Chapter 5

Neutral Difference Equations-I

5.1 Introduction

This chapter is concerned with a class of nonlinear neural fourth order difference equations of the form

\[
A^2(a_n A^2(y_n + p_n y_{n-k})) + f(n, y_{\sigma(n)}) = 0, \quad n \in \mathbb{N}(n_0),
\]

and assume that

1. \(\{a_n\}\) is a positive sequence of real numbers for \(n \in \mathbb{N}(n_0)\) such that
   \[
   \sum_{n=n_0}^{\infty} \frac{n}{a_n} = \infty;
   \]
2. \(\{p_n\}\) is a real sequence such that \(0 \leq p_n < p < 1\) for all \(n \in \mathbb{N}(n_0)\);
3. \(k\) is a nonnegative integer and \(\{\sigma(n)\}\) is a sequence of positive integers with \(\lim_{n \to \infty} \sigma(n) = \infty;\)
(C₄) \( f : \mathbb{N}(n_0) \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( f(n, u) \) is nondecreasing in \( u \) with \( uf(n,u) > 0 \) for all \( u \neq 0 \) and for all \( n \in \mathbb{N}(n_0) \), and \( f(n, \cdot) \neq 0 \) eventually.

By a solution of equation (5.1.1) we mean a real sequence \( \{y_n\} \) which is defined for \( n \geq \min_{m \geq n_0} \{m - k, \sigma(m)\} \) and satisfies (5.1.1) for all \( n \in \mathbb{N}(n_0) \). Compared to ordinary difference equations the study of neutral equations and in particular higher order neutral equations, has received considerably less attention, see, for example [6, 11, 12, 13, 15, 26, 27, 41, 42, 47, 45] and references cited therein. Most of the results obtained for the higher order equations in these papers are for the case \( a_n \equiv 1 \) for all \( n \in \mathbb{N}(n_0) \). Therefore our purpose here is to establish some necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (5.1.1) that exhibit certain type of asymptotic behavior. In addition, we obtain some sufficient conditions for equation (5.1.1) to be oscillatory. Examples are included to illustrate our results.
5.2 Some Preliminary Results

In this section we state and prove some lemmas, which are useful in establishing the main results. For the sake of convenience we will use the following notations.

\[ R(n) = \sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} \frac{t}{a_t} \]

and

\[ R(n, N) = \sum_{s=N}^{n-1} \sum_{t=N}^{s-1} \frac{(t - N)}{a_t}. \]

In terms of \( R(n), (C_1) \) is equivalent to saying that \( R(\infty) = \infty \).

Let \( \{y_n\} \) be a real sequence. We will also define a companion or associated sequence \( \{z_n\} \) of it by

\[ z_n = y_n + p_n y_{n-k}, \quad n \in \mathbb{N}(n_0). \] (5.2.1)

where \( \{p_n\} \) and \( k \) have been defined above. First we give some relation between the sequences \( \{y_n\} \) and \( \{z_n\} \).

**Lemma 5.2.1.** Let \( \{y_n\}_{n=n_0}^{\infty} \) be a positive sequence and \( \{z_n\} \) be defined by (5.2.1).

(i) If \( \lim_{n \to \infty} y_n = \infty \), then \( \lim_{n \to \infty} z_n = \infty \).

(ii) If \( \{z_n\} \) converges to zero then so does \( \{y_n\} \).
(iii) If \( \lim_{n \to \infty} p_n = p^* \in [0, 1) \) and \( \lim_{n \to \infty} z_n = c \neq 0 \), then \( \lim_{n \to \infty} y_n = \frac{c}{1 + p^*} \).

and

(iv) If \( \lim_{n \to \infty} \frac{p_n R(n - k)}{R(n)} = q^* \in [0, 1) \) and \( \lim_{n \to \infty} \frac{z_n}{R(n)} = c \neq 0 \),
then \( \lim_{n \to \infty} \frac{y_n}{R(n)} = \frac{c}{1 + q^*} \).

Proof. The first assertion is clear since \( z_n > y_n \). The second follows from the definition of \( \{z_n\} \). To prove (iii) assume without loss of generality that \( y_n > 0 \) for \( n \geq n_0 \).

Let

\[ Q = \limsup_{n \to \infty} y_n \text{ and } q = \liminf_{n \to \infty} y_n. \]

Then there exist subsequences of nonnegative integers \( \{n_i\} \) and \( \{m_i\} \) such that

\[ \lim_{i \to \infty} y_{n_i} = Q, \text{ and } \lim_{i \to \infty} y_{m_i} = q. \]

Since \( \lim_{n \to \infty} p_n = p^* \in [0, 1) \), we have

\[ c = \lim_{i \to \infty} Z_{n_i} = \lim_{i \to \infty} (y_{n_i} + p_{n_i} y_{n_i-k}) \geq Q + p^* q \]

and

\[ c = \lim_{i \to \infty} Z_{m_i} = \lim_{i \to \infty} (y_{m_i} + p_{m_i} y_{m_i-k}) \leq q + p^* Q. \]

Hence, \( q + p^* Q \geq Q + p^* q \) or \( q \geq Q \). Since \( q \leq Q \) by definition, we see that \( q = Q \). Thus, \( \lim_{n \to \infty} y_n \) exists and the result follows from the definition of \( z_n \).
To prove (iv), let

\[ w_n = \frac{Z_n}{R(n)} = \frac{y_n}{R(n)} + \frac{p_n y_{n-k}}{R(n)} \]

\[ = \frac{y_n}{R(n)} + \frac{p_n R(n-k)}{R(n)} \frac{y_{n-k}}{R(n-k)} \]

\[ = x_n + q_n x_{n-k}, \]

where \( x_n = \frac{y_n}{R(n)} \) and \( q_n = \frac{p_n R(n-k)}{R(n)}. \)

Given that \( \lim_{n \to \infty} w_n = c \) and \( \lim_{n \to \infty} q_n = q^* \in [0, 1) \), an application of (iii) immediately gives

\[ \lim_{n \to \infty} x_n = \frac{c}{1 + q^*}. \]

Now the proof is complete.

Lemma 5.2.2. If \( \{y_n\}_{n=n_0}^\infty \) is an eventually positive solution of equation (5.1.1), then there are only the following two cases for \( n \) large enough.

(I) \( y_n > 0, z_n > 0, \Delta z_n > 0, a_n \Delta^2 z_n > 0, \Delta(a_n \Delta^2 z_n) > 0; \)

(II) \( y_n > 0, z_n > 0, \Delta z_n > 0, a_n \Delta^2 z_n < 0, \Delta(a_n \Delta^2 z_n) > 0. \)

Proof. Let \( \{y_n\} \) be an eventually positive solution of equation (5.1.1). Then there exists an integer \( n_1 \in \mathbb{N}(n_0) \) such that \( y_{\sigma(n)} > 0 \) for all \( n \in \mathbb{N}(n_1) \). Now \( \Delta^2(a_n \Delta^2 z_n) < 0 \) for all \( n \in \mathbb{N}(n_1) \), so \( \{\Delta(a_n \Delta^2 z_n)\} \), \( \{(a_n \Delta^2 z_n)\} \) and \( \{z_n\} \) are eventually monotonic and of one sign, say \( n \geq n_2 \). Suppose that \( \Delta(a_{n_3} \Delta^2 z_{n_3}) = -c_1 \leq 0 \) for some \( n_3 \geq n_2 \). Note that since \( f(n, \cdot) \neq 0 \), we can assume that \( c_1 \neq 0 \). It follows that \( \Delta(a_n \Delta^2 z_n) \leq -c_1, \) for \( n \in \mathbb{N}(n_3) \).
Summing both sides of the last inequality, we see that there exists \( n_4 \in \mathbb{N}(n_3) \) and \( c_2 > 0 \) such that \( \Delta^2 z_n \leq -\frac{c_2 n}{a_n} \) for \( N \in \mathbb{N}(n_4) \). Summing again from \( N_4 \) to \( n - 1 \) we obtain

\[
\Delta z_n \leq \Delta z_{n_4} - c_2 \sum_{s=n_4}^{n-1} \frac{s}{a_s}.
\]

As condition \( (C_1) \) holds, there exists \( n_5 \geq n_4 \), and \( c_3 > 0 \) such that

\[
\Delta z_n \leq -c_3 \text{ for } n \in \mathbb{N}(n_5).
\]

A final summation from \( n_5 \) to \( n - 1 \) of the last inequality yields

\[
z_n \leq z_{n_5} - c_3(n - n_5),
\]

which implies \( \lim_{n \to \infty} z_n = -\infty \). This contradiction implies that

\[
\Delta(a_n \Delta^2 z_n) > 0 \text{ for } n \in \mathbb{N}(n_2).
\]

Now if \( a_n \Delta^2 z_n > 0 \) for all \( n \in \mathbb{N}(n_3) \), then we have

\[
a_n \Delta^2 z_n \geq a_{n_3} \Delta^2 z_{n_3} = c_4 > 0 \text{ for } n \in \mathbb{N}(n_3).
\]

Multiply the above inequality by \( \frac{n}{a_n} \) and sum from \( n_3 \) to \( n - 1 \) to obtain

\[
n \Delta z_n - n_3 \Delta z_{n_3} - \sum_{s=n_3}^{n-1} \Delta z_{s+1} > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s}.
\]

If \( \Delta z_n < 0 \) for all \( n \geq n_3 \), then

\[
n \Delta z_n - n_3 \Delta z_{n_3} + z_{n_3+1} > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s} \to \infty.
\]

as \( n \to \infty \). Thus \( \{z_n\} \) is eventually positive and so is \( \{y_n\} \). Thus Case (ii) is verified.
Lemma 5.2.3. If \( N \ge n_0 \) then \( \lim_{n \to \infty} \frac{R(n, N)}{R(n)} = 1. \)

Proof. Since \( R(n) \) and \( R(n, N) \to \infty \) as \( n \to \infty \), we have from Stolz's theorem [8]

\[
\lim_{n \to \infty} \frac{R(n, N)}{R(n)} = \lim_{n \to \infty} \frac{\sum_{s=N}^{n-1} \sum_{t=N}^{s-1} \left( \frac{t-N}{a_t} \right)}{\sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} \left( \frac{1}{a_t} \right)}
\]

\[= \lim_{n \to \infty} \frac{\Delta \left( \sum_{s=N}^{n-1} \sum_{t=N}^{s-1} \left( \frac{t-N}{a_t} \right) \right)}{\Delta \left( \sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} \left( \frac{1}{a_t} \right) \right)}
\]

\[= \lim_{n \to \infty} \frac{\sum_{t=N}^{n-1} \frac{t-N}{a_t}}{\sum_{t=n_0}^{n-1} \frac{1}{a_t}}
\]

\[= \lim_{n \to \infty} \frac{\Delta \left( \sum_{t=N}^{n-1} \frac{t-N}{a_t} \right)}{\Delta \left( \sum_{t=n_0}^{n-1} \frac{1}{a_t} \right)}
\]

\[= \lim_{n \to \infty} \frac{n-N}{n} = \lim_{n \to \infty} \frac{n-N}{n} = 1. \]

\[\square\]

Lemma 5.2.4. If \( \{y_n\}_{n=n_0}^{\infty} \) is an eventually positive solution of equation (5.1.1), then there exists an integer \( N \in \mathbb{N}(n_0) \) and a constant \( k_1 > 0 \) such that

\[\frac{1}{2} \Delta(a_n \Delta^2 z_n) R(n) \leq z_n \leq k_1 R(n), \quad n \geq N.\]
Proof. Let \( \{y_n\}_{n=n_0}^{\infty} \) be a positive solution of (5.1.1). Then there exists an integer \( n_1 \in \mathbb{N}(n_0) \) such that \( \sigma(n) > 0 \) for \( n \in \mathbb{N}(n_1) \). Now from Lemma 5.2.4 we have \( \Delta z_n > 0 \) for \( n \in \mathbb{N}(n) \) for some \( N \geq n_1 \). Now

\[
\Delta^2(a_n \Delta^2 z_n) < 0.
\]

Summing the above inequality twice from \( n_0 \) to \( n - 1 \) we have

\[
\Delta^2 z_n < \frac{A_0 n}{a_n} + \frac{A_1}{a_n}; n \in \mathbb{N}(n_0),
\]

where \( A_0 \) and \( A_1 \) are constants. Summing the last inequality again from \( n_0 \) to \( n - 1 \), we have

\[
\Delta z_n < A_0 \sum_{s=n_0}^{n-1} \frac{s}{a_s} + A_1 \sum_{s=n_0}^{n-1} \frac{1}{a_s} + A_2, \quad n \in \mathbb{N}(n_0),
\]

where \( A_2 \) is a constant.

A final summation of the last inequality yields

\[
z_n < A_0 \left( \sum_{t=n_0}^{n-1} \frac{t}{a_t} \right) + A_1 \sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} \frac{1}{a_t} + A_2(n - n_0); \quad n \in \mathbb{N}(n_0)
\]

for some constant \( A_3 \). It is easy to see that there exist \( k_1 > 0 \) and a positive integer \( N \) such that

\[
z_n \leq k_1 R(n) \quad \text{for } n \geq N.
\]

To prove the other side of the inequality let \( N \in \mathbb{N}(n_0) \) be large enough so that \( \{y_n\} \) satisfies Case I or Case II of Lemma 5.2.4. Assume that Case I holds. Since \( \{\Delta(a_n \Delta^2 z_n)\} \) is positive and nonincreasing, it
follows that

\[ a_n \Delta^2 z_n \geq a_n \Delta^2 z_n - a_N \Delta^2 z_N \]
\[ = \sum_{s=N}^{n-1} \Delta a_n \Delta^2 z_n \]
\[ \geq (\Delta(a_{n-1} \Delta^2 z_{n-1}))(n - N), \]

or

\[ \Delta^2 z_n \geq \frac{n - N}{a_n} \Delta(a_n \Delta^2 z_n), \quad n \in \mathbb{N}(N). \]

Summing the last inequality from \( N \) to \( n - 1 \) we obtain,

\[ \Delta z_n \geq \sum_{s=N}^{n-1} \frac{s - N}{a_s} \Delta(a_s \Delta^2 z_s) \geq \Delta(a_n \Delta^2 z_n) \sum_{s=N}^{n-1} \frac{s - N}{a_s}, \]

or

\[ z_n \geq \sum_{s=N}^{n-1} \Delta(a_s \Delta^2 z_s) \sum_{t=N}^{s-1} \frac{t - N}{a_t} \]
\[ \geq \Delta(a_n \Delta^2 z_n) \sum_{s=N}^{n-1} \sum_{t=N}^{s-1} \frac{t - N}{a_t} \]
\[ \geq \Delta(a_n \Delta^2 z_n) R(n, N). \]

Using Lemma 5.2.5 we get

\[ z_n \geq \Delta(a_n \Delta^2 z_n) \frac{1}{2} R(n). \]

Next, assume that Case II holds. Multiplying equation (5.1.1) by \( R(n + 2, N) \) and summing from \( N \) to \( N - 1 \), we obtain

\[ \sum_{s=N}^{n-1} R(s+2,N) \Delta^2(a_s \Delta^2 z_s) + \sum_{s=N}^{n-1} R(s+2,N) f(s, y_{\sigma(s)}) = 0. \]
By a repeated summation by parts and using $\Delta(a_{n-1}\Delta^2 R_{(n-1,N)}) = 1$, we obtain

$$R_{(n+1,N)}(a_n\Delta^2 z_n) - \Delta R(n, N)a_n\Delta^2 z_n + a_{n-1}(\Delta^2 R_{(n-1,N)})\Delta y_n - z_n + \sum_{s=N}^{n-1} R(s + 2, N)f(s, y_{\sigma(s)}) \leq 0.$$ 

Now, Case II of Lemma 2.1 implies that

$$z_n = R(n + 1, N)\Delta(a_n\Delta^2 z_n) \geq R(n, N)\Delta(a_n\Delta^2 z_n), n \in \mathbb{N}(N).$$

This completes the proof of the lemma.

Lemma 5.2.5. Let $\{y_n\}_{n=n_0}^{\infty}$ be an eventually positive solution of equation (5.1.1), then there exists an integer $n_1 \in \mathbb{N}(n_0)$ such that for any integer $N \geq n_1$ we have

$$z_n \geq \sum_{s=N}^{n-1} R(s, N)f(s, y_{\sigma(s)}), \quad n \geq N.$$ 

Proof. Assume that $\{y_n\}$ is a nonoscillatory solution of (5.1.1). Without loss of generality we may assume that $y_{\sigma(s)} > 0$ for all $n \in \mathbb{N}(n_1), n_1 \geq n_0$. By Lemma 5.2.4 $\{\Delta(a_n\Delta^2 z_n)\}$ is positive and decreasing. Summing equation (5.1.1), we obtain

$$\Delta(a_n\Delta^2 z - n) \geq \sum_{s=n}^{\infty} f(s, y_{\sigma(s)}), n \in \mathbb{N}(n_1).$$

From Lemma 5.2.4, we have

$$z_n \geq R(n, N)\Delta(a_n\Delta^2 z_n) \text{ for } n \in \mathbb{N}(n), N \geq n_1.$$
Combining the last two inequalities, we obtain

\[ z_n \geq R(n < N) \sum_{s=n}^{\infty} f(s, y_{\sigma(s)}), n \geq N. \]

Hence the proof is complete. \( \square \)

**Lemma 5.2.6.** If \( \{y_n\}_{n=n_0}^{\infty} \) is an eventually positive solution of equation (5.1.1), then there exists an integer \( N \in \mathbb{N}(n_0) \) such that

\[ \Delta z_n \geq \frac{1}{2} \Delta(a_n \Delta^2 z_n) \Delta R(n) \text{ for } n \geq N. \]

Also if \( \sigma(n) \leq n \), then \( \Delta z_{\sigma(n)} \geq \frac{1}{2} \Delta(a_n \Delta^2 z_n) \Delta R(\sigma(n)) \) for \( n \geq N. \)

**Proof.** From Lemma 5.2.2 we have for \( n \geq n_1 \in \mathbb{N}(n_0), z_n > 0, \Delta z_n > 0 \) and \( \Delta^2(a_n \Delta^2 z_n) < 0 \). Hence

\[
\Delta z_n \geq \sum_{s=n_1}^{n-1} \Delta^2 z_s \\
= \sum_{s=n_1}^{n-1} \frac{1}{a_s} a_s \Delta^2 z_s \\
\geq \sum_{s=n_1}^{n-1} \frac{1}{a_s} \left( \sum_{t=n_1}^{s-1} \Delta(a_t \Delta^2 z_t) \right) \\
\geq \Delta(a_n \Delta^2 z_n) \sum_{s=n_1}^{n-1} \frac{s - n_1}{a_s} \\
= \Delta(a_n \Delta^2 z_n) \Delta R(n, n_1).
\]

From Lemma 5.2.3, we conclude that there exists an integer \( N \geq n_1 \) such that

\[ \Delta R(n, n_1) \geq \frac{1}{2} \Delta R(n) \text{ for } n \geq N, \]
and hence
\[ \Delta z_n \geq \frac{1}{2} \Delta (a_n \Delta^2 z_n) \Delta R(n) \text{ for } n \geq N. \] (5.2.2)

Since \( \Delta^2(a_n \Delta^2 z_n) < 0 \) and \( \sigma(n) \leq n \), we have
\[ \Delta z_{\sigma(n)} \geq \frac{1}{2} \Delta (a_n \Delta^2 z_n) \Delta R(\sigma(n)) \text{ for } n \geq N. \] (5.2.3)

The proof is now complete. \( \square \)

Lemma 5.2.7. If \( \{y_n\} \) is an eventually positive solution of equation 5.1.1, then there exists an integer \( N \in \mathbb{N}(n_0) \) such that
\[ (1 - p_n) z_n \leq y_n \leq z_n \text{ for } n \geq N. \]

Proof: Let \( \{y_n\} \) be an eventually positive solution of equation (5.1.1), for \( n \geq N \). Then from definition of \( z_n \), we have \( z_n \geq y_n \) for \( n \geq N \).

From Lemma 5.2.2, we have \( z_n > 0 \) and \( \Delta z_n > 0 \) for \( n \geq N \). Hence
\[ y_n = z_n - p_n y_{n-k} \geq z_n - p_n z_{n-k} \geq (1 - p_n) z_n \text{ for } n \geq N. \] The proof is now complete. \( \square \)

5.3 Existence of Nonoscillatory Solutions

In this section we derive necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (5.1.1) with some specified asymptotic behavior. We begin with the following theorem which provides an existence of a nonoscillatory solution \( \{y_n\} \) of equation (5.1.1)
such that \( \lim_{n \to \infty} \frac{z_n}{R(n)} = \alpha \neq 0 \).

**Theorem 5.3.1.** A necessary and sufficient condition for the equation (5.1.1) to have a nonoscillatory solution \( \{y_n\} \) such that \( \lim_{n \to \infty} \frac{z_n}{R(n)} = \alpha \neq 0 \) is that

\[
\sum_{n=n_0}^{\infty} f(n, c(1 - p_{\sigma(n)})R(\sigma(n))) < \infty, \quad (5.3.1)
\]

for some \( c \neq 0 \)

**Proof. Necessity:** Let \( \{y_n\} \) be a nonoscillatory solution of equation (5.1.1) such that \( \lim_{n \to \infty} \frac{z_n}{R(n)} = \alpha \neq 0 \). We may assume without loss of generality that \( \{y_n\} \) is eventually positive. Then there exists an integer \( N \in \mathbb{N}(n_0) \) and constants \( \alpha_1 > 0, \alpha_2 > 0 \) such that

\[
\alpha_1 R(\sigma(n)) \leq z_{\sigma(n)} \leq \alpha_2 R(\sigma(n)) \text{ for } n \geq N. \quad (5.3.2)
\]

In view of (5.2.2) and (5.3.2), we obtain

\[
y_{\sigma(n)} \geq (1 - p_{\sigma(n)})\alpha_1 R(\sigma(n)) \text{ for } n \geq N. \quad (5.3.3)
\]

Since \( \Delta(a_n \Delta^2 z_n) > 0 \) by Lemma 5.2.2, on summing equation (5.1.1) we obtain

\[
\sum_{n=N}^{\infty} f(n, y_{\sigma(n)}) < \infty. \quad (5.3.4)
\]

From (5.3.3) and (5.3.4) we conclude that

\[
\sum_{n=N}^{\infty} f(n, \alpha_1(1 - p_{\sigma(n)})R(\sigma(n))) < \infty.
\]
Sufficiency: Suppose that (5.3.1) holds for some \( c \neq 0 \). We may assume that \( c > 0 \), since a similar proof holds if \( c < 0 \). Let \( d > 0 \) be such that \( \frac{4d}{1-p} < c \) and choose \( N \in \mathbb{N}(n_0) \) so large that

\[
\sum_{n=N}^{\infty} f(n, c(1 - p_{\sigma(n)})R(\sigma(n))) < \frac{(1-p)d}{8},
\]

and

\[
N_0 = \min\{N, N - k, \inf_{n \geq N} \sigma(n)\} \geq n_0.
\]

Let \( B_{N_0} \) be the linear space of all real sequences \( \{y_n\}_{n=N_0}^{\infty} \) such that

\[
\sup_{n \geq N_0} \frac{|y_n|}{R(n)} < \infty.
\]

It is not difficult to see \( B_{N_0} \) is endowed with the norm

\[
\| y \| = \sup_{n \geq N_0} \frac{|y_n|}{R(n)}
\]

is a Banach space. Consider the following subset

\[
S = \{y \in B_{N_0} : 2(1-p)dR(n) \leq y_n \leq 4dR(n)
\]

for \( n \geq N \) and \( y_n = 0 \) for \( N_0 \leq n < N \}

of the Banach space \( B_{N_0} \). Define an operator \( \mathcal{F} : S \to B_{N_0} \) by

\[
(\mathcal{F}y)_n = \left\{ \begin{array}{ll}
(3 + p)dR(n) - p_n y_{n-k} & \quad n \geq N, \\
\sum_{s_3=N}^{n-1} \sum_{s_2=N}^{s_3-1} \frac{1}{a_{s_3}} \sum_{s_1=N}^{s_2-1} \sum_{s=s_1}^{\infty} f(s, y_{\sigma(s)}), & \quad N_0 \leq n < N.
\end{array} \right.
\]

We assert that the assumptions of Lemma 2.3.1 are satisfied. First of all, as one can easily verify that \( S \) is a closed, bounded and convex subset of \( B_{N_0} \). Next, \( \mathcal{F} \) is an invariant mapping. Indeed, for any \( \{y_n\} \) in \( S \), we
have

\[(\mathcal{F}y)_n \leq (3 + p)dR(n) + \sum_{s_3=N}^{n-1} \sum_{s_2=N}^{s_3-1} \sum_{s_1=N}^{s_2-1} \frac{(1 - p)d}{8} s_1 \]

\[\leq (3 + p)dR(n) + \frac{(1 - p)d}{8} R(n)\]

\[\leq 4dR(n),\]

and

\[(\mathcal{F}y)_n \geq (3 + p)dR(n) - p_n y_{n-k}\]

\[\geq (3 + p)dR(n) - 4pdR(n)\]

\[\geq 2(1 - p)dR(n)\].

Next we shall show that \(F\) is continuous. Let \(\{y^{(i)}\}\) be a sequence in \(S\) such that \(y^{(i)} \to y = \{y_n\}\). Since \(S\) is closed, \(y \in S\). Further more,

\[| (\mathcal{F}y^{(i)})_n - (\mathcal{F}y)_n | \]

\[\leq p_n \|y^{(i)} - y\| + \sum_{s_3=N}^{n-1} \sum_{s_2=N}^{s_3-1} \sum_{s_1=N}^{s_2-1} \sum_{s=N}^{\infty} f(s, y^{(i)}_{\sigma(s)}) - f(s, y_{\sigma(s)}) |\]

\[\leq p \|y^{(i)} - y\| + \sum_{s_3=N}^{\infty} \sum_{s_2=N}^{s_3-1} \sum_{s_1=N}^{s_2-1} \sum_{s=N}^{\infty} f(s, y^{(i)}_{\sigma(s)}) - f(s, y_{\sigma(s)}) |.

Since by continuity

\[\lim_{i \to \infty} f(s, y^{(i)}_{\sigma(s)}) - f(s, y_{\sigma(s)}) | = 0,\]

and

\[|f(s, y^{(i)}_{\sigma(s)}) - f(s, y_{\sigma(s)})| < \varepsilon,\]
for \( n \geq n_0 \). We see from Lebesgue's dominated convergence theorem that

\[
\lim_{i \to \infty \left\{ \sup_{n \geq N_0} \left| \frac{(Fy^{(i)})_n}{R(n)} - \frac{(Fy)_n}{R(n)} \right| \right\} = 0,
\]

that is

\[
\lim_{i \to \infty} \left\| (Fy^{(i)})_n - (Fy)_n \right\| = 0.
\]

Thus \( F \) is continuous.

Finally, we prove that \( FS \) is uniformly Cauchy. Indeed, let \( \{y_n\} \in B_{N_0} \).

Then for \( m, n \geq N \),

\[
| (Fy)_n - (Fy)_m | = | p_n y_{n-k} - p_m y_{m-k} | + \sum_{s_1=N}^{m-1} \sum_{s_2=N}^{s_1-1} \sum_{s_3=n}^{s_2} \sum_{s_4=N}^{s_3} \frac{1}{a_{s_2}} f(s, y_{\sigma(s)}) \leq | p_n y_{n-k} - p_m y_{m-k} | + \sum_{s_4=N}^{m} \sum_{s_3=n}^{s_4-1} \sum_{s_2=N}^{s_3} \sum_{s_1=N}^{s_2} f(s, y_{\sigma(s)}) \leq 8pdR(n) + \frac{(1-p)d}{8}R(n).
\]

Hence there exists an integer \( N_1 \) such that \( \left| \frac{(Fy)_n}{R(n)} - \frac{(Fy)_m}{R(m)} \right| < \varepsilon \) whenever \( m, n \geq N_1 \). Since \( \varepsilon \) is independent of \( \{y_n\} \), we see that \( FS \) is uniformly Cauchy.

By means of Lemma 2.3.1, there is an element \( y^* = \{y^{*}_n\} \) in \( S \) such that \( y^* = Fy^* \). We may easily verify that \( y^* \) is an eventually positive
solution of (5.1.1). Furthermore from Stolz's [8] theorem

\[
\lim_{n \to \infty} \frac{z_n}{R(n)} = \lim_{n \to \infty} \frac{\Delta z_n}{\Delta R(n)} = \lim_{n \to \infty} \frac{a_n \Delta^2 z_n}{a_n \Delta^2 R(n)} = \lim_{n \to \infty} \frac{a_n \Delta^2 z_n}{n} = \lim_{n \to \infty} \Delta(a_n \Delta^2 z_n) = (3 + p)d.
\]

Thus, \( \{y_n\} \) is a nonoscillatory solution of equation (5.1.1) with the desired asymptotic property. The proof is now complete. \( \square \)

**Theorem 5.3.2.** A necessary and sufficient condition for the equation (5.1.1) to have a nonoscillatory solution \( \{y_n\} \) such that \( \lim_{n \to \infty} z_n = \beta \neq 0 \) is that

\[
\sum_{n=n_0}^{\infty} R(n) |f(n, c(1 - p_{\sigma(n)}))| < \infty, \quad (5.3.5)
\]

for some \( c \neq 0 \).

**Proof.** Necessity: Let \( \{y_n\} \) be a nonoscillatory solution of equation (5.1.1) such that \( \lim_{n \to \infty} z_n = \beta \neq 0 \). We may assume without loss of generality that \( \{y_n\} \) is eventually positive. Then there exist an integer \( N \in \mathbb{N}(n_0) \) and constants \( \beta_1 > 0, \beta_2 > 0 \) for which \( \beta_1 \leq z_{\sigma(n)} \leq \beta_2 \) for \( n \geq N \). Hence from (5.2.2), we have

\[
y_{\sigma(n)} \geq \beta_1(1 - p_{\sigma(n)}) \text{ for } n \geq N. \quad (5.3.6)
\]

Multiply equation (5.1.1) by \( R(n) \) and summing it from \( N \) to \( n - 1 \), we
obtain
\[
\sum_{s=N}^{n-1} R(s) f(s, y_{\sigma(s)}) = -\sum_{s=N}^{n-1} R(s) \Delta^2(a_s \Delta^2 z_s)
\]
\[
= -R(n) \Delta(a_n \Delta^2 z_n) + \Delta R(n) a_{n+1} \Delta^2 z_{n+1} - (n+1) \Delta z_{n+1} + z_{n+3} + \lambda,
\]  
(5.3.7)
where \(\lambda\) is a constant. Observe that \(z_n\) is subject to the Case II of Lemma 5.2.2, we deduce from (5.3.7) that
\[
\sum_{n=N}^{\infty} R(n) f(n, y_{\sigma(n)}) < \infty.
\]  
(5.3.8)
From (5.3.6) and (5.3.8), we obtain
\[
\sum_{s=N}^{\infty} R(n) f(n, \beta_1 (1 - p_{\sigma(n)})) < \infty,
\]  
and hence the result follows.

**Sufficiency:** Suppose that (5.3.5) holds for some \(c > 0\). The case of negative \(c\) can be treated similarly. Let \(d \leq \frac{(1-p)}{2} d\), and take \(N \in \mathbb{N}(n_0)\) so large that
\[
\sum_{n=N}^{\infty} R(n) f(n, c(1 - p_{\sigma(n)})) \leq \frac{(1-p)}{4} d,
\]  
and
\[
N_0 = \min\{N, N - k, \inf_{n \geq N} \sigma(n)\} \geq n_0.
\]
Let \(B_{N_0}\) be the space considered in the proof of Theorem 5.3.1 with norm \(\| y_n \| = \sup_{n \geq N} |y_n|\). We define a bounded, closed and convex subset \(S\) of
$B_{N_0}$ by

$$S = \{ y \in B_{N_0} : (1 - p)d \leq y_n \leq 2d \text{ for } n \geq N \text{ and } y_n = y_N \text{ for } N_0 \leq n < N \}$$

and an operator $\mathcal{F} : S \rightarrow B_{N_0}$ by

$$(\mathcal{F}y)_n = \begin{cases} 
(1 + p)d - p_n y_{n-k} \\
+ \sum_{s_3 = N}^{n-1} \sum_{s_2 = s_3}^{\infty} \sum_{s_1 = s_2}^{\infty} \sum_{s = s_1}^{\infty} f(s, y_{\sigma(s)}), & n \geq N \\
(\mathcal{F}y)_N, & N_0 \leq n < N.
\end{cases}$$

Arguing as in Theorem 5.3.1, we can easily show that $\mathcal{F}$ satisfies all conditions of Lemma 2.3.1. Therefore, there exists $y \in S$ such that $y = \mathcal{F}y$, that is, $\{y_n\}$ is a nonoscillatory solution of equation (5.1.1).

Since

$$\Delta z_n = \sum_{s_2 = n}^{\infty} \sum_{s_1 = s_2}^{\infty} \sum_{s = s_1}^{\infty} f(s, y_{\sigma(s)}) > 0,$$

it follows that $\lim_{n \to \infty} z_n = \beta \in [(1 - p)d, 2d]$. This completes the proof of Theorem 5.3.2.

Now we find conditions under which conditions of Theorem 5.3.1 (respectively Theorem 5.3.2) yields

$$\lim_{n \to \infty} \frac{y_n}{R(n)} = \text{constant} \neq 0 (\text{respectively } \lim_{n \to \infty} y_n = \text{constant} \neq 0).$$

(5.3.9)
On combining Theorems 5.3.1 and 5.3.2 with Lemma 5.2.1 (iv) and (iii), we have the following result, which ensure the existence of nonoscillatory solution of (5.1.1) having property (5.3.9).

**Theorem 5.3.3.** Assume that condition (iv) of Lemma 5.2.1 holds. Then equation (5.1.1) has a nonoscillatory solution \( \{y_n\} \) such that \( \lim_{n \to \infty} \frac{y_n}{R(n)} = \text{constant} \neq 0 \), if and only if (5.3.5) is satisfied.

**Theorem 5.3.4.** Assume that condition (iii) of Lemma 5.2.1 holds. Then equation (5.1.1) has a nonoscillatory solution \( \{y_n\} \) such that \( \lim_{n \to \infty} y_n = \text{constant} \neq 0 \), if and only if (5.3.5) is satisfied.

**Example 5.3.1.** Consider the neutral difference equation of the form

\[
\Delta^2 \left( (n+1)(n+2) \Delta^2 \left( y_n + \frac{n-k}{2(n-k-1)} y_{n-k} \right) \right) + \frac{4(n-\ell)^3}{n(n+1)(n+2)(n-\ell-1)^3} y_{n-\ell}^3 = 0,
\]

where \( n \geq \max\{k+\ell+1, k+2\} \) where \( k \) and \( \ell \) are positive integers. It is easy to check that condition (5.3.5) is satisfied but the condition (5.3.1) is not satisfied for the equation (5.3.10). Therefore by Theorems 5.3.1 and 5.3.3, there exists no nonoscillatory solution of (5.3.10) such that \( \lim_{n \to \infty} \frac{y_n}{R(n)} = \text{constant} \neq 0 \), and by Theorem 5.3.2 and 5.3.4, there exists a nonoscillatory solution of (5.3.10) such that \( \lim_{n \to \infty} y_n = \text{constant} \neq 0 \). In fact \( \{y_n\} = \left\{ \frac{n-1}{n} \right\} \) is such a solution of equation (5.3.10).
5.4 Oscillation Theorems

In this section, we establish conditions for the oscillation of all solutions of equation (5.1.1) when \( f \) is strongly sublinear or strongly superlinear as given in Definition 2.1.1.

**Theorem 5.4.1.** Let \( f \) be strongly sublinear and \( \sigma(n) \leq n \). A necessary and sufficient condition for all solutions of equation (5.1.1) are oscillatory is that

\[
\sum_{n=n_0}^{\infty} |f(n, c(1 - p_{\sigma(n)})R(\sigma(n)))| = \infty,
\]

(5.4.1)

for all \( c \neq 0 \)

**Proof.** The necessity of condition (5.4.1) follows from the sufficiency part of Theorem 5.3.1. Now we prove the sufficiency of condition (5.4.1).

Assume that there exists a nonoscillatory solution \( \{y_n\} \) of equation (5.1.1). Without loss of generality we may assume that \( \{y_n\} \) is eventually positive. From Lemmas 5.2.2, 5.2.4 and 5.2.7 there exists \( N \geq n_0 \) and \( k > 0 \) such that \( z_n > 0, \Delta z_n > 0 \) and \( \Delta(a_n \Delta^2 z_n) > 0 \) for \( n \geq N \);

\[
y_{\sigma(n)} \geq (1 - p_{\sigma(n)})z_{\sigma(n)} \text{ for } n \geq N,
\]

(5.4.2)

and

\[
\frac{1}{2} \Delta(a_n \Delta^2 z_n)R(n) \leq z_n \leq kR(n); \quad n \geq N.
\]

(5.4.3)

Since \( \sigma(n) \leq n \) and \( \Delta^2(a_n \Delta^2 z_n) < 0 \) we have from Lemma 5.2.4,
$$z_{\sigma(n)} \geq \frac{1}{2} \Delta(a_n \Delta^2 z_n) R(\sigma(n)); \quad n \geq N. \quad (5.4.4)$$

From (5.4.2)-(5.4.4) and strong sublinearity of \( f \), we have

$$\Delta(-\Delta(a_n \Delta^2 z_n))^{1-\beta} = (1 - \beta) \xi^{-\beta} f(n, y_{\sigma(n)}) ,$$

where

$$\Delta(a_{n+1} \Delta^2 z_{n+1}) < \xi < \Delta(a_n \Delta^2 z_n).$$

Thus,

$$\Delta(-\Delta(a_n \Delta^2 z_n))^{1-\beta}$$

$$= (1 - \beta) \xi^{-\beta} f(n, y_{\sigma(n)})$$

$$\geq (1 - \beta)(\Delta(a_n \Delta^2 z_n))^{-\beta} \frac{f(n, y_{\sigma(n)})}{y_{\sigma(n)}}$$

$$\geq (1 - \beta)(\Delta(a_n \Delta^2 z_n))^{-\beta} \frac{f(n, (1 - p_{\sigma(n)})z_{\sigma(n)})}{((1 - p_{\sigma(n)})z_{\sigma(n)})^\beta}$$

$$\geq (1 - \beta)(\Delta(a_n \Delta^2 z_n))^{-\beta} \frac{f(n, (1 - p_{\sigma(n)})kR(\sigma(n))}{k^\beta(R(\sigma(n)))^\beta}$$

$$\geq \frac{(1 - \beta)}{(2k)^\beta} f(n, (1 - p_{\sigma(n)})kR(\sigma(n))).$$

Summing the last inequality from \( N \) to \( n - 1 \), we obtain

$$\frac{(1 - \beta)}{(2k)^\beta} \sum_{s=N}^{n-1} f(s, k(1 - p_{\sigma(s)})R(\sigma(s))) \leq (\Delta(a_N \Delta^2 z_N))^{1-\beta} ,$$

which leads to

$$\sum_{s=N}^{\infty} f(s, k(1 - p_{\sigma(s)})R(\sigma(s))) < \infty,$$
and hence contradicts (5.4.1). This completes the proof.

\[ \square \]

**Theorem 5.4.2.** Let $f$ be strongly superlinear and $\sigma(n) \geq n$. Then a necessary and sufficient condition for all solutions of equation (5.1.1) are oscillatory is that

\[
R(n)|f(n, c(1 - p_{\sigma(n)}))| = \infty, \quad (5.4.5)
\]

for all $c \neq 0$.

**Proof.** The necessity of condition (5.4.5) follows from the sufficiency part of Theorem 5.3.2. We prove the sufficiency of condition (5.4.5). Assume that there exists a nonoscillatory solution \( \{y_n\} \) of equation (5.1.1). Without loss of generality we may assume that \( \{y_n\} \) is eventually positive. From Lemmas 5.2.2, 5.2.5 and 5.2.7 there exists $N_1 \geq n_0$ such that

\[
z_n > 0, \Delta z_n > 0, \Delta (a_n \Delta^2 z_n) > 0 \quad n \geq N_1;
\]

\[
y_{\sigma(n)} \geq (1 - p_{\sigma(n)}) z_{\sigma(n)};
\]

and

\[
z_n \geq \sum_{s=N}^{n-1} R(s, N) f(s, y_{\sigma(s)}), \quad n \geq N \geq N_1. \quad (5.4.6)
\]

We have from $z_n > 0, \Delta z_n > 0$, there exists a $k_1 > 0$ such that $z_n \geq k_1$ for $n \geq N$ and hence $y_n \geq k_1(1 - p_n)$ for $n \geq N$. From strong superlinearity, and $\sigma(n) \geq n$, we have

\[
f(n, y_{\sigma(n)}) \geq \frac{f(n, k_1(1 - p_{\sigma(n)}))}{(k_1(1 - p_{\sigma(n)}))^\alpha} ((1 - p_{\sigma(n)}) z_{\sigma(n)})^\alpha 
\geq k_1^{-\alpha} z_n^\alpha f(n, k_1(1 - p_{\sigma(n)})). \quad (5.4.7)
\]
From (5.4.6) and (5.4.7) it follows that
\[
    z_n \geq \sum_{s=N}^{n-1} k_1^{-\alpha} z_n^\alpha R(s, N) f(s, 1 - p_{\sigma(s)})
\]
and hence
\[
    \Delta \left( \left( \sum_{s=N}^{n-1} k_1^{-\alpha} z_n^\alpha R(s, N) f(s, 1 - p_{\sigma(s)}) \right)^{1-\alpha} \right)
    = \frac{(1 - \alpha) k_1^{-\alpha} z_n^\alpha R(n, N) f(n, 1 - p_{\sigma(n)})}{\left( \sum_{s=N}^{n-1} k_1^{-\alpha} z_n^\alpha R(n, N) f(n, 1 - p_{\sigma(n)}) \right)^{1-\alpha}}
    \leq (1 - \alpha) z_n^{-\alpha} k_1^{-\alpha} z_n^\alpha R(n, N) f(n, 1 - p_{\sigma(n)})
    = (1 - \alpha) k_1^{-\alpha} R(n, N) f(n, 1 - p_{\sigma(n)}),
\]
which implies that
\[
    R(n, N) f(n, 1 - p_{\sigma(n)}) \leq \frac{k_1^\alpha}{1 - \alpha} \Delta \left( \left( \sum_{s=N}^{n-1} k_1^{-\alpha} z_n^\alpha R(s, N) f(s, 1 - p_{\sigma(s)}) \right)^{1-\alpha} \right).\]

Summing the last inequality from \(N_2\) to \(n\) and using \(\alpha > 1\) we obtain
\[
    \sum_{s=N_2}^{n} R(s, N) f(s, 1 - p_{\sigma(s)}) \leq \frac{k_1^\alpha}{\alpha - 1} \left( \sum_{s=N}^{N_2} k_1^{-\alpha} z_n^\alpha R(s, N) f(s, 1 - p_{\sigma(s)}) \right)^{1-\alpha}.
\]
Hence \( \sum_{s=N_2}^{\infty} R(s) f(s, 1 - p_{\sigma(s)}) \) \(<\infty\), a contradiction to 5.4.5. This completes the proof. \(\square\)
Theorem 5.4.3. Assume that there exists a real sequence \( \{q_n\} \) such that
\[
\frac{f(n, u)}{u} \geq Mq_n \quad \text{for all } u \neq 0, \ n \geq n_0
\]
and
\[
\sigma(n) = n - \ell, \text{ where } \ell \text{ is a nonnegative integer less than } n \text{ for } n \geq n_0.
\]

If there exists a positive sequence \( \{\rho_n\} \) such that
\[
\limsup_{n \to \infty} \sum_{s=n_0}^{n} \rho_s \left[ (1 - p_{s-\ell})q_s - \frac{(\Delta \rho_s)^2}{2M\Delta R(s-\ell)\rho_s^2} \right] = \infty,
\]
then all solutions of equation (5.1.1) are oscillatory.

Proof: Let \( \{y_n\} \) be a nonoscillatory solution of equation (5.1.1) and assume without loss of generality that \( \{y_n\} \) is eventually positive. From Lemmas 5.2.2 and 5.2.7 we have \( z_n > 0, z_{n-\ell} > 0, \Delta z_n > 0, \) and \( \Delta(a_n\Delta^2 z_n) > 0 \) for \( n \geq N \) and \( y_{n-\ell} \geq (1 - p_{n-\ell})z_{n-\ell}. \) Define
\[
\omega_n = \frac{\rho_n\Delta(a_n\Delta^2 z_n)}{z_{n-\ell}}; n \geq N.
\]
Then in view of Lemma 5.2.6, (5.4.8) and (5.4.9) we have
\[
\Delta \omega_n \leq \frac{\rho_n\Delta(a_n\Delta^2 z_n) + \Delta(a_n\Delta^2 z_n)\Delta \rho_n}{z_{n-\ell}} - \frac{\rho_n\Delta(a_n\Delta^2 z_n)\Delta z_{n-\ell}}{(z_{n-\ell})^2}
\]
\[
\leq -Mq_n(1 - p_{n-\ell})\rho_n + \Delta \rho_n \frac{\omega_n}{\rho_n} - \frac{1}{2\rho_n} \omega_n^2 \Delta R(n-\ell)
\]
\[
\leq -Mq_n(1 - p_{n-\ell})\rho_n + \frac{(\Delta \rho_n)^2}{2\rho_n \Delta R(n-\ell)}.
\]
Summing the last inequality from \( N \) to \( n \geq N, \) we obtain
\[
\sum_{s=N}^{n} \rho_s \left[ (1 - p_{s-\ell})q_s - \frac{(\Delta \rho_s)^2}{2M\Delta R(s-\ell)\rho_s^2} \right] \leq \frac{\omega_N}{M}
\]
and this contradicts (5.4.10). Thus the proof is complete.

For the linear equation

\[ \Delta^4(y_n + p_n y_{n-\tau}) + q_n y_{n-\sigma} = 0, \]

(5.4.11)

where \( \tau \) and \( \sigma \) are nonnegative integers less than \( n \), we obtain from Theorem 5.4.3, the following corollary.

**Corollary 5.4.4.** Suppose \( q_n \geq 0 \) for all \( n \geq n_0 \) and there exists a positive sequence \( \{\rho_n\} \) such that

\[ \lim_{n \to \infty} \sup_{s=n_0}^{n} \rho(s) \left[ (1 - p_{s-\sigma})q_s - \left( \frac{\Delta \rho_s}{\rho_s} \right)^2 \right] = \infty, \]

then all solutions of equation (5.4.11) are oscillatory.

We conclude this chapter with the following examples.

**Example 5.4.1.** Consider the neutral difference equation

\[ \Delta^2 \left( n(n+1) \Delta^2 \left( y_n + \frac{1}{\sqrt{n-1}} y_{n-1} \right) \right) + n y_{n-1}^{\frac{3}{2}} = 0; \quad n \geq 3. \]  

(5.4.12)

It is easy to see that all conditions of Theorem 5.4.2 are satisfied and hence all solutions of (5.4.12) are oscillatory.

**Example 5.4.2.** Theorem 5.4.3 implies that all solutions of the neutral difference equation

\[ \Delta^2 \left( \frac{1}{n+3} \Delta^2 \left( y_n + \frac{1}{n+3} y_{n-3} \right) \right) + n(n+1) y_{n+3}^2 = 0; \quad n \geq 3. \]

are oscillatory.