CHAPTER 3

QUANTUM COORDINATE ALGEBRAS
AND STEINBERG MODULES
0. INTRODUCTION

In this chapter, we attempt to study, in a limited sense, the cohomology for quantum groups. Thanks to the earlier works of Lusztig [24], Tanisaki [29] and Andersen, Polo and Kexin [2], this study has already been neatly developed to the extent of deriving results analogous to the classical case such as Serre duality, Borel-Weil-Bott theory, Kempf vanishing theorem, Serre's theorem etc.

We shall take the cue from Andersen et al, and prove certain results in the $0^{th}$ cohomology groups $H^0(\mathfrak{g}/\mathfrak{g}_{\mathfrak{B}}, \lambda)$, $\lambda \in \mathfrak{P}$, in quantum setup which compare with similar results for classical cases (see [10]).

We shall also study the Steinberg module for the quantum group. But first, we need the concept of quantum coordinate algebras, which in a sense, is dual to the quantum group (algebra) and which possesses all the (usual) required properties to define the cohomology in the present context.
1. QUANTUM COORDINATE ALGEBRA AND COHOMOLOGY GROUP

Let us fix some notations and definitions. We shall be following Andersen, Polo and Kexin [2].

Let us also fix a prime p which will come into play when we are dealing with Steinberg modules.

Let \( Z \) denote the local ring \( Z[\mathfrak{m}] \), where \( q \) is an indeterminate and \( m \) is the maximal ideal in \( Z[q] \) generated by \( q^{-1} \) and \( p \).

Let \( U \) be a quantum group over \( Q(q) \), the fraction field of \( q \), and \( U_\mathfrak{a} \) be an \( \mathfrak{a} \)-subalgebra of \( U \) defined in chapter 1 (sec. 1). Let \( U_\mathfrak{a}^o, U_\mathfrak{a}^+, U_\mathfrak{a}^- \) be the subalgebras of \( U_\mathfrak{a} \) defined earlier. (Note that \( U_\mathfrak{a}^o \) is commutative). Let \( U_\mathfrak{a}^- U_\mathfrak{a}^o = U_\mathfrak{a} B^- \) and \( U_\mathfrak{a}^o U_\mathfrak{a}^+ = U_\mathfrak{a} B^+ \), which are subalgebras of \( U_\mathfrak{a} \). (\( U_\mathfrak{a} B^- \) is the quantum group corresponding to the negative Borel subalgebras).

Let \( W \) be the weyl group corresponding to the root system \( R \) and \( P \), the set of weights. The action of the \( \varphi_\alpha \in W \) is given by \( \varphi_\alpha(\lambda) = \lambda - (\lambda|\alpha)_\alpha \), for any \( \alpha \in R, \lambda \in P \), from which one can get the action of any element \( w \in W \). One also needs the 'dot action' of \( W \) on \( P \) given by
\( w \cdot \lambda = w(\lambda + \rho) - \rho \), for any \( w \in W, \lambda \in \Gamma \). Let \( w_0 \) be the unique element of \( W \) such that \( w_0(R^+) = -R^+ \). This implies that \( w_0^2 \) fixes \( R^+ \) and hence \( w_0^2 = 1 \).

It has been known (see Rosso [28] p. 585, and Andersen et al [2] p. 5) that one can associate a "character" (algebra homomorphism) \( \chi_\lambda : U^{\text{\mathcal{O}}}_{\mathfrak{g}} \rightarrow \mathcal{A} \) for a given weight \( \lambda \) as follows.

\[ \chi_\lambda(K) = q^{d\lambda}. \]

If \( M \) is a \( U_{\mathfrak{g}} \) - module and \( \lambda \in \Gamma \), the \( \chi_\lambda \) weight space \( M_\lambda \) of \( M \) is given by

\[ M_\lambda = \left\{ x \in M \mid ux = \chi_\lambda(u)x, \text{ for all } u \in U^{\text{\mathcal{O}}}_{\mathfrak{g}} \right\}. \]

Let \( M^* = \bigoplus_{\lambda \in \Gamma} M_\lambda \) be the direct sum of the weight spaces.

We define \( F(M) = \left\{ x \in M^* \mid E_1^{(r)}x = F_1^{(r)}x = 0, 1 \leq i \leq n, r > 0 \right\}. \)

\( M \) is said to be integrable of type 1 if \( M = F(M) \) (see Chapter 1, sec. 2, p. 5).

From now onwards we consider the integrable \( U_{\mathfrak{g}} \) - module of type one only.
It is a classical fact that in order to study the cohomology group, we need the "induction functor". It has been successfully defined in our context of quantum groups by Andersen et al. Let us briefly recount this notion which is essential for our study of quantum coordinate algebras and cohomology groups.

Let $\mathcal{B}$ denote the category of $\mathcal{A}$ - modules and let $\mathcal{E}$ denote the category of integrable $U_{\mathcal{A}}$ - modules. We define the "induction functor" $H : \mathcal{B} \rightarrow \mathcal{E}$ as follows:

Let $\mathcal{I}$ be the set of two sided ideals of $U_{\mathcal{A}}$ which satisfy the following conditions:

i. $U_{\mathcal{A}}/I$ is an $\mathcal{A}$ - module of finite type.

ii. $I \cap U_{\mathcal{A}}^{o}$ contains a finite intersection of ideals $\left\{ \text{Ker}(\chi_{\lambda}) \right\}$ for all $\chi_{\lambda} : U_{\mathcal{A}}^{o} \rightarrow \mathcal{A}$, $(\lambda \in \mathcal{P})$.

For an $\mathcal{A}$ - module $M$, we define

$$H(M) = \left\{ f \in \text{Hom}_{\mathcal{A}}(U_{\mathcal{A}}, M) \mid f(I) = 0 \text{ for some } I \in \mathcal{I} \right\}.$$
The following facts are to be noted here:

i. \( \text{Hom}_{{\mathfrak{sl}_n}}(U_\mathfrak{g}, M) \), which we shall sometimes denote by \( \mathcal{H}(M) \), is a \( U_\mathfrak{g} \) - module by the action:

\[
(u, \theta)(x) = \theta(xu), \quad u, x \in U_\mathfrak{g}, \quad \theta \in \mathcal{H}(M). \quad ...(3.1.1)
\]

ii. \( H(M) \) coincides with the \( U_\mathfrak{g} \) - submodule \( F(\mathcal{H}(M)) \) of \( \mathcal{H}(M) \) and hence \( H(M) \) is integrable and belongs to the category \( \mathcal{C} \).

iii. The functor \( H \) from \( \mathcal{B} \) to \( \mathcal{C} \) is exact and that \( \mathfrak{g}[U_\mathfrak{g}] \) is a free \( \mathfrak{g} \) - module.

Definition: 3.1.2

We define the quantum coordinate algebra \( \mathfrak{g}[U_\mathfrak{g}] \) to be \( H(\mathfrak{g}) \).

We now proceed to define the \( U_\mathfrak{g} \) - module \( H^i(U_\mathfrak{g}/U_\mathfrak{g}B^-, M) \), for all \( i \geq 0 \).

Let \( I' \), \( I \) and \( J \) be subset of \( \{1, 2, \ldots, n\} \) such that \( I' \subseteq I \). Let \( U^1 \) and \( U^2 \) denote the subalgebras of \( U_\mathfrak{g} \) defined by

\( U^1 \) = the subalgebra generated by \( U_\mathfrak{g}B^- \) and

\[
\left\{ E_i^{(r)} \mid i \in I, \quad r \geq 0 \right\} \quad \text{and}
\]
$U^2$ = the subalgebra generated by $U_B$ and 
\[ \left\{ E_i^{(r)} \mid i \in I', r \geq 0 \right\}, \text{so that } U^2 \subseteq U^1. \]

Let $\mathcal{E}^1$ denote the category of $U^1$-module $V$ such that
\[
V = F^I(V) = \left\{ x \in \bigoplus \sum_{\lambda} V_{\lambda} \mid E_i^{(r)} x = 0 = F_i^{(r)} x, \right. \\
\left. \lambda \in \mathbb{P} \right. \\
\text{for all } i \in I, r \gg 0 \right\}.
\]

and $\mathcal{E}^2$ denote the category of $U^2$-module $V$ such that
\[
V = F^{I'}(V) = \left\{ x \in \bigoplus \sum_{\lambda} V_{\lambda} \mid E_i^{(r)} x = 0 = F_i^{(r)} x, \right. \\
\left. \lambda \in \mathbb{P} \right. \\
\text{for all } i \in I', r \gg 0 \right\}.
\]

For a module $M \in \mathcal{E}^2$, we can see that $\text{Hom}_{U^2}(U^1, M)$ is a $U^1$-submodule of $\text{Hom}_{U^2}(U^1, M)$ in the sense of (3.1.1).

We now set
\[
H^0(U^1/U^2, M) = F(\text{Hom}^I_{U^2}(U^1, M)) ... (3.1.3)
\]
which belongs to $\mathcal{E}^1$. It is seen that $H^0(U^1/U^2, -)$ is a covariant left exact functor from $\mathcal{E}^2$ to $\mathcal{E}^1$.

The following results concerning injective objects in the categories $\mathcal{E}^1$ and $\mathcal{E}^2$ enable us to define the right derived functor $H^i(U^1/U^2, M)$ of induction.
I. The induction functor \( \varepsilon^2 \rightarrow \varepsilon^1 \) takes injective objects to injective objects.

II. The category \( \varepsilon^1 \) has enough injective objects.

For our purpose, we shall take \( U^1 = U \) and \( U^2 = U B^- \) and simply denote \( H^i(U U_B^-, M) \) by \( H^i(M) \).

Now let \( \lambda \in P^+ \), the set of dominant weights. If we take \( M \) to be the \( U B^- \) module corresponding to the (one-dimensional) character \( \lambda \), we denote \( H^i(M) \) by \( H^i(\lambda) \).

2. BASE CHANGE

Let \( \mathfrak{g} \rightarrow \Gamma \) be a homomorphism of \( \mathfrak{g} \) into a field \( \Gamma \), so that \( \Gamma \) becomes an \( \mathfrak{g} \) - algebra. We denote the image of \( \mathfrak{g} \) in \( \Gamma \) by \( \varepsilon \), which we have called a "specialization" (see chapter 1 [Sec. 4]).

If \( \varepsilon \) is a primitive \( \ell \)th root of unity, we know from Andersen et al. ([2], p. 38) that \( \ell \) must be a power of \( P \).

We set \( U_\Gamma = U \otimes \Gamma \) which is a quantum group on \( \Gamma \).

We write \( H^i_\Gamma = H^i(U_\Gamma / U_\Gamma B^-, -) \).
3. SOME KNOWN RESULTS (see [2])

3.3.1 (Lemma 1.13, p. 8)

Let $M \in \mathcal{C}$. If $\lambda$ is a weight of $M$, then so is $w\lambda$, for any $w \in W$.

3.3.2 (Section 2.9, p. 19)

If $M$ is a $U_\mathfrak{g}$ - module, we set:

$$M^{\mathfrak{g}} = \left\{ x \in M \mid ux = \nu(u)x \text{ for all } u \in U_\mathfrak{g} \right\}$$

where $\nu$ is the Counit. This is called the space of $U_\mathfrak{g}$ - invariants in $M$.

3.3.3 (Borel-Weil-Bott-theory, p.36, Proposition 6.1)

The module $H^i(\lambda) = H^i(U_\mathfrak{g} / U_\mathfrak{g}^+ \lambda), \lambda \in P, i \geq 0$.

Let $\lambda \in P$,

i. $H^0(\lambda) \neq 0$ if and only if $\lambda \in P^+$.

ii. If $\lambda \in P^+$ then $H^0(\lambda)^{U_\mathfrak{g}^+}$ is a free $\mathfrak{g}$ - submodule of rank 1. In fact $H^0(\lambda)^{U_\mathfrak{g}^+} = H^0(\lambda)$.

iii. If $\lambda \in P^+$ then $\lambda$ is the unique maximal weight of $H^0(\lambda)$.
3.3.4  (Corollary 6.2, p. 37)

Let $\lambda \in P^+$. The $H^0_\lambda(\lambda)$ contains a unique simple $U_\lambda$-
module, $L_\lambda(\lambda)$. It has highest weight $\lambda$.

3.3.5  (Theorem 6.4, p. 37)

Suppose $e$ is not a root of 1. Then we have for all
$\lambda \in P$ with $\lambda + \rho \in P^+$ and all $w \in W$.

$$H^1_\lambda(w, \lambda) \cong \begin{cases} H^0_\lambda(\lambda) & \text{if } i = \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

where $\ell(w)$ is a length function defined for $w \in W$.

3.3.6  (Serre duality section 7, p. 39)

$$H^0_\lambda(\lambda) = H^N_\lambda(-\lambda - 2\rho)^*$$

where $N$ is the number of positive roots and $\rho$ is the
half the sum of positive roots.

4.  SOCLE OF $H^0(\lambda)$

We obtain certain useful result concerning the socle of
$H^0(\lambda)$ and the socle of $H^0_\lambda(\lambda)$.

Lemma : 3.4.1

Each weight $\mu$ of $H^0(\lambda)$ satisfies $w_\lambda \mu \leq \mu \leq \lambda$, where $w_\lambda$
is the longest element in the weylgroup $W$. 

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Proof:

From 3.3.3 (iii), we have if $\mu$ is any weight of $H^0(\lambda)$ then $\mu \leq \lambda$ ... (3.4.2)

Also we have if $\mu$ is a weight of $H^0(\lambda)$ then by 3.3.1, $w_0\mu$ is a weight of $H^0(\lambda)$.

Hence by (3.3.3(iii)) $w_0\mu \leq \lambda$ ... (3.4.3)

Clearly $\lambda \leq \mu \iff w_0\mu \leq w_0\lambda$ for all $\lambda, \mu \in P$.

(Since $w_0\lambda = -\lambda$, $w_0\mu = -\mu$)

Moreover $w_0$ is an element of order 2, which gives from (3.4.3) that, $w_0\lambda \leq \mu$. Hence each weight $\mu$ of $H^0(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$. Hence the Lemma. □

Corollary: 3.4.4

Let $\lambda \in P^+$. Then

i. $U_l B^+ \subseteq \text{Soch}^0(\lambda) \subseteq (\text{Soch}^0(\lambda))_\lambda$ and $\dim (\text{Soch}^0(\lambda))_\lambda = 1$.

ii. Any weight $\mu$ of $\text{Soch}^0(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$.

Proof:

i. Follows immediately from 3.3.3 (ii).

ii. Follows from Lemma 3.4.1 □
Lemma : 3.4.5

Let $\lambda \in P^+$. The module dual to $Soc^0(\lambda)$ is $Soc^0(-w_0 \lambda)$.

Proof :

From Dixmier [6] (7.2.8) we get that for any finite dimensional module $V$, the highest weight of $V^*$ is $-\mu$, when $\mu$ is the highest weight of $V$.

In the case of $Soc^0(\lambda)$, $\lambda \in P^+$ and $Soc^0(\lambda)$ is simple implies $Soc^0(\lambda)$ is finite dimensional. This implies that the weight $-\mu$ of $(Soc^0(\lambda))^*$ satisfies $w_0 \lambda \leq -\mu \leq \lambda$.

Hence $-\lambda \leq -w_0 \lambda$.

As $(Soc^0(\lambda))^*$ is simple it has to be isomorphic to $Soc^0(-w_0 \lambda)$. □

Theorem : 3.4.6

Suppose $\epsilon$ is not a root of 1 and $\lambda + \rho \in P^+$, then $Soc^0_1(\lambda) = H^0_1(\lambda)$.

Proof :

We have $w_0(\rho) = -\rho$.

Using dot action $w_0(-w_0 \lambda) = w_0(-w_0 \lambda + \rho) - \rho$

$= -\lambda - 2\rho$. 

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Hence by Serre duality 3.3.6.

\[ H^*_N (w_o(-w_\lambda)) \cong H^*_T(\lambda)^* \]  \hspace{1cm} \ldots \hspace{1cm} (3.4.7)

Where \( N = |R^+| = \ell(w_o) \).

Also 3.3.5, implies for each \( \lambda + \rho \in P^+ \)

\[ H^*_N (w_o(-w_\lambda)) \cong H^*_T(-w_\lambda) \]  \hspace{1cm} \ldots \hspace{1cm} (3.4.8)

From (3.4.7) and (3.4.8), we get

\[ H^*_T(\lambda)^* \cong H^*_T(-w_\lambda) \]

\( \text{Soc} H^*_T(\lambda) \cong H^*_T(\lambda)/M \), where \( M \) is the maximal submodule.

Since any composition factor of \( M \) has highest weight less than \( \lambda \). As \( \text{Soc} H^*_T(\lambda) \) occurs with multiplicity 1 as a composition factor in \( H^*_T(\lambda) \).

\( \text{Since, } \dim (\text{Soc} H^*_T(\lambda))_\lambda = 1 = \dim H^*_T(\lambda)_\lambda \)

This implies \( M = 0 \)

and \( \text{Soc} H^*_T(\lambda) = H^*_T(\lambda) \).

Suppose that \( I \) is a non-empty subset of \( \Pi \) (simple roots).

Let \( R_I = R \cap \mathbb{Z}I \) and \( R_I^+ = R^+ \cap \mathbb{Z}I \), \( R'_I = R^+ - R_I \).

Also let \( N' = |R_I| \) and \( n' = |I|, N' = |R'| \).
We define the following subalgebra of $U_q$.

Let $U_I$ be the subalgebra of $U_q$, which is generated by

$$\left\{ \begin{array}{c} r_1, \ldots, E^{N''}, j_1, \ldots, j, F^{N''}, \prod_{i=1}^n k_i, r_i, j_i \in \mathbb{Z}_+, \\
\alpha(N''), \alpha(i) \in \mathbb{R}^+ \end{array} \right\}$$

$$\alpha(i) \in \mathbb{R}^+, \quad (m_1, \ldots, m_n) \in \mathbb{Z}^n$$

Let $U_I^+$ be the subalgebra of $U_I$ generated by

$$\left\{ \begin{array}{c} r_1, \ldots, E^{N''}, r_i \in \mathbb{Z}, \\
\alpha(N''), \alpha(i) \in \mathbb{R}^+ \end{array} \right\}$$

Let $L_I$ be the subalgebra of $U_q$ generated by

$$\left\{ \begin{array}{c} r_1, \ldots, E^{N''}, n m_i, r_i \in \mathbb{Z}_{+}, \\
\alpha(N''), \prod_{i=1}^n k_i, j_i \in \mathbb{Z}_{+}, \\
\alpha(i) \in \mathbb{R}_{+}, \ldots, \alpha(N''), j_i \in \mathbb{Z}_{+} \end{array} \right\}$$

$$\alpha(i) \in \mathbb{R}_{+}, \quad (m_1, \ldots, m_n) \in \mathbb{Z}^n$$

Proposition : 3.4.9

$$U_I^+ \left( \text{Soc}H^0(\lambda) \right) = \bigoplus_{\gamma \in \mathbb{Z}I} \text{Soc}H^0(\lambda)_{\lambda - \gamma}$$

Proof :

From 3.3.2, we get

$$U_I^+ \left( \text{Soc}H^0(\lambda) \right) = \left\{ x \in (\text{Soc}H^0(\lambda)) \mid u \cdot x = v(u) \cdot x, \text{ for all } u \in U_I^+ \right\}$$

($v$ being the counit)
But $\nu(E_i) = 0$.

Therefore,

$$(\text{Soc} H^0(\lambda))^I_U = \left\{ x \in (\text{Soc} H^0(\lambda)) \mid u.x = 0, \text{ for all } u \in U^+_I \right\}$$

For any $\alpha \in \mathbb{R}^+ - R_I$, any $n > 0$ and any $\gamma \in \mathbb{Z}I$, the weight $\lambda - \gamma + n\alpha > \lambda$, because there is a simple root $\beta \in I$ occurring in $\alpha$, hence $n\alpha - \gamma$ with a positive coefficients.

So the element with weight $\lambda - \gamma + n\alpha$ is not in $(\text{Soc} H^0(\lambda))$.

This implies $E^n_\alpha(\text{Soc} H^0(\lambda))_{\lambda - \gamma} = 0 \quad \ldots(3.4.10)$

Let $x \in (\text{Soc} H^0(\lambda))_{\lambda - \gamma}$ and $u \in U^+_I$, then $u.x = 0$ by (3.4.10).

So $x \in (\text{Soc} H^0(\lambda))^I_U$.

Hence $(\text{Soc} H^0(\lambda))_{\lambda - \gamma} \subseteq (\text{Soc} H^0(\lambda))^I_U$.

For the reverse inclusion, we first note that $U^+_I (\text{Soc} H^0(\lambda))^I_U$ is an $L_I$ - submodule. Also it is the sum of its weight spaces of the form $(\text{Soc} H^0(\lambda))^I_U$ and for each
the sum $\mu = \sum_{\gamma \in \mathbb{Z}I} \left((\text{Soc}^0(\lambda))^{U_+^I}\right)_{\mu+\gamma}$ is an $L_I$ submodule. Now, if there is a $\mu \in \lambda + \mathbb{Z}I$ with $\left((\text{Soc}^0(\lambda))^{U_+^I}\right)_{\mu} \neq 0$, then

$$\phi = \sum_{\gamma \in \mathbb{Z}I} \left((\text{Soc}^0(\lambda))^{U_+^I}\right)_{\mu+\gamma}$$

contains a non-zero vector $v$ invariant under $U_+^\mathfrak{g} \cap L_I$, in the sense of (3.3.2). As $U_+^\mathfrak{g} = (U_+^\mathfrak{g} \cap L_I) U_I^+$ and as $U_I^+$ fixes $(\text{Soc}^0(\lambda))^{U_+^I}$, $v$ is invariant under $U_+^\mathfrak{g}$ hence is contained in $(\text{Soc}^0(\lambda))^{U_+^\mathfrak{g}} = (\text{Soc}^0(\lambda))_\lambda$. This implies $\mu - \lambda \in \mathbb{Z}I$, a contradiction.

Therefore $(\text{Soc}^0(\lambda))^{U_I^+} = \phi \sum_{\gamma \in \mathbb{Z}I} \left((\text{Soc}^0(\lambda))\right)_{\lambda-\gamma}$.

\[\square\]

5. THE STEINBERG MODULE

The Steinberg module, which was originally studied by Steinberg for a finite Chevalley group $G$ as the 'unique' irreducible summand of the regular representation of $G$ has dimension $p^a$, where $p^a$ is the highest power of $p$ dividing the order of $G$, and it is known that $a$ is the total number of positive roots. This module remains irreducible upon reduction mod $p$. One usually denotes this module as $St$. $St$ is also an irreducible summand of the corresponding
(restricted) universal enveloping algebra $u$ for $G$. Later $St$ was generalized to $St_n$, for every $n$ for the hyperalgebra $u_n$ of $G$, and $\dim St_n = p^n$.

It is also known that $St_n$ is

i. the weyl module $V ((p^n - 1)p)$

ii. the socle of $H^0((p^n - 1)p)$ ($\rho$ is half the sum of positive roots)

(see [10], p. 226).

One can naturally expect to define a Steinberg module for a quantum group. Andersen et al have apparently taken the cue from Jantzen and defined the Steinberg module $St$ to be the simple $U_{\lambda - \rho} -$ module $H^0((\ell-1)p)$ of highest weight $(\ell - 1)p)$, where $U_{\lambda} = U_\otimes \otimes \lambda$.

3.5.1 STEINBERG MODULE IN QUANTUM GROUP

We let $M_{\alpha} (\lambda), M_{\alpha, \rho} (\lambda)$ to denote the Verma modules over $U_{\lambda}$ and $L_{\alpha} (\lambda), L_{\alpha, \rho} (\lambda)$ the corresponding (unique) simple factor modules. Also $U_{\rho}, U_{\rho}^+, U_{\rho}^-$ are subalgebras of $U_{\lambda}$ (see Chapter 2, sec [2] and [5]).

From the corollary (2.7.8) of Chapter 2, we get

$$\text{Soc} M_{\alpha, \rho} (\lambda) \cong L_{\alpha, \rho} (\lambda + 2\rho)$$

... (3.5.2)
Now we take $\lambda = (\ell - 1)\rho$ which is in $\mathfrak{p}^\times$.

Then (5.1.1) implies that

$$\text{Soc}M_{\varepsilon,r}((\ell - 1)\rho) \cong L_{\varepsilon,r}(w_0((\ell - 1)\rho + 2\rho))$$

$$= L_{\varepsilon,r}(w_0(\ell\rho + \rho))$$

$$= L_{\varepsilon,r}((\ell - 1)\rho)$$

(Since $\varepsilon^{-\rho} = \varepsilon^{\ell\rho - \rho} = \varepsilon^{(\ell - 1)\rho}$)

for all $r > 0$

There is some non-zero vector $v$ in $\text{Soc}M_{\varepsilon,r}((\ell - 1)\rho)$ with weight $(\ell - 1)\rho$.

But $M_{\varepsilon,r}((\ell - 1)\rho)_{(\ell - 1)\rho} = \mathbb{C}v_\lambda$. So $v_\lambda \in \text{Soc}M_{\varepsilon,r}((\ell - 1)\rho)$

and $v_\lambda$ generates $M_{\varepsilon,r}((\ell - 1)\rho)$.

Hence $M_{\varepsilon,r}((\ell - 1)\rho) = \text{Soc}M_{\varepsilon,r}((\ell - 1)\rho) \cong L_{\varepsilon,r}((\ell - 1)\rho)$

for all $r \in \mathbb{N}$.

We call this the Steinberg module $S_{\varepsilon,r}$, which is of dimension $\mathfrak{l}^{r\mathbb{N}}$, where $N = |\mathbb{R}^\times|$.

At the same time, we know that there exists a natural injective $U_r$ - homomorphism.

$$f_r : M_{\varepsilon,r}((\ell - 1)\rho) \rightarrow M_{\varepsilon,r}((\ell - 1)\rho) \text{ (see (2.5.3))}$$
Hence we conclude that

\[ \text{St}_r = \text{M}_{\epsilon, r}((l-1)\rho) = \text{Soc M}_{\epsilon, r}((l-1)\rho) \]

\[ \subset \text{Soc M}_{\epsilon}((l-1)\rho) \]

\[ \cong L_{\epsilon}((l-1)\rho) \text{ by Theorem 2.7.10} \]

We call \( L_{\epsilon}((l-1)\rho) \) the Universal Steinberg module.