Chapter 3

Oscillation and Asymptotic Behavior of Solutions

3.1 Introduction

In this chapter we consider a more general fourth order difference equation of the form

\[ \Delta(a_n \Delta(b_n \Delta(c_n \Delta y_n))) + q_n f(y_{n+1}) = h_n, \quad n \in \mathbb{N}_0, \]  

(3.1.1)

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{q_n\}, \{h_n\} \) are sequences of real numbers and \( f: \mathbb{R} \to \mathbb{R} \) is continuous with \( uf(u) > 0 \) for \( u \neq 0 \). By a solution of equation (3.1.1), we mean a real sequence \( \{y_n\} \) satisfying equation (3.1.1) so that \( \sup_{n \geq m} |y_n| > 0 \) for any \( m \in \mathbb{N}_0 \). We always assume that such solutions exist.
Our purpose in this chapter is to obtain conditions for the oscillation of all solutions of equation (3.1.1) and nonoscillatory solutions of equation (3.1.1) tend to zero as \( n \to \infty \). In Section 3.2 we obtain conditions for the oscillation of all solutions of equation (3.1.1) when \( h_n \equiv 0 \) and Section 3.3 contains sufficient conditions which ensure that all nonoscillatory solutions of (3.1.1) tend to zero as \( n \to \infty \). Since, our equation is quite general and therefore the results of this paper even in some special cases complement and generalize some of the results in the literature [9, 14, 29, 43, 49]. Results obtained here are partially motivated by that of in [23, 28]. Examples are provided to illustrate the results.

3.2 Oscillation Results

In this section we study the oscillatory behavior of equation (3.1.1) under the following additional conditions:

\((c_1)\) \( h_n \equiv 0; \)

\((c_2)\) \( \{a_n\}, \{b_n\}, \{c_n\} \) and \( \{q_n\} \) are real and positive sequences such that

\[
\sum_{n=0}^{\infty} \frac{1}{a_n} = \sum_{n=0}^{\infty} \frac{1}{b_n} = \sum_{n=0}^{\infty} \frac{1}{c_n} = \infty; \quad (3.2.1)
\]
(c₃) \( f \) is nondecreasing and \( \frac{f(u)}{u} \geq M > 0 \) for \( u \neq 0 \)

**Theorem 3.2.1.** Let conditions (c₁) — (c₃) hold and suppose that each of (H₁), (H₂) and (H₃) is true.

(H₁) \[ \sum_{n=0}^{\infty} \left( \sum_{s=0}^{n-1} \frac{1}{a_s} \left( \sum_{t=0}^{s-1} \frac{1}{b_t} \left( \sum_{r=0}^{t-1} \frac{1}{c_r} \right) \right) \right) q_n = \infty. \]

(H₂) If \[ \sum_{n=0}^{\infty} q_n < \infty \] and

\[ \sum_{n=0}^{\infty} \left( \frac{1}{a_n} \sum_{s=n}^{\infty} q_s \right) < \infty \]

then

\[ \sum_{n=0}^{\infty} \left( \frac{1}{b_n} \sum_{s=n}^{\infty} \left( \frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right) \right) = \infty. \]

(H₃)

\[ \sum_{n=0}^{\infty} \left( \sum_{s=0}^{n-1} \frac{1}{c_s} \sum_{t=0}^{s-1} \frac{1}{b_t} \right) q_n = \infty. \]

Then every solution of equation (3.1.1) is oscillatory.

**Proof.** Let \( \{y_n\} \) be a nonoscillatory solution of equation (3.1.1). Without loss of generality we may assume that \( \{y_n\} \) is eventually positive since the proof is similar when \( \{y_n\} \) is eventually negative. Therefore there is an integer \( n_0 \in \mathbb{N}_0 \) such that \( y_n > 0 \) for all \( n \geq n_0 \). For \( n \geq n_0 \), let
\( u_n = y_n, v_n = c_n \Delta u_n, w_n = b_n \Delta v_n \) and \( z_n = a_n \Delta w_n \). Now the system,

\[
\begin{align*}
\Delta u_n &= \frac{v_n}{c_n} \\
\Delta v_n &= \frac{w_n}{b_n} \\
\Delta w_n &= \frac{z_n}{a_n} \\
\Delta z_n &= -q_n f(u_{n+1})
\end{align*}
\]

(3.2.4)

is satisfied. Clearly, \( \{z_n\} \) is nonincreasing. If there is an integer \( n_1 \geq n_0 \) such that \( z_{n_1} < 0 \), then

\[
\begin{align*}
w_n &= w_{n_0} + \sum_{s=n_0}^{n-1} \frac{z_s}{a_s} \\
v_n &= v_{n_0} + \sum_{s=n_0}^{n-1} \frac{w_s}{b_s} \\
u_n &= u_{n_0} + \sum_{s=n_0}^{n-1} \frac{v_s}{c_s}
\end{align*}
\]

(3.2.5)

and from (3.2.1) we have

\[
\begin{align*}
w_n \\
v_n \\
u_n
\end{align*}
\]

\( n \to \infty \)

a contradiction. Thus \( z_n \geq 0 \) for all \( n \geq n_0 \), so \( \lim_{n \to \infty} z_n = z_\infty \) exists and \( z_\infty \geq 0 \). Also, \( z_{n_1} > 0 \) if \( n_1 > n_0 \), then \( z_n = 0 \) whenever \( n \geq n_1 \). Thus, from (3.2.4), \( \Delta z_n = 0 \) and \( q_n = 0 \) whenever \( n \geq n_1 \). But this contradicts \( (H_1) \), so \( z_n > 0 \) for \( n \geq n_0 \). Thus \( \{w_n\} \) is increasing for \( n \geq n_0 \). Now we take different cases. Suppose \( w_n < 0 \) for \( n \geq n_0 \). Now \( w_\infty \leq 0 \)
and if $w_\infty < 0$ then (3.2.5) again gives a contradiction so $w_\infty = 0$. Now $v_n$ is decreasing for $n \geq n_0$, and $v_\infty < 0$ is impossible, so $v_\infty \geq 0$. If $j \geq n \geq n_0$, then

$$z_j - z_n = -\sum_{s=n}^{j-1} q_s f(u_{s+1}),$$

so

$$z_\infty - z_n = -\sum_{s=n}^{\infty} q_s f(u_{s+1}),$$

or

$$z_n \geq \sum_{s=n}^{\infty} q_s f(u_{s+1}) \geq \sum_{s=n}^{\infty} q_s f(u_s).$$

Since $v_n > 0$, $u_n$ is increasing, so

$$z_n \geq f(u_{n_0}) \sum_{s=n}^{\infty} q_s,$$

for $n \geq n_0$. If (3.2.2) fails, this is a contradiction, hence assume (3.2.2) holds. Since $w_\infty = 0$, we have

$$w_n = -\sum_{s=n}^{\infty} \frac{z_s}{a_s},$$

for $n \geq n_0$. But the last inequality says that if (3.2.3) fails this is a contradiction, so assume (3.2.3) holds. If $n \geq n_0$, then

$$v_n - v_{n_0} = \sum_{s=n_0}^{n-1} \frac{w_s}{b_s} = -\sum_{s=n_0}^{n-1} \frac{1}{b_s} \left( \sum_{t=s}^{\infty} \frac{z_t}{a_t} \right),$$
and so

\[- v_{n_0} \leq - \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left( \sum_{t=s}^{\infty} z_t \right),\]

\[v_{n_0} \geq \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left( \sum_{t=s}^{\infty} z_t \right),\]

\[\geq f(u_0) \sum_{s=n_0}^{n-1} \frac{1}{b_s} \left( \sum_{t=s}^{\infty} \frac{1}{a_t} \left( \sum_{i=t}^{\infty} q_i \right) \right).\]

This however contradicts \((H_2)\) and we are through the case \(w_n < 0\) for \(n \geq n_0\).

Since \(\{w_n\}\) is increasing and \(w_n < 0\) is false ensure that there is an integer \(n_1 \in \mathbb{N}\) such that \(n_1 \geq n_0\) and \(w_n > 0\) for all \(n > n_1\). Now \(\{v_n\}\) is increasing for all \(n \geq n_1\). If \(v_n < 0\) for all \(n \geq n_1\), then \(\{u_n\}\) is bounded. But \((H_1)\) and a result in [47] say that every bounded solution of equation (3.1.1) is oscillatory, so there is an integer \(n_2 \geq n_1\) such that \(v_n > 0\) for all \(n \geq n_2\). Now if \(n \geq n_2\), then

\[u_n = u_{n_2} + \sum_{s=n_2}^{n-1} \frac{v_s}{c_s},\]

\[\geq \sum_{s=n_2}^{n-1} \frac{v_s}{c_s},\]

\[= \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left( v_{n_2} + \sum_{t=n_2}^{s-1} \frac{w_t}{b_t} \right),\]

\[\geq \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left( \sum_{t=n_2}^{s-1} \frac{w_t}{b_t} \right),\]

\[\geq w_{n_2} \sum_{s=n_2}^{n-1} \frac{1}{c_s} \left( \sum_{t=n_2}^{s-1} \frac{1}{b_t} \right).\]
If \( n \geq n_2 \),

\[
0 < z_n = z_{n_2} + \sum_{s=n_2}^{n-1} \Delta z_s = z_{n_2} - \sum_{s=n_2}^{n-1} q_s f(u_{s+1}).
\]

So

\[
z_{n_2} \geq \sum_{s=n_2}^{n-1} q_s f(u_s) \geq M w_{n_2} \sum_{s=n_2}^{n-1} q_s \left( \sum_{t=n_2}^{s-1} \frac{1}{c_t} \sum_{j=n_2}^{t-1} \frac{1}{b_j} \right). \tag{3.2.6}
\]

But, according to Stolz’s Theorem [8], we have

\[
\lim_{s \to \infty} \frac{\sum_{t=n_2}^{s-1} \frac{1}{c_t} \sum_{j=n_2}^{t-1} \frac{1}{b_j}}{\sum_{t=0}^{s-1} \frac{1}{c_t} \sum_{j=0}^{t-1} \frac{1}{b_j}} = 1,
\]

so \((H_3)\) implies the divergence of the summations in (3.2.6) as \( n \to \infty \).

This contradiction completes the proof of the theorem. \(\square\)

**Corollary 3.2.2.** Assume \((H_3)\) holds and

\[
\sum_{n=0}^{\infty} \left( \sum_{s=0}^{n-1} \frac{1}{a_s} \left( \sum_{t=0}^{s-1} \frac{1}{b_t} \right) \right) q_s = \infty. \tag{3.2.7}
\]

Then every solution of equation (3.1.1) is oscillatory.

**Proof.** Let \( \{y_n\} \) be a nonoscillatory solution of equation (3.1.1). Without loss of generality we may assume that \( \{y_n\} \) is eventually positive. If (3.2.2) and (3.2.3) hold, and \( n \in \mathbb{N} \), then two successive applications of
summation by parts give

\[
\sum_{s=0}^{\infty} \frac{1}{b_s} \left( \sum_{t=s}^{\infty} \frac{1}{a_t} \left( \sum_{j=t}^{\infty} q_j \right) \right)
\]

\[
= \left( \sum_{t=0}^{n-1} \frac{1}{b_t} \right) \left( \sum_{t=n}^{\infty} \frac{1}{a_t} \sum_{j=t}^{\infty} q_j \right) + \sum_{s=0}^{n-1} \frac{1}{a_s} \left( \sum_{t=0}^{s} \frac{1}{b_t} \right) \sum_{j=s}^{\infty} q_j
\]

\[
\geq \sum_{s=0}^{n-1} \left( \frac{1}{a_s} \sum_{t=0}^{s-1} \frac{1}{b_t} \right) \left( \sum_{j=s}^{\infty} q_j \right) + \sum_{s=0}^{n-1} q_s \left( \sum_{t=0}^{s} \frac{1}{a_t} \sum_{j=0}^{t-1} \frac{1}{b_j} \right)
\]

Thus (3.2.7) implies \((H_2)\). Now (3.2.1) and two applications of Stolz’s Theorem imply that

\[
\lim_{i \to \infty} \frac{\sum_{s=0}^{i-1} \frac{1}{a_s} \sum_{t=0}^{s-1} \frac{1}{b_t}}{\sum_{s=0}^{i-1} \frac{1}{a_s} \sum_{t=0}^{s-1} \frac{1}{b_t} \sum_{j=0}^{t-1} \frac{1}{c_j}} = 0,
\]

so there is an integer \(N \in \mathbb{N}\) such that

\[
\sum_{s=0}^{i-1} \frac{1}{a_s} \sum_{t=0}^{s-1} \frac{1}{b_t} \sum_{j=0}^{t-1} \frac{1}{c_j} \geq \sum_{s=0}^{i-1} \frac{1}{a_s} \sum_{t=0}^{s-1} \frac{1}{b_t},
\]

whenever \(i \geq N\), and we see that (3.2.7) implies \((H_1)\), and the result now follows from Theorem 3.2.1.

\[\square\]

**Remark 3.2.1.** If \(a_n \equiv c_n\) then (3.2.7) is same as \((H_3)\), so in this case, (3.2.7) implies that every solution of equation (3.1.1) is oscillatory. If
\[ a_n = c_n = 1 \text{ and } b_n = r_n, \text{ then (3.2.7) is equivalent to} \]
\[
\sum_{n=0}^{\infty} \sum_{s=0}^{n-1} \frac{(n-s-1)}{r_s} q_n = \infty, \text{ and hence Corollary 3.2.2 implies that every solution of equation (3.1.1) is oscillatory. This is Theorem 6.11 of S.S. Cheng [9].}

**Example 3.2.1.** Consider the difference equation

\[
\Delta \left( (n+1) \Delta \left( \frac{1}{n} \Delta (n \Delta y_n) \right) \right) + \left( 8n+14 + \frac{(2n+1)}{n(n+1)} \right) y_{n+1} (1 + |y_{n+1}|) = 0,
\]

(3.2.8)

for \( n \geq 1 \). Here \( a_n = n+1, b_n = \frac{1}{n}, c_n = n, q_n = 8n+14+\frac{2n+1}{n(n+1)} \) and \( f(u) = u(1+|u|) \). It is easy to see that all conditions of Corollary 3.2.2 are satisfied and hence every solution of equation (3.2.8) is oscillatory. In fact, \( \{y_n\} = \{(−1)^n\} \) is such a solution of equation (3.2.8).

### 3.3 Asymptotic Behavior of Nonoscillatory Solutions

Here we discuss the asymptotic behavior of nonoscillatory solutions of equation (3.1.1) under the following conditions:

\[(c_4) \ \{a_n\}, \{b_n\}, \{c_n\}, \{q_n\} \text{ and } \{h_n\} \text{ are real and positive sequences such that} \]
\[
\sum_{n=0}^{\infty} \frac{1}{a_n} < \infty, \sum_{n=0}^{\infty} \frac{1}{b_n} < \infty, \sum_{n=0}^{\infty} \frac{1}{c_n} < \infty.
\]

(3.3.1)
\( (c_5) \lim_{n \to \infty} \rho_i(n) = 0 \) where \( \rho_i(n) = \sum_{s=n+1}^{\infty} \frac{\rho_{i-1}(s)}{r_i(s)} \), \( i = 1, 2, 3 \).

\( (\rho_0(n) \equiv 1), r_1(n) = c_n, r_2(n) = b_n \) and \( r_3(n) = a_n \).

We begin with two lemmas that will be needed in the proof of our main result of this section.

**Lemma 3.3.1.** Consider the difference equation

\[ \Delta u_n - \frac{\Delta \rho(n)}{\rho(n)} u_n + \frac{\Delta \rho(n)}{\rho(n)} \phi_n = 0, \quad (3.3.2) \]

where \( \{\phi_n\} \) and \( \{\rho(n)\} \) are real sequences defined for \( n \geq N \in \mathbb{N} \) and

\( \rho(n) > 0, \Delta \rho(n) < 0, \lim_{n \to \infty} \rho(n) = 0. \)

Let \( \{u_n\} \) be the solution of (3.3.2) defined for \( n \geq N \in \mathbb{N} \) satisfying

\( u_N = 0. \) Then,

\[ \lim_{n \to \infty} \phi_n = \infty \Rightarrow \lim_{n \to \infty} u_n = \infty, \]

\[ \lim_{n \to \infty} \phi_n = -\infty \Rightarrow \lim_{n \to \infty} u_n = -\infty. \]

**Proof.** The solution \( \{u_n\} \) of equation (3.3.2) is given by the formula

\[ u_n = -\rho(n) \sum_{s=N}^{n-1} \frac{\Delta \rho(s)}{\rho(s) \rho(s+1)} \phi_s, \quad n \geq N. \]

If \( \lim_{n \to \infty} \phi_n = \pm \infty \), then it is obvious that

\[ \lim_{n \to \infty} \left( -\sum_{s=N}^{n-1} \frac{\Delta \rho(s)}{\rho(s) \rho(s+1)} \phi_s \right) = \begin{cases} +\infty \\ -\infty. \end{cases} \]
Hence, by Stolz's theorem,
\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{\Delta \left( - \sum_{s=N}^{n-1} \frac{\Delta \rho(s)}{\rho(s) \rho(s+1)} \phi_s \right)}{\Delta \left( \frac{1}{\rho(n)} \right)} = \lim_{n \to \infty} \phi_n = \begin{cases} +\infty \\ -\infty. \end{cases}
\]
and the lemma is proved. \[\square\]

**Theorem 3.3.2.** Let conditions (c₄) and (c₅) hold. Assume that
\[
\liminf_{u \to \infty} f(u) > 0 \quad \text{and} \quad \limsup_{u \to \infty} f(u) < 0.\]
If
\[
\sum_{n=N}^{\infty} \rho_3(n) q_n = \infty, \quad (3.3.3)
\]
and
\[
\sum_{n=N}^{\infty} \rho_3(n) |h_n| < \infty, \quad (3.3.4)
\]
then all nonoscillatory solutions of equation (3.1.1) are bounded and tend to zero as \( n \to \infty. \)

**Proof.** Let \( \{y_n\} \) be a nonoscillatory solution of (3.1.1). We may suppose that \( y_n > 0 \) \( n \geq N_1 \in \mathbb{N}. \) Define
\[
G_0(n) = y_n, \quad G_i(n) = r_i(n) \Delta G_{i-1}(n), \quad (i = 1, 2, 3), \quad (3.3.5)
\]
\[
u_k(n) = \sum_{s=N+1}^{n} \rho_{3-k}(s) \Delta G_{3-k}(s), \quad (k = 0, 1, 2, 3). \quad (3.3.6)
\]
We shall first show that \( \{y_n\} \) is bounded above. From equation (3.1.1), we obtain
\[
G_3(n) - G_3(N_1) + \sum_{s=N_1}^{n-1} q_s f(y_{s+1}) = \sum_{s=n_1}^{n-1} h_s. \quad (3.3.7)
\]
Since the first sum of (3.3.7) is positive and by (3.3.4), the second sum is bounded, there exists a constant $k_3$ such that

$$G_3(n) = r_3(n) \Delta G_2(n) \leq k_3 \text{ for } n \geq N_1.$$  

Dividing the last inequality by $r_3(n)$ and summing from $N_1$ to $n - 1$, we obtain

$$G_2(n) - G_2(N_1) \leq k_3 \sum_{s=N_1}^{n-1} \frac{1}{r_3(n)}, \text{ for } n \geq N_1,$$

which shows, in view of (3.3.1), that there exists a constant $k_2$ such that

$$G_2(n) = r_2(n) \Delta G_1(n) \leq k_2, \text{ for } n \geq N.$$  

Applying the above arguments repeatedly, we obtain

$$G_1(n) \leq k_1 \quad \{ n \geq N_1 \}$$

$$G_0(n) \leq k_0$$

where $k_1$ and $k_0$ are constants. It follows that $\{y_n\}$ is bounded above for $n \geq N_1$.

A summation by parts yields

$$u_{k-1}(n)$$

$$= \sum_{s=N_1+1}^{n} \rho_{4-k}(s) \Delta G_{4-k}(s)$$

$$= \rho_{4-k}(n+1)G_{4-k}(n+1) - \rho_{4-k}(N_1+1)G_{4-k}(N_1+1)$$

$$+ \sum_{s=N_1+1}^{n} \frac{\rho_{3-k}(s)}{r_{4-k}(s)} G_{4-k}(s)$$
This shows that \( \{u_k(n)\} \) satisfies the difference equation

\[
\frac{\rho_{4-k}(n)}{\Delta \rho_{4-k}(n)} u_k(n) - u_k(n) + \phi_k(n) = 0, \quad (3.3.8)
\]

or equivalently

\[
\Delta u_k(n) = \frac{\Delta \rho_{4-k}(n)}{\rho_{4-k}(n)} u_k(n) + \frac{\Delta \rho_{4-k}(n)}{\rho_{4-k}(n)} \phi_k(n) = 0, \quad (3.3.9)
\]

where \( \phi_k(n) = u_{k-1}(n) + 2\rho_{4-k}(N_1 + 1)G_{4-k}(N_1 + 1) \).

Since \( u_k(N_1) = 0 \) by (3.3.6) and since

\[
\begin{align*}
\rho_{4-k}(n) &> 0 \\
\Delta \rho_{4-k}(n) &< 0 \\
\lim_{n \to \infty} \rho_{4-k}(n) &> 0
\end{align*}
\]

by condition \( (c_5) \), we can apply Lemma 3.3.1 to (3.3.9) to conclude that \( \lim_{n \to \infty} u_{k-1}(n) = \infty (\text{or} \ -\infty) \) which in turn implies that \( \lim_{n \to \infty} u_k(n) = \infty (\text{or} \ -\infty) \).

Multiply both sides of equation (3.1.1) by \( \rho_3(n) \) and summing from \( N_1 + 1 \) to \( n \), we have

\[
\sum_{s=N_1+1}^{n} \rho_3(s) \Delta G_3(s) + \sum_{s=N_1+1}^{n} \rho_3(s)q_s f(y_{s+1}) = \sum_{s=N_1+1}^{n} \rho_3(s)h_s. \quad (3.3.10)
\]
We consider the following two cases:

\[
\sum_{n=N_1+1}^{\infty} \rho_3(s) q_s f(y_{s+1}) = \infty, \quad (3.3.11)
\]

\[
\sum_{n=N_1+1}^{\infty} \rho_3(s) q_s f(y_{s+1}) < \infty. \quad (3.3.12)
\]

Suppose (3.3.11) holds. In view of (3.3.4) the right hand side of (3.3.10) tends to a finite limit as \( n \to \infty \), so from (3.3.10) we see that \( \lim_{n \to \infty} u_0(n) = -\infty \). Hence, by Lemma 3.3.1 applied to (3.3.9) with \( k = 1 \), we have

\[
\lim_{n \to \infty} u_1(n) = -\infty. \quad \text{Applying Lemma 3.3.1 again to (3.3.9) with } k = 2, \text{ we find } \lim_{n \to \infty} u_2(n) = -\infty. \quad \text{Repeating this procedure we can conclude that } \lim_{n \to \infty} u_3(n) = -\infty, \quad \text{which implies } \lim_{n \to \infty} y_n = -\infty. \quad \text{This, however, contradicts the positivity of } y_n. \quad \text{Hence (3.3.11) is impossible. Next, letting } n \to \infty \text{ in (3.3.10) and using (3.3.12), we see that } \lim_{n \to \infty} u_0(n) \text{ is finite. From (3.3.8) with } k = 1, \text{ we have}
\]

\[
\frac{\rho_3(n)}{\Delta \rho_3(n)} \Delta u_1(n) - u_1(n) + \phi_1(n) = 0,
\]

or

\[
u_1(n) = \frac{\rho_3(n)}{\rho_3(N_1)} \left( u_1(N_1) - \rho_3(N_1) \sum_{s=N_1}^{n-1} \frac{\Delta \rho_3(s)}{\rho_3(s) \rho_3(s+1)} \phi_1(s) \right).
\]

Taking the limit \( n \to \infty \) and using \((c_5)\), we obtain

\[
\lim_{n \to \infty} u_1(n) = -\lim_{n \to \infty} \rho_3(n) \sum_{s=N_1}^{n-1} \frac{\Delta \rho_3(s)}{\rho_3(s) \rho_3(s+1)} \phi_1(s).
\]

This limit must be finite, since \( \lim_{n \to \infty} u_1(n) = -\infty \) would imply \( \lim_{n \to \infty} y_0(n) = -\infty \) which is a contradiction, and \( \lim_{n \to \infty} u_1(n) = \infty \) would
imply \( \lim_{n \to \infty} y_n = \infty \), a contradiction to the boundedness of \( y_n \). Continuing in this way, we see that \( \lim_{n \to \infty} u_3(n) \) is finite. Hence, \( \lim_{n \to \infty} y_n \) exists as a finite number. On the other hand, from (3.3.3) and ((3.3.12)) we see that \( \lim \inf_{n \to \infty} y_n = 0 \). Therefore, we conclude that \( y_n \) tends to zero as \( n \to \infty \).

We conclude this chapter with the following example and a remark.

**Example 3.3.1.** Consider the difference equation

\[
\Delta(2^n \Delta(2^n \Delta y_n))) + 8^n y_{n+1} = \frac{1}{8},
\]

(3.3.13)

for \( n \geq 0 \). In this case \( \rho_1(n) = \frac{1}{2^n} \), \( \rho_2(n) = \frac{1}{3} \left( \frac{1}{4^n} \right) \), \( \rho_3(n) = \frac{1}{21} \left( \frac{1}{8^n} \right) \).

Since all conditions of Theorem 3.3.2 are satisfied, every nonoscillatory solution of (3.3.13) tends to zero as \( n \to \infty \). This equation has a nonoscillatory solution \( \{y_n\} = \left\{ \frac{1}{2^n} \right\} \) which tends to zero as \( n \to \infty \).

**Remark 3.3.1.** It is interesting to obtain that equation (3.1.1) has a nonoscillatory solution under the assumption of Theorem 3.3.2.