy open) set in $Y$ is an $(i, j)$-fuzzy $\alpha$-open(($(i, j)$-fuzzy preopen set )in $X$ for $i \neq j$ and $i, j = 1, 2$.

**Definition 5.1.3.** A mapping $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is said to be pairwise fuzzy pairwise fuzzy $\alpha$-open (pairwise fuzzy preopen) if the image of each $T_i$-fuzzy open $(T_i$-fuzzy open) set in $X$ is an $(i, j)$-fuzzy $\alpha$-open($(i, j)$-fuzzy preopen) set in $Y$ for $i \neq j$ and $i, j = 1, 2$.

In a similar manner one can define pairwise fuzzy $\alpha$-closed and pairwise fuzzy preclosed mappings.

**Remark 5.1.2.** It is clear that a pairwise fuzzy continuous (fuzzy open, fuzzy closed) mapping is also a pairwise fuzzy $\alpha$ continuous (fuzzy $\alpha$- continuous, fuzzy $\alpha$- closed) mapping but the converse is not true in general as shown in the example 5.1.2.

**Example 5.1.2.** Let $X = \{a, b\}, T_1 = \{0, 1, \lambda_1\}, T_2 = \{0, 1, \lambda_2\}, Y = \{p, q\}, S_1 = \{0, 1, \mu_1\}, S_2 = \{0, 1, \mu_2\}$, where $\lambda_1, \lambda_2 : X \to [0, 1]$ and
\( \mu_1, \mu_2 : Y \rightarrow [0, 1] \) are defined as follows:

\[
\begin{align*}
\lambda_1(x) &= 0.4, \quad x = a, \quad \lambda_1(x) = 0, \quad x = b; \\
\lambda_2(x) &= 0.8, \quad x = a, \quad \lambda_2(x) = 0, \quad x = b; \\
\mu_1(y) &= 0.7, \quad y = p, \quad \mu_1(y) = 0, \quad y = q; \\
\mu_2(y) &= 0.9, \quad y = p, \quad \mu_2(y) = 0, \quad y = q.
\end{align*}
\]

Then \((X, T_1, T_2)\) and \((Y, S_1, S_2)\) are fuzzy bitopological spaces. Define a mapping \(f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)\) by \(f(a) = p, f(b) = q\), then \(f\) is pairwise fuzzy \(\alpha\)-continuous but it is not pairwise fuzzy continuous.

Also a mapping \(f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)\) is pairwise fuzzy \(\alpha\)-continuous if and only if it is pairwise fuzzy semicontinuous and pairwise fuzzy precontinuous.

### 5.2 \((i, j)\)-Fuzzy \(\beta\)-Open and \((i, j)\)-Fuzzy \(\beta\)-Closed Sets

**Definition 5.2.1.** A fuzzy set \(\lambda\) in \(fbts (X, T_1, T_2)\) is said to be

(i) \((i, j)\)-fuzzy \(\beta\)-open set if \(\lambda \leq Cl_{T_i}(Int_{T_j}(Cl_{T_i}(\lambda)))\)

(ii) \((i, j)\)-fuzzy \(\beta\)-closed set if its complement is \((i, j)\)-fuzzy \(\beta\)-open equivalently \(\lambda\) is \((i, j)\)-fuzzy \(\beta\)-closed if \(\lambda \geq Int_{T_i}(Cl_{T_j}(Int_{T_i}(\lambda)))\) for \(i \neq j\) and \(i, j = 1, 2\).

One can refer to [7, 8] for the properties of fuzzy \(\beta\)-open sets and fuzzy \(\beta\)-continuous mappings.
Definition 5.2.2. A fuzzy set $\lambda$ in a \textit{fbs} $(X, T_1, T_2)$ is called a pairwise fuzzy $\beta$-open ($\beta$-closed) if it is both $(1, 2)$-fuzzy $\beta$-open ($\beta$-closed) and $(2, 1)$-fuzzy $\beta$-open ($\beta$-closed).

Remark 5.2.1. It is clear that if $\lambda$ is a $(j, i)$-fuzzy semiopen (resp. $(j, i)$-fuzzy semiclosed) then it is an $(i, j)$-fuzzy $\beta$-open, (resp. $(i, j)$-fuzzy $\beta$-closed) set. But the converse is not true in general for $i \neq j$ and $i, j = 1, 2$, as shown by the following example 5.2.1.

Example 5.2.1. Let $X = \{a, b, c\}, T_1 = \{0, 1, \lambda_1, \lambda_2\}, T_2 = \{0, 1, \mu_1\}$, where $\lambda_1, \lambda_2, \mu_1 : X \rightarrow [0, 1]$ are defined as follows: $\lambda_1(x) = 0.2, x = a, \lambda_1(x) = 0, x = b, c; \lambda_2(x) = 0.5, x = a, b, \lambda_2(x) = 0, x = c; \mu_1(x) = 0.7, x = c, \mu_1(x) = 0, x = a, b$. Clearly $(X, T_1, T_2)$ is a fuzzy bitopological space. Let $\nu$ be a fuzzy set defined on $X$ defined as $\nu : X \rightarrow [0, 1]$ such that $\nu(x) = 0.9, x = a, c, \nu(x) = 0, x = b$. Then $\nu$ is an $(i, j)$-fuzzy $\beta$-open set but not $(j, i)$-fuzzy semiopen set in $(X, T_1, T_2)$ for $i \neq j$ and $i, j = 1, 2$.

Remark 5.2.2. Also we can prove that if $\lambda$ is a $(j, i)$-fuzzy preopen (resp., $(j, i)$-fuzzy preclosed) set it is also an $(i, j)$-fuzzy $\beta$-open (resp., $(i, j)$-fuzzy $\beta$-closed) set for $i \neq j$ and $i, j = 1, 2$. But the converse need not be true in general as shown by the following example 5.2.2.

Example 5.2.2. Let $X = \{a, b, c, d\}, T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}$, $T_2 = \{0, 1, \lambda_4, \lambda_5, \lambda_6\}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 : X \rightarrow [0, 1]$ are defined
as

\[ \lambda_1(x) = 1, x = a, \lambda_1(x) = 0, x = b, c, d; \]
\[ \lambda_2(x) = 1, x = b, \lambda_2(x) = 0, x = a, c, d; \]
\[ \lambda_3(x) = 1, x = a, b, \lambda_3(x) = 0, x = c, d; \]
\[ \lambda_4(x) = 1, x = b, \lambda_4(x) = 0, x = a, c, d; \]
\[ \lambda_5(x) = 1, x = c, \lambda_5(x) = 0, x = a, b, d; \]
\[ \lambda_6(x) = 1, x = b, c, \lambda_6(x) = 0, x = a, d. \]

Then \((X, T_1, T_2)\) is a fuzzy bitopological space. Let \(\mu : X \to [0, 1]\) be a fuzzy set defined as \(\mu(x) = 1, x = b, d, \mu(x) = 0, x = a, c\). Then \(\mu\) is an \((i, j)\)-fuzzy \(\beta\)-open set but not \((j, i)\)-fuzzy preopen set in \(fbs \ (X, T_1, T_2)\) for \(i \neq j\) and \(i, j = 1, 2\).

**Proposition 5.2.1.** In a \(fbs \ (X, T_1, T_2)\) each \((i, j)\)-fuzzy \(\beta\)-open set which is \((i, j)\)-fuzzy semiclosed is \((j, i)\)-fuzzy semiopen for \(i \neq j\) and \(i, j = 1, 2\).

**Proof.** Let \(\lambda\) be an \((i, j)\)-fuzzy \(\beta\)-open set and \((i, j)\) fuzzy semiclosed. Then we have

\[ \lambda \leq Cl_{T_i}(Int_{T_j}(Cl_{T_i}(\lambda))) \quad (1) \]

and

\[ \lambda \geq Int_{T_j}(Cl_{T_i}(\lambda)) \quad (2) \]
respectively. Now (2) gives $\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda)) \leq \lambda$. Therefore

$$\text{Int}_{T_j}(\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda))) \leq \text{Int}_{T_j}\lambda$$

which implies

$$\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda)) \leq \text{Int}_{T_j}\lambda. \quad (3)$$

Now using (3) in (1) we get $\lambda \leq \text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda))$ which implies that $\lambda$ is $(j, i)$-fuzzy semiopen set. \qed

**Corollary 5.2.2.** In a fbts $(X, T_1, T_2)$ each $(i, j)$-fuzzy $\beta$-closed set which is also $(i, j)$-fuzzy semiopen is $(j, i)$-fuzzy semiclosed for $i \neq j$ and $i, j = 1, 2$.

**Proposition 5.2.3.** In a fbts $(X, T_1, T_2)$ each $(i, j)$-fuzzy $\beta$-open and $(i, j)$-fuzzy $\alpha$-closed set is $T_i$-fuzzy closed for $i \neq j$ and $i, j = 1, 2$.

**Proof.** Let $\lambda$ be both $(i, j)$-fuzzy $\beta$-open set and $(i, j)$-fuzzy $\alpha$-closed set. Then we have

$$\lambda \leq \text{Cl}_{T_i}(\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda))). \quad (1)$$

and

$$\lambda \geq \text{Cl}_{T_i}\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda)). \quad (2)$$

From (1) and (2) we get $\lambda = \text{Cl}_{T_i}\text{Int}_{T_j}(\text{Cl}_{T_i}(\lambda))$. Thus $\lambda$ is a $T_i$-fuzzy closed set for $i \neq j$ and $i, j = 1, 2$. \qed

**Corollary 5.2.4.** In a fbts $(X, T_1, T_2)$ each $(i, j)$-fuzzy $\beta$-closed and $(i, j)$-fuzzy $\alpha$-open set is $T_i$-fuzzy open for $i \neq j$ and $i, j = 1, 2$. 
Proposition 5.2.5. In a fbts $(X, T_1, T_2)$, let $\lambda$ be any $(j, i)$-fuzzy preopen set and $\mu$ be any fuzzy set such that $\lambda \leq \mu \leq Cl_{T_i}Int_{T_j}(\lambda)$. Then $\mu$ is $(i, j)$-fuzzy $\beta$-open set.

Proof. Since $\lambda$ is $(j, i)$-fuzzy preopen set, we have

$$\lambda \leq Int_{T_j}(Cl_{T_i}(\lambda)).$$  \hfill (1)

Also by hypothesis

$$\mu \leq Cl_{T_i}Int_{T_j}(\lambda).$$ \hfill (2)

Making use of (1) in (2) we get

$$\mu \leq Cl_{T_i}Int_{T_j}(Int_{T_j}(Cl_{T_i}(\lambda))) = Cl_{T_i}Int_{T_j}(Cl_{T_i}(\lambda)).$$

This implies $\mu \leq Cl_{T_i}Int_{T_j}(Cl_{T_i}(\mu))$. This shows $\mu$ is $(i, j)$-fuzzy $\beta$-open set.

Proposition 5.2.6. In a fbts $(X, T_1, T_2)$, let $\mu$ be any fuzzy set and let $\lambda$ be $(j, i)$-fuzzy semiopen set such that $Int_{T_j}Cl_{T_i}(\lambda) \geq \mu \geq \lambda$. Then $\mu$ is $(j, i)$-fuzzy $\alpha$-open set.

Proof. Since $\lambda$ is $(j, i)$-fuzzy semiopen set, we have

$$\lambda \leq Cl_{T_i}Int_{T_j}(\lambda).$$ \hfill (1)

By hypothesis

$$\mu \leq Int_{T_j}Cl_{T_i}(\lambda).$$ \hfill (2)
Making use of (1) in (2) we get
\[ \mu \leq \text{Int}_{T_j} \text{Cl}_{T_i}(\lambda) \leq \text{Int}_{T_j} \text{Cl}_{T_i}[\text{Cl}_{T_i} \text{Int}_{T_j}(\lambda)]. \]

This implies \( \mu \leq \text{Int}_{T_j} \text{Cl}_{T_i} \text{Int}_{T_j}(\lambda) \) which shows that \( \mu \) is \((j, i)\)-fuzzy \( \alpha \)-open set. \( \Box \)

**Proposition 5.2.7.** If \((X, T_1, T_2)\) and \((Y, S_1, S_2)\) are fber's such that \((X, T_1, T_2)\) is product related to \((Y, S_1, S_2)\). Then the product \( \lambda \times \mu \) of a pairwise fuzzy \( \beta \)-open set \( \lambda \) of \( X \) and pairwise fuzzy \( \beta \)-open set \( \mu \) of \( Y \) is a pairwise fuzzy \( \beta \)-open set of the fuzzy product space \((X \times Y, T_1 \times S_1, T_2 \times S_2)\).

**Proof.** Suppose \( \lambda \) is \((i, j)\)-fuzzy \( \beta \)-open in \((X, T_1, T_2)\) and \( \mu \) is \((i, j)\)-fuzzy \( \beta \)-open in \((Y, S_1, S_2)\). We shall prove \( \lambda \times \mu \) is also \((i, j)\)-fuzzy \( \beta \)-open in \((X \times Y, T_1 \times S_1, T_2 \times S_2)\).

By hypothesis, for \( i \neq j, i, j = 1, 2 \), we have
\[ \lambda \leq \text{Cl}_{T_i} \text{Int}_{T_j} \text{Cl}_{T_i}(\lambda) \]  \hspace{1cm} (1)
and
\[ \mu \leq \text{Cl}_{S_i} \text{Int}_{S_j} \text{Cl}_{S_i}(\mu). \]  \hspace{1cm} (2)

We claim \( \lambda \times \mu \leq \text{Cl}_{T_i \times S_i} \text{Int}_{T_j \times S_j} \text{Cl}_{T_i \times S_i}(\lambda \times \mu) \). For take
\[ \text{Cl}_{T_i \times S_i} \text{Int}_{T_j \times S_j} \text{Cl}_{T_i \times S_i}(\lambda \times \mu) = \text{Cl}_{T_i \times S_i} \text{Int}_{T_j \times S_j} [\text{Cl}_{T_i}(\lambda) \times \text{Cl}_{T_i}(\mu)] \]
\[ = \text{Cl}_{T_i \times S_i} \{ \text{Int}_{T_j}[\text{Cl}_{T_i}(\lambda)] \times \text{Int}_{S_j}[\text{Cl}_{T_i}(\mu)] \} \]
\[ = \text{Cl}_{T_i} \text{Int}_{T_j} \text{Cl}_{T_i}(\lambda) \times \text{Cl}_{S_i} \text{Int}_{S_j} \text{Cl}_{S_i}(\mu). \]  \hspace{1cm} (3)
Using (1) and (2) in (3) we get $\text{Cl}_{T_1 \times S_1} \text{Int}_{T_3 \times S_3} \text{Cl}_{T_1 \times S_1}(\lambda \times \mu) \geq \lambda \times \mu$, that is $\lambda \times \mu \leq \text{Cl}_{T_1 \times S_1} \text{Int}_{T_3 \times S_3} \text{Cl}_{T_1 \times S_1}(\lambda \times \mu)$. Hence the proposition is proved.

5.3 Pairwise Fuzzy $\beta$-Continuous Mappings

Definition 5.3.1. A mapping $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ is said to be pairwise fuzzy $\beta$-continuous if $f^{-1}(\mu)$ is a $(j, i)$-fuzzy $\beta$-open set in $X$ for each $S_i$-fuzzy open set $\mu$ of $Y$, for $i \neq j$ and $i, j = 1, 2$.

Remark 5.3.1. It is easy to prove that if $f$ is pairwise fuzzy semicontinuous mapping then $f$ is pairwise fuzzy $\beta$-continuous. The converse of this statement is not necessarily true as shown by the following example 5.3.1.

Example 5.3.1. Let $X = \{x_1, x_2, x_3\}, T_1 = \{0, 1, \mu_1, \mu_2\}, T_2 = \{0, 1, \mu_3\}$, where $\mu_1, \mu_2, \mu_3 : X \rightarrow [0, 1]$ are defined as follows:

$$\mu_1(x) = 1, x = x_2, \mu_1(x) = 0, x = x_1, x_3;$$

$$\mu_2(x) = 1, x = x_1, x_2, \mu_2(x) = 0, x = x_3;$$

$$\mu_3(x) = 1, x = x_1, \mu_3(x) = 0, x = x_2, x_3.$$

Then $(X, T_1, T_2)$ is a $\rho$-bets. Also let $Y = \{y_1, y_2, y_3\}, S_1 = \{0, 1, \lambda_1\}, S_2 = \{0, 1, \lambda_2\}$ where $\lambda_1, \lambda_2 : Y \rightarrow [0, 1]$ are defined as $\lambda_1(y) = 1, y = y_1, \lambda_1(y) = 0, y = y_2, y_3; \lambda_2(y) = 1, y = y_2, \lambda_2(y) = 0, y = y_1, y_3$. Then $(Y, S_1, S_2)$...
is a fbt's. Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be a mapping defined by $f(x_1) = y_1, f(x_2) = y_2$, and $f(x_3) = y_3$. Then $f$ is pairwise fuzzy $\beta$-continuous but not pairwise fuzzy semicontinuous.

Similarly, we can easily see that if $f$ is pairwise fuzzy precontinuous then $f$ is pairwise fuzzy $\beta$-continuous. However the converse is not true as shown in the following example 5.3.2.

**Example 5.3.2.** Let $X = Y = \{a, b, c, d\}$ and $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}, T_2 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \mu_4, \mu_5\}, S_1 = \{0, 1, \nu\}$ and $S_2 = \{0, 1, \eta\}$, where $\lambda_1, \lambda_2, \lambda_3, \mu_4, \mu_5 : X \rightarrow [0, 1]$ and $\nu, \eta : Y \rightarrow [0, 1]$ are defined as follows:

- $\lambda_1(x) = 1, x = a, \lambda_1(x) = 0, x = b, c, d; \lambda_2(x) = 1, x = b, \lambda_2(x) = 0, x = a, c, d; \lambda_3(x) = 1, x = a, b, \lambda_3(x) = 0, x = c, d; \mu_4(x) = 1, x = a, c, \mu_4(x) = 0, x = b, d; \mu_5(x) = 1, x = a, b, c, \mu_5(x) = 0, x = d; \nu(y) = 1, y = a, c, \nu(y) = 0, y = b, d, \eta(y) = 1, y = b, d, \eta(y) = 0, y = a, c.$

Clearly $(X, T_1, T_2)$ and $(Y, S_1, S_2)$ are fuzzy bitopological spaces.

Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be the identity mapping. Then $f$ is pairwise fuzzy $\beta$-continuous mapping but not pairwise fuzzy precontinuous mapping.

The interrelations between pairwise fuzzy $\beta$-continuous function and other types of pairwise fuzzy continuities are discussed and counter examples to show the non-coincidences of all the fuzzy continuities are also given in the following section 5.4.
5.4 Interrelations

Let \((X, T_1, T_2)\) and \((Y, S_1, S_2)\) be two \(fbs's\) and let \(f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)\) be a function defined between the two \(fbs's\). In the previous sections we have defined

1. pairwise fuzzy continuous function, (definition 0.5.34),
2. pairwise fuzzy \(\alpha\)-continuous function, (definition 5.1.2),
3. pairwise fuzzy semicontinuous function, (definition 0.5.34),
4. pairwise fuzzy precontinuous function, (definition 5.1.2) and
5. pairwise fuzzy \(\beta\)-continuous function, (definition 5.3.1),

between two fuzzy bitopological spaces \((X, T_1, T_2)\) and \((Y, S_1, S_2)\). Now the interrelations between them are given as follows:

(i) Suppose that \(f\) is pairwise fuzzy continuous function. Then it is easy to see that (1) \(\Rightarrow\) (2), (1) \(\Rightarrow\) (3), (1) \(\Rightarrow\) (4) and (1) \(\Rightarrow\) (5).

(ii) Suppose that \(f\) is pairwise fuzzy \(\alpha\)-continuous (i.e.) (2) is true. Then, from example 5.1.2 it is easy to see that (2) \(\not\Rightarrow\) (1). Also we can prove, (2) \(\Rightarrow\) (3), (2) \(\Rightarrow\) (4) and (2) \(\Rightarrow\) (5).

(iii) Suppose that \(f\) is pairwise fuzzy semicontinuous (i.e.) (3) holds true. Then (3) \(\not\Rightarrow\) (1) is seen from the following example 5.4.1.
Example 5.4.1. Let \( X = \{a, b, c\}, T_1 = \{0, 1, \mu_1, \mu_2\}, T_2 = \{0, 1, \mu_3\} \)
where \( \mu_1, \mu_2 : X \to [0, 1] \) are defined as
\[ \mu_1(x) = 1, \mu_2(x) = 0, x = a, c; \mu_2(x) = 1, x = a, b; \mu_2(x) = 0, x = c \text{ and } \mu_3(x) = 1, x = a, \mu_3(x) = 0, x = b, c. \]
Then \((X, T_1, T_2)\) is \(fbts\). Let \( Y = \{p, q, r\}, S_1 = \{0, 1, \lambda_1\}, S_2 = \{0, 1, \lambda_2\} \)
where \( \lambda_1, \lambda_2 : Y \to [0, 1] \) are defined as
\[ \lambda_1(y) = 1, y = p, \lambda_1(y) = 0, y = q, r \text{ and } \lambda_2(y) = 1, y = p, q, \lambda_2(y) = 0, y = r. \]
Then \((Y, S_1, S_2)\) is \(fbts\). Let \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) be a mapping defined by
\[ f(a) = q, f(b) = p \text{ and } f(c) = r. \]
From the above example it is also clear that \((3) \not\Rightarrow (2)\). Also \((3) \not\Rightarrow (4)\), as seen from example 5.4.2.

Example 5.4.2. Let \( X = \{a, b, c\}, T_1 = \{0, 1, \mu_1\}, T_2 = \{0, 1, \mu_2\} \)
where \( \mu_1, \mu_2 : X \to [0, 1] \) are defined as
\[ \mu_1(x) = 0.3, \mu_1(x) = 0, x = b, c; \mu_2(x) = 0.6, x = b, \mu_2(x) = 0, x = a, c. \]
Then \((X, T_1, T_2)\) is \(fbts\). Let \( Y = \{p, q, r\}, S_1 = \{0, 1, \lambda_1\}, S_2 = \{0, 1, \lambda_2\} \)
where \( \lambda_1, \lambda_2 : Y \to [0, 1] \) are defined as
\[ \lambda_1(y) = 0.3, y = p, \lambda_1(y) = 0, y = q, r \text{ and } \lambda_2(y) = 0.7, y = q, \lambda_2(y) = 0, y = p, r. \]
Then \((Y, S_1, S_2)\) is \(fbts\). Let \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) be a mapping defined by
\[ f(a) = p, f(b) = q \text{ and } f(c) = r. \]
Then \( f \) is pairwise fuzzy semicontinuous but not pairwise fuzzy precontinuous.

Now \((3) \Rightarrow (5)\) is obvious.

(iv) Suppose that \( f \) is pairwise fuzzy precontinuous (i.e.) \((4)\) holds true.

Then from example 5.4.3 we can easily see that \((4) \not\Rightarrow (1), (4) \not\Rightarrow (2)\)
and \((4) \not\Rightarrow (3)\).
Example 5.4.3. Let $X = \{a, b, c\}$, $T_1 = \{0, 1, \mu_1, \mu_2\}$, $T_2 = \{0, 1, \mu_3\}$ where $\mu_1, \mu_2, \mu_3 : X \to [0, 1]$ are defined as $\mu_1(x) = 1, x = b, \mu_1(x) = 0, x = a, c$; $\mu_2(x) = 1, x = a, b, \mu_2(x) = 0, x = c$; $\mu_3(x) = 1, x = a, \mu_3(x) = 0, x = b, c$. Then $(X, T_1, T_2)$ is fBTS.

Let $Y = \{p, q, r\}$, $S_1 = \{0, 1, \lambda_1\}$, $S_2 = \{0, 1, \lambda_2\}$ where $\lambda_1, \lambda_2 : Y \to [0, 1]$ are defined as $\lambda_1(y) = 1, y = p, \lambda_1(y) = 0, y = q, r$ and $\lambda_2(y) = 1, y = q, \lambda_2(y) = 0, y = p, r$. Then $(Y, S_1, S_2)$ is fBTS. Let $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ be a mapping defined by $f(a) = p, f(b) = q$ and $f(c) = r$. Then $f$ is pairwise fuzzy precontinuous but not pairwise fuzzy continuous (semicontinuous, $\alpha$-continuous). Now (4) $\Rightarrow$ (5) is obvious.

(v) Assume that $f$ is pairwise fuzzy $\beta$-continuous (i.e.) (5) holds true. Then (5) $\Rightarrow$ (1), (2), (3) as shown by the following example 5.4.4

Example 5.4.4.: Let $X = \{a, b, c\}$, $T_1 = \{0, 1, \mu_1, \mu_2\}$, $T_2 = \{0, 1, \mu_3\}$ where $\mu_1, \mu_2, \mu_3 : X \to [0, 1]$ are defined as $\mu_1(x) = 1, x = b, \mu_1(x) = 0, x = a, c$; $\mu_2(x) = 1, x = a, b, \mu_2(x) = 0, x = c$; $\mu_3(x) = 1, x = a, \mu_3(x) = 0, x = b, c$. Then $(X, T_1, T_2)$ is fBTS. Let $Y = \{p, q, r\}$, $S_1 = \{0, 1, \lambda_1\}$, $S_2 = \{0, 1, \lambda_2\}$ where $\lambda_1, \lambda_2 : Y \to [0, 1]$ are defined as $\lambda_1(y) = 1, y = p, \lambda_1(y) = 0, y = q, r$ and $\lambda_2(y) = 1, y = q, \lambda_2(y) = 0, y = p, r$. Then $(Y, S_1, S_2)$ is fBTS.

Let $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ be a mapping defined by $f(a) = p, f(b) = q$ and $f(c) = r$. Then $f$ is pairwise fuzzy $\beta$-continuous but not
pairwise fuzzy continuous (semicontinuous, $\alpha$-continuous). The implication of (5) $\Rightarrow$ (4) is obvious from example 5.3.2.

The following proposition gives several characterizations of pairwise fuzzy $\beta$-continuous mappings.

**Proposition 5.4.1.** Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be a mapping. Then the following statements are equivalent for $i \neq j$ and $i, j = 1, 2$.

(i) $f$ is pairwise fuzzy $\beta$-continuous.

(ii) The inverse image of each $S_i$-fuzzy closed set in $Y$ is a $(j, i)$-fuzzy $\beta$-closed set in $X$.

(iii) $\text{Int}_{T_j}[\text{Cl}_{T_i}[\text{Int}_{T_j}[f^{-1}(\mu)]]] \subseteq f^{-1}(\text{Cl}_{S_i}(\mu))$ for each fuzzy set $\mu$ of $Y$.

(iv) $f(\text{Int}_{T_j}[\text{Cl}_{T_i}[\text{Int}_{T_j}[\lambda]]]) \subseteq \text{Cl}_{S_i}[f(\lambda)]$ for each fuzzy set $\lambda$ of $X$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that (i) is true. Let $\mu$ be a $S_i$-fuzzy closed set of $Y$. Then by definition $1_Y - \mu = \mu'$ is $S_i$-fuzzy open set of $Y$. Hence by (i) $f^{-1}(\mu')$ is a $(j, i)$-fuzzy $\beta$-open set of $X$. Thus we have $f^{-1}(\mu') = f^{-1}(1_Y - \mu) = 1_X - f^{-1}(\mu)$ which is a $(j, i)$-fuzzy $\beta$-open set of $X$. This implies $f^{-1}(\mu)$ is a $(j, i)$-fuzzy $\beta$-closed set in $X$. Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Suppose (ii) is true. Let $\mu$ be any fuzzy set of $Y$. Then $f^{-1}(\text{Cl}_{S_i}[\mu])$ is a $(j, i)$-fuzzy $\beta$-closed set in $X$. Hence we have

$$f^{-1}(\text{Cl}_{S_i}(\mu)) \supseteq \text{Int}_{T_j}(f^{-1}(\text{Cl}_{S_i}(\mu)))$$
\[ \geq \text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(f^{-1}(\mu)))) \]

\[ \Rightarrow \text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(f^{-1}(\mu)))) \leq f^{-1}(\text{Cl}_{S_i}(\mu)). \]

Hence (ii) \( \Rightarrow \) (iii).

(iii) \( \Rightarrow \) (iv). Assume that (iii) is true. Let \( \lambda \) be any fuzzy set of \( X \) and \( f(\lambda) = \mu \). Now by (iii) we have

\[ f^{-1}(\text{Cl}_{S_i}(\mu)) \geq \text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(f^{-1}(\mu)))) \]

i.e. \[ f^{-1}(\text{Cl}_{S_i}(f(\lambda))) \geq \text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(f(\lambda)))) \]

\[ \geq \text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda))). \]

Taking image of \( f \) on either side we get

\[ f[f^{-1}(\text{Cl}_{S_i}(f(\lambda)))] \geq f[\text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda)))] \]

i.e. \[ \text{Cl}_{S_i}f(\lambda) \geq f[\text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda)))] \]

since \( f[f^{-1}(\text{Cl}_{S_i}(f(\lambda)))] \leq \text{Cl}_{S_i}(f(\lambda)). \) Therefore,

\[ f[\text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(\lambda)))] \leq \text{Cl}_{S_i}(f(\lambda)). \]

Hence (iii) \( \Rightarrow \) (iv).

(iv) \( \Rightarrow \) (i). Assume that (iv) is true. Let \( \mu \) be a \( S_i \)-fuzzy open set of \( Y \). Then \( 1_Y - \mu = \mu' \) is a \( S_i \)-fuzzy closed set. Then by (iv) we have

\[ f[\text{Int}_{T_j}(\text{Cl}_{T_i}(\text{Int}_{T_j}(f^{-1}(\mu'))))] \leq \text{Cl}_{S_i}(f(f^{-1}(\mu'))). \]
\[ \leq \text{Cl}_{C_i}(\mu') = \mu'. \]

So that

\[ \text{Int}_{C_i}(\text{Cl}_{C_i}(\text{Int}_{C_i}(f^{-1}(\mu')))) \leq f^{-1}(\mu'). \]

(i.e.) \( f^{-1}(\mu') \) is a \((j,i)\)-fuzzy \( \beta \)-closed set in \( X \).

(i.e.) \( f^{-1}(\mu') = f^{-1}(1_Y - \mu) = 1_X - f^{-1}(\mu) \) is \((j,i)\)-fuzzy \( \beta \)-closed set in \( X \) and therefore \( f^{-1}(\mu) \) is a \((j,i)\)-fuzzy \( \beta \)-open set in \( X \) and consequently \( f \) is a pairwise fuzzy \( \beta \)-continuous mapping. Hence (iv) \( \Rightarrow \) (i). \( \square \)

**Proposition 5.4.2.** Let \( f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \) be a pairwise fuzzy \( \beta \)-continuous and pairwise fuzzy \( \alpha \)-open mapping. Then the inverse image of each \((j,i)\)-fuzzy \( \beta \)-open set of \( Y \) is \((j,i)\)-fuzzy \( \beta \)-open set in \( X \).

**Proof.** Follows from definitions 5.3.1 and 5.1.3. \( \square \)

**Remark 5.4.1.** The composition of two pairwise fuzzy \( \beta \)-continuous mappings need not be pairwise fuzzy \( \beta \)-continuous in general, as the following example shows.

**Example 5.4.5.** Let \((X, T_1, T_2)\) and \((Y, S_1, S_2)\) be two fbt's described as in example 5.3.2 and let \( Z = X = Y = \{a, b, c, d\}, U_1 = \{0, 1, \theta\}, U_2 = \{0, 1, \phi\} \) where \( \theta, \phi : Z \rightarrow [0, 1] \) are defined as \( \theta(x) = 1, x = c, d, \theta(x) = 0, x = a, b, \) and \( \phi(x) = 1, x = b, c, d, \phi(x) = 0, x = a \). Then \((Z, U_1, U_2)\) is clearly a fbt's.
Let \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) and \( g : (Y, S_1, S_2) \to (Z, U_1, U_2) \) be identity mappings. Then clearly \( f \) and \( g \) are pairwise fuzzy \( \beta \)-continuous but the composition \( g \circ f \) is not pairwise fuzzy \( \beta \)-continuous.

**Proposition 5.4.3.** Let \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) and \( g : (Y, S_1, S_2) \to (Z, U_1, U_2) \) be two mappings. If \( f \) is pairwise fuzzy \( \beta \)-continuous and pairwise fuzzy \( \alpha \)-open and \( g \) is pairwise fuzzy \( \beta \)-continuous, then \( g \circ f \) is pairwise fuzzy \( \beta \)-continuous.

**Proposition 5.4.4.** Let \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) be a mapping and \( g : (X, T_1, T_2) \to (X \times Y, V_1, V_2) \) (where \( V_i (i = 1, 2) \) is the product topology generated by \( T_i \) and \( S_i \)) given by \( g(x) = (x, f(x)) \), its graph mapping. Then \( f \) is pairwise fuzzy \( \beta \)-continuous if and only if \( g \) is pairwise fuzzy \( \beta \)-continuous.

**Proposition 5.4.5.** Let \( \{X_m, T_1^m, T_2^m\} : m \in \Lambda \} \) and \( \{(Y_m, S_1^m, S_2^m) : m \in \Lambda \} \) be two arbitrary families of fuzzy bitopological spaces. Let \( f_m : X_m \to Y_m \) be pairwise fuzzy \( \beta \)-continuous for each \( m \in \Lambda \) and \( f : (\prod X_m, V_1, V_2) \to (\prod Y_m, W_1, W_2) \) where \( V_i \) (resp., \( W_i \)) is the product fuzzy topology generated by \( T_i^m \) (resp., \( S_i^m \)) for \( i = 1, 2 \) be a mapping defined by \( f((x_m)) = (f_m(x_m)) \) for each \( x_m \in \prod X_m \). Then \( f \) is pairwise fuzzy \( \beta \)-continuous.

**Proposition 5.4.6.** Let \( f : (X, P_1, P_2) \to (\prod X_m, V_1^m, V_2^m) \) be a pairwise fuzzy \( \beta \)-continuous mapping for each \( m \in \Lambda \) and define \( f_m : (X, P_1, P_2) \to \)
(\(X_m, T^m_1, T^m_2\)) by setting \(f_m(x) = (f(x))_m\). Then \(f_m\) is pairwise fuzzy \(\beta\)-continuous.

5.5 Pairwise Fuzzy \(\beta\)-Open and \(\beta\)-Closed Mappings

**Definition 5.5.1.** A mapping \(f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)\) is said to be pairwise fuzzy \(\beta\)-open (pairwise fuzzy \(\beta\)-closed) if the image of each \(T_i\)-fuzzy open \((T_i\)-fuzzy closed) set in \(X\) is a \((j, i)\)-fuzzy \(\beta\)-open \(((j, i)\)-fuzzy \(\beta\)-closed) set in \(Y\) for \(i \neq j\) and \(i, j = 1, 2\).

**Remark 5.5.1.** It is clear that pairwise fuzzy semiopen (pairwise fuzzy semi-closed) mapping is pairwise fuzzy \(\beta\)-open (pairwise fuzzy \(\beta\)-closed) mapping. The converse of these statements is not necessarily true as shown by the following example 5.5.1.

**Example 5.5.1.** Let \(X = \{a, b, c\}, T_1 = \{0, 1, \lambda_1\}, T_2 = \{0, 1, \lambda_2\}\) where \(\lambda_1, \lambda_2 : X \rightarrow [0, 1]\) are defined as \(\lambda_1(x) = 1, x = a, \lambda_1(x) = 0, x = b, c; \lambda_2(x) = 1, x = c, \lambda_2(x) = 0, x = a, b\). Then \((X, T_1, T_2)\) is a fbts. Let \(Y = \{p, q, r\}, S_1 = \{0, 1, \mu_1\}, S_2 = \{0, 1, \mu_2\}\) where \(\mu_1, \mu_2 : Y \rightarrow [0, 1]\) are defined as \(\mu_1(y) = 1, y = q, \mu_1(y) = 0, y = p, r\) and \(\mu_2(y) = 1, y = p, \mu_2(y) = 0, y = q, r\). Then \((Y, S_1, S_2)\) is a fbts. Let \(f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)\) be a mapping defined by \(f(a) = p, f(b) = q\) and \(f(c) = r\). Then \(f\) is pairwise fuzzy \(\beta\)-open but not pairwise fuzzy semiopen.
Remark 5.5.2. It is clear that pairwise fuzzy preopen (preclosed) mapping is pairwise fuzzy $\beta$-open ($\beta$-closed) mapping. However the converse is not true in general as the following example 5.5.2 shows.

Example 5.5.2. Let $X = \{a, b, c\}, T_1 = \{0, 1, \lambda\}, T_2 = \{0, 1, \mu\}$ where $\lambda, \mu : X \to [0, 1]$ are defined as $\lambda(x) = 1, x = b, c, \lambda(x) = 0, x = a$ and $\mu(x) = 1, x = b, \mu(x) = 0, x = a, c$. Then $(X, T_1, T_2)$ is a $fbts$. Also let $Y = \{p, q, r\}, S_1 = \{0, 1, \nu\}, S_2 = \{0, 1, \eta\}$ where $\nu, \eta : Y \to [0, 1]$ are defined as $\nu(y) = 1, y = r, \nu(y) = 0, y = p, q$ and $\eta(y) = 1, y = p, q, \eta(y) = 0, y = q, r$. Clearly $(Y, S_1, S_2)$ is a $fbts$. Let $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ be a mapping defined by $f(a) = p, f(b) = q$ and $f(c) = r$. Then $f$ is pairwise fuzzy $\beta$-open but not pairwise fuzzy preopen.

Proposition 5.5.1. Let $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ be a pairwise fuzzy $\beta$-open (pairwise fuzzy $\beta$-closed) mapping. If $\nu$ be any fuzzy set in $Y$ and $\lambda'$ is a $T_i$-fuzzy closed ($T_i$-fuzzy open) set such that $f^{-1}(\nu) \leq \lambda'$, then there exists a $(j, i)$-fuzzy $\beta$-closed ($(j, i)$-fuzzy $\beta$-open) set $\mu$ in $Y$ with $\nu \leq \mu$ such that $f^{-1}(\mu) \leq \lambda'$.

Proof. Let $\mu = 1_Y - f(1_X - \lambda)$. Since $f^{-1}(\nu) \leq \lambda'$, we have

$$1_X - f^{-1}(\nu) \geq 1_X - \lambda',$$ where $\lambda' = 1 - \lambda$.

$$f^{-1}(1_Y - \nu) \geq 1_X - \lambda'$$

$$1_Y - \nu \geq f(1_X - \lambda')$$
(i.e.) \( f(1_X - \lambda') \leq 1_Y - \nu \)

Since \( f \) is pairwise fuzzy \( \beta \)-open (pairwise fuzzy \( \beta \)-closed), then \( \mu \) is \((j, i)\)-fuzzy \( \beta \)-closed ((\( j, i \))-fuzzy \( \beta \)-open) set of \( Y \) and

\[
f^{-1}(\mu) = f^{-1}(1_Y - f(1_X - \lambda')).
\]

Therefore

\[
f^{-1}(\mu) = 1_X - f^{-1}(f(1_X - \lambda')) \leq 1_X - (1_X - \lambda') = \lambda'
\]

\[\Rightarrow f^{-1}(\mu) \leq \lambda'. \text{ Hence the proposition is proved.} \]

**Corollary 5.5.2.** If \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) is pairwise fuzzy \( \beta \)-open mapping then \( f^{-1}[\text{Int}_{S_j}(C1_{S_i}(\text{Int}_{S_j}(\mu)))] \leq C_{T_j}f^{-1}(\mu) \) for fuzzy set \( \mu \) of \( Y \) for \( i \neq j \) and \( i, j = 1, 2 \).

**Proof.** Since \( C_{T_i}(f^{-1}(\mu)) \) is a \( T_i \)-fuzzy closed set in \( X \) such that \( f^{-1}(\mu) \leq C_{T_i}(f^{-1}(\mu)) \) for each fuzzy set \( \mu \) of \( Y \), it follows from proposition 5.5.1, that there exists a \((j, i)\) fuzzy \( \beta \)-closed set \( \nu \) in \( Y \) with \( \mu \leq \nu \) such that

\[
f^{-1}(\nu) \leq C_{T_j}(f^{-1}(\mu)). \tag{1}
\]

Since \( \mu \leq \nu \),

\[
f^{-1}[\text{Int}_{S_j}(C1_{S_i}(\text{Int}_{S_j}(\mu)))] \leq f^{-1}[\text{Int}_{S_j}(C1_{S_i}(\text{Int}_{S_j}(\nu)))]. \tag{2}
\]

But \( \nu \) is \((j, i)\)-fuzzy \( \beta \)-closed. This implies

\[
\nu \geq \text{Int}_{S_j}(C1_{S_i}(\text{Int}_{S_j}(\nu))). \tag{3}
\]
Using (3) in (2) we get

\[ f^{-1}[\text{Int}_{S_j}(C_{1_{S_j}}(\text{Int}_{S_j}(\mu)))] \leq f^{-1}(\nu). \] (4)

Using (1) in (4) to get

\[ f^{-1}[\text{Int}_{S_j}(C_{1_{S_j}}(\text{Int}_{S_j}(\mu)))] \leq C_{1_{T_j}}(f^{-1}(\mu)). \]

Hence the corollary is proved.

5.6 Pairwise Fuzzy \( \beta \)-irresolute Mappings

Based on the definition of pairwise fuzzy irresolute mappings [73] from a \( f\text{bts} (X, T_1, T_2) \) to \( f\text{bts} (Y, S_1, S_2) \) here we define pairwise fuzzy \( \beta \)-irresolute mapping and study its relationship with various forms of pairwise fuzzy \( \beta \)-continuous mappings.

**Definition 5.6.1.** A mapping \( f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \) is said to be pairwise fuzzy \( \beta \)-irresolute if \( f^{-1}(\mu) \) is a \((j, i)\)-fuzzy \( \beta \)-open set in \( X \) for every \((j, i)\)-fuzzy \( \beta \)-open set \( \mu \) in \( Y \) for \( i \neq j \), and \( i, j = 1, 2 \).

**Remark 5.6.1.** Pairwise fuzzy \( \beta \)-irresolute mapping implies pairwise fuzzy \( \beta \)-continuous. However the converse is not true as shown by the following example 5.6.1.

**Example 5.6.1.** Let \( X = \{a, b, c\} \), \( T_1 = \{0, 1, \lambda_1, \lambda_2\} \), \( T_2 = \{0, 1, \mu_1, \mu_2\} \) where \( \lambda_1, \lambda_2, \mu_1, \mu_2 : X \rightarrow [0, 1] \) defined as \( \lambda_1(x) = 1, x = a, \lambda_1(x) = 0, x = b, \lambda_1(x) = 1, x = c \), and \( \mu_1(x) = 0, x = a, \mu_1(x) = 1, x = b, \mu_1(x) = 0, x = c \), and \( \mu_2(x) = 1, x = a, \mu_2(x) = 0, x = b, \mu_2(x) = 1, x = c \).
0, \ x = b, c; \ \lambda_2(x) = 1, \ x = a, c, \ \lambda_2(x) = 0, \ x = b; \ \mu_1(x) = 1, \ x = b, \ \mu_1(x) = 0, \ x = a. \ \text{Then} \ \ (X, T_1, T_2) \ \text{is a fbts. Let} \ Y = \{p, q, r\}, \ S_1 = \{0, 1, \nu\}, \ S_2 = \{0, 1, \eta\}, \ \text{where} \ \nu, \ \eta: Y \rightarrow [0, 1] \ \text{are defined as} \ \nu(y) = 1, \ y = p, \ \nu(y) = 0, \ y = q, r \ \text{and} \ \eta(y) = 1, \ y = q, \ \eta(y) = 0, \ y = p, r. \ \text{Clearly} \ (Y, S_1, S_2) \ \text{is a fbts.}

Let \ f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \ \text{be a mapping defined by} \ f(a) = p, f(b) = q, f(c) = r. \ \text{Then} \ f \ \text{is pairwise fuzzy} \ \beta- \ \text{continuous but not pairwise fuzzy} \ \beta- \ \text{irresolute.}

From proposition 5.4.2 and definition 5.6.1 we get if \ f \ \text{is pairwise fuzzy} \ \alpha- \ \text{open and pairwise fuzzy} \ \beta- \ \text{continuous then} \ f \ \text{is pairwise fuzzy} \ \beta- \ \text{irresolute. However this implication is not equivalent as shown by the following example 5.6.2.}

**Example 5.6.2.** Let \ X = Y = \{a, b, c\}, \ T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}, \ T_2 = \{0, 1, \lambda_2, \lambda_4, \lambda_5\}, \ S_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_4\}, \ S_2 = \{0, 1, \lambda_4, \lambda_5\}, \ \text{where} \ \lambda_i : X = Y \rightarrow [0, 1] \ \text{for} \ i = 1, 2, 3, 4 \ \text{and} \ 5 \ \text{are defined as follows:} \ \lambda_1(x) = 1, \ x = a, \ \lambda_1(x) = 0, \ x = b, c; \ \lambda_2(x) = 1, \ x = a, b, \ \lambda_2(x) = 0, \ x = c; \ \lambda_3(x) = 1, \ x = a, c, \ \lambda_3(x) = 0, \ x = b; \ \lambda_4(x) = 1, \ x = b, \ \lambda_4(x) = 0, \ x = a, c \ \text{and} \ \lambda_5(x) = 1, \ x = b, c, \ \lambda_5(x) = 0, \ x = a. \ \text{Then} \ (X, T_1, T_2) \ \text{and} \ (Y, S_1, S_2) \ \text{are fbts'.}

Let \ f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \ \text{be the identity mapping. Then we can easily see that} \ f \ \text{is pairwise fuzzy} \ \beta-\text{irresolute but not pairwise fuzzy}
α-open.

**Proposition 5.6.1.** If \( f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \) is pairwise fuzzy \( β \)-irresolute and \( g : (Y, S_1, S_2) \rightarrow (Z, Q_1, Q_2) \) is pairwise fuzzy \( β \)-continuous, then the composition \( g \circ f : (X, T_1, T_2) \rightarrow (Z, Q_1, Q_2) \) is pairwise fuzzy \( β \)-continuous.

The above Proposition 5.6.1 is an improved statement of proposition 5.4.3.

**Definition 5.6.2.** A mapping \( f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2) \) is said to be pairwise fuzzy irresolute (pairwise fuzzy preirresolute) if \( f(a) \) is an \((i, i)\)-fuzzy semiopen ((\(i, j\))-fuzzy preopen) set in \((X, T_1, T_2)\) for every \((i, j)\)-fuzzy semi open ((\(i, j\))-fuzzy preopen) set \( a \) in \( Y \) for \( i \neq j \) and \( i, j = 1, 2 \).

**Remark 5.6.2.** If \( f \) is pairwise fuzzy irresolute or pairwise fuzzy preirresolution then it is pairwise fuzzy \( β \)-irresolute. But the converse is not true. This can be shown by the following example 5.6.3.

**Example 5.6.3.** Let \( X = \{a, b, c\}, T_1 = \{0, 1, \lambda_1\}, T_2 = \{0, 1, \lambda_2\} \) where \( \lambda_1, \lambda_2 : X \rightarrow [0, 1] \), are defined as \( \lambda_1(x) = 1, x = a, \lambda_1(x) = 0, x = b, c \) and \( \lambda_2(x) = 1, x = b, \lambda_2(x) = 0, x = a, c \). Then \((X, T_1, T_2)\) is a fbts.

Let \( Y = \{p, q, r\}, S_1 = \{0, 1, \mu_1, \mu_2\}, S_2 = \{0, 1, \mu_3\} \) where \( \mu_1, \mu_2, \mu_3 : Y \rightarrow [0, 1] \), defined as \( \mu_1(y) = 1, y = p, \mu_1(y) = 0, y = q, r; \mu_2(y) = 1, y = p, q; \mu_2(y) = 0, y = r, \mu_3(y) = 1, y = r; \mu_3(y) = 0, y = p, q \). Then
(Y, S_1, S_2) is fbts.

Let f : (X, T_1, T_2) → (Y, S_1, S_2) be a mapping defined by f(a) = p, f(b) = q, f(c) = r. Then f is pairwise fuzzy β- irresolute but is neither pairwise fuzzy irresolute nor pairwise fuzzy preirresolute.

5.7 Pairwise Fuzzy β-Extremally Disconnected Spaces

In this section we define the concept of pairwise fuzzy β- extremally disconnected spaces and several characterizations of them are also studied.

Definition 5.7.1. Let λ be a fuzzy set in a fbts (X, T_1, T_2). Then T_1-β-fuzzy closure and T_1-β - fuzzy interior of λ, i = 1, 2, denoted by β- Cl_{T_i}(λ) and β - Int_{T_i}(λ) respectively, are defined as follows:

β-Cl_{T_i}(λ) = \bigwedge \{δ/δ is T_i-fuzzy β- closed set and δ ≥ λ\}

β-Int_{T_i}(λ) = \bigvee \{δ/δ is T_i-fuzzy β-open set and δ ≤ λ\}.

Definition 5.7.2. Let (X, T_1, T_2) be any fbts. Then X is called pairwise fuzzy β-extremally disconnected if the T_1-β- closure of each T_2-β-fuzzy open set is T_2-fuzzy β-open and T_2-β- closure of each T_1-fuzzy β-open set is T_1- fuzzy β- open.

The following proposition gives several characterizations of pairwise fuzzy β- extremally disconnected spaces.
Proposition 5.7.1. For any $fbts(X, T_1, T_2)$ the following are equivalent

(i) $X$ is pairwise fuzzy $\beta$-extremally disconnected.

(ii) For each $T_1$-fuzzy $\beta$-closed set $\lambda$ (say) $\beta-Int_{T_2}(\lambda)$ is $T_1$-fuzzy $\beta$-closed set. (Similarly, whenever $\mu$ is a $T_2$-fuzzy $\beta$-closed set then $\beta-Int_{T_1}(\mu)$ is $T_2$-fuzzy $\beta$-closed set).

(iii) For each $T_1$-fuzzy $\beta$-open set $\lambda$ (say) $\beta-Cl_{T_1}[1 - \beta-Cl_{T_2}(\lambda)]$ 
\[= 1 - \beta-Cl_{T_1}(\lambda). \] (Similar statement holds for a $T_2$-fuzzy $\beta$-open set $\eta$).

Proof: (i) $\Rightarrow$ (ii) Assume that (i) is true. Let $\lambda$ be a $T_1$-fuzzy $\beta$-closed set. Then, by definition $(1 - \lambda)$ is a $T_1$-fuzzy $\beta$-open set. Then from (i) $\beta-Cl_{T_2}(1 - \lambda)$ is $T_1$-fuzzy $\beta$-open set. Clearly, $1 - \beta-Cl_{T_2}(1 - \lambda)$ is a $T_1$-fuzzy $\beta$-closed set.

But $1 - \beta-Cl_{T_2}(1 - \lambda) = \beta-Int_{T_2}(\lambda)$ and so $\beta - Int_{T_2}(\lambda)$ is $T_1$-fuzzy $\beta$-closed set. Hence (i) $\Rightarrow$ (ii). (Similar proof for a $T_2$-fuzzy $\beta$-closed set $\mu$ also holds).

(ii) $\Rightarrow$ (iii). Assume that (ii) is true. Suppose that $\lambda$ is a $T_1$-fuzzy $\beta$-open set. Then $(1 - \lambda)$ is a $T_1$-fuzzy $\beta$-closed set. Now $\beta-Cl_{T_2}(\lambda)$ is $T_1$-fuzzy $\beta$-open set and therefore $1 - \beta-Cl_{T_2}(\lambda)$ is a $T_1$-fuzzy $\beta$-closed set.

Therefore, $1 - \beta-Cl_{T_2}(\lambda) = 1 - \lambda$.

Now $\beta-Cl_{T_1}[1 - \beta-Cl_{T_2}(\lambda)] = \beta - Cl_{T_1}[1 - \lambda] = 1 - \beta-Cl_{T_1}(\lambda)$. Hence we get $1 - \beta-Cl_{T_1}[1 - \beta-Cl_{T_2}(\lambda)] = 1 - \beta-Cl_{T_1}(\lambda)$. 
(Similarly, we can show that \( \beta - Cl_{T_2}[1 - \beta - Cl_{T_1}(\lambda)] = 1 - \beta - Cl_{T_2}(\lambda) \) for a \( T_2 \)-fuzzy \( \beta \)-open set \( \lambda \)).

(iii) \( \Rightarrow \) (i). Assume that (iii) is true. Then for a \( T_1 \)-fuzzy \( \beta \)-open set \( \lambda \) we have

\[
\beta - Cl_{T_1}[1 - \beta - Cl_{T_2}(\lambda)] = 1 - \beta - Cl_{T_1}(\lambda).
\] (1)

To prove \( X \) is pairwise fuzzy \( \beta \)-extremally disconnected, we have to show that \( \beta - Cl_{T_2}(\lambda) \) is \( T_1 \)-fuzzy \( \beta \)-open set. From (1) we have \( 1 - \beta - Cl_{T_2}(\lambda) \) is \( T_1 \)-fuzzy \( \beta \)-closed set and therefore \( \beta - Cl_{T_2}(\lambda) \) is obviously \( T_1 \)-fuzzy \( \beta \)-open set, which we want to show. Hence (iii) \( \Rightarrow \) (i).

(Similarly, we can show that whenever \( \mu \) is a \( T_2 \)-fuzzy \( \beta \)-open set then \( \beta - Cl_{T_1}(\mu) \) is \( T_2 \)-fuzzy \( \beta \)-open set). \( \Box \)