CHAPTER VI
On Non-oscillatory Functions
Over Hardy Fields

6.1 Introduction

In this Chapter, we define a class of non-oscillatory functions and study their behaviour. This problem is of independent interest. Further by extending the notion of Boshernitzan's admissible pair of solutions for differential equations (Definition 11 of Chapter I) to general functions, we introduce non-oscillatory functions pair. The properties of non-oscillatory functions pairs in a Hardy field are very interesting and more general. Further some theorems proved in Chapter I, and Chapter II for differential equations can be deduced from the theorems of this chapter. Throughout this Chapter we assume that $K$ stands for a perfect Hardy field.

6.2 Non-oscillatory functions

Definition 1

A real valued continuous function $y = g(x)$ on $I = (a, \infty)$ ($a \geq 0$) is called a non-oscillatory function if $g(x) \neq 0$ but $g(x_k) = 0$ for a sequence $(x_k)$ such that $\lim_{k \to \infty} x_k = \infty$; otherwise called oscillatory.

It is clear that $y \in B$, the ring of germs of continuous real valued functions on deleted neighbourhoods of $+\infty$ in $R$. Examples of non-oscillatory functions are rational functions, exponential functions and logarithmic functions. Functions such as $\exp(\sin x)$ and $\exp(\cos x)$ are non-oscillatory but they do not belong to any Hardy field.
Next we shall study the properties of non-oscillatory functions in $K$. The proofs are obvious and so we omit them.

6.3 Properties of Non-oscillatory Functions in $K$

1. The linear combination of a finite number of non-oscillatory functions in $K$ are non-oscillatory in $K$. But the linear combination of finite number of non-oscillatory functions in $c'(a, \mathfrak{a})$ need not be non-oscillatory. For example, if $f(x) = e^x + \sin x$ and $g(x) = -e^x$ then $f(x) + g(x)$ is not non-oscillatory.

2. If $f(x)$ and $g(x)$ are non-oscillatory in $K$ then $f(x)/g(x)$ ($g(x) \neq 0$) and $f(x)g(x)$ are non-oscillatory in $K$.

3. If $f(x) \in K$ is non-oscillatory, then
$$\int f(t) \, dt \in K \text{ and is non-oscillatory.}$$

4. If $f(x) \in K$ is non-oscillatory, then $\exp f(x)$ and $\log |f(x)| \in K$ and they are non-oscillatory.

5. If $f(x) \in K$, then $f'(x) \in K$ and is non-oscillatory.

6.4 Non-oscillatory Functions Pair

According to Boshernitzan (Definition 11 of Chapter I), a pair $[y_1, y_2]$ of solutions of $y'' + \phi(x) y = 0$ is called an admissible pair if i) $y_1, y_2 > 0$ and ii) $V(y_1) > V(y_2)$.

Extending this definition to any general function of the Hardy field $K$ we have the notion of non-oscillatory functions pair in $K$. Our definition is more general and is valid for any two functions of $K$. When they are germs of differential
equations, non-oscillatory functions pair become Boshernitzan's admissible pair of solutions.

**Definition 2**

The non-oscillatory functions $u(x)$ and $v(x)$ in $K$ are said to be a non-oscillatory functions pair (or simply functions pair) if i) $V(v) > V(u)$ and ii) $W(u,v) = uv' - vu' 
eq 0$ ($u,v > 0$).

In this case

$$v(x) = u(x) \int \frac{W(v,u)}{u^2} \, dt$$

We shall denote it by $[u,v]$. If $[u,v]$ is a non-oscillatory functions pair, $v/u$ is monotonic since $(v/u)' 
eq 0$. Further, there exists an infinite number of non-oscillatory functions pairs for a given Wronskian. For instance, consider the following example.

**Example 6.1**

$$[x,1], [x^2, \frac{1}{3x}], \ldots, [x^n, \frac{1}{(2n-1)x^{n-1}}]$$

are non-oscillatory functions pairs with Wronskian Unity.

**Remark 6.1**

From the above example, it is clear that if $[u,v]$ is a non-oscillatory functions pair, $[v,u]$ need not be a non-oscillatory functions pair.

**Theorem 1**

If $[u,v_1]$ and $[u,v_2]$ are functions pairs in $K$, then $[u, v_1 + v_2]$ is also a functions pair.

The proof is obvious.
Theorem 2

If \([u,v]\) is a non-oscillatory functions pair in \(K\) then \(W(v,u)\) is positive and \(v'/v < u'/u\).

Proof

Since \(V(v) > V(u)\), \((v/u)\) is a positive function which decreases to zero. So it follows that \((v/u)' < 0\). This gives the desired results.

Remark 6.2

When \(v(x)\) is of the form \(v(x) = a u(x) + b\) where \(a, b\) are real constants, the condition \(V(v) > V(u)\) needn't be satisfied. In this case the Wronskian \(W(v,u)\) is positive if \(bu' > 0\). For example, if \(v(x) = 3x^2 + 4\), \(u(x) = x^2\), \(W(v,u) = 8x\) which is positive for \(x > 0\). (Note that the condition \(V(v) > V(u)\) is not satisfied).

Theorem 3

If \([g,f]\) is a non-oscillatory functions pair in \(K\) then the following integrals are convergent.
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\(i) \int_{c}^{d} \frac{w(f, g)}{f^2 + g^2} \, dt\)

\(ii) \int_{c}^{d} \frac{w(f, g)}{g^2} \exp(f/g) \, dt\)

\(iii) \int_{c}^{d} \frac{w(f, g)}{g^2} \sin(f/g) \, dt\)

\(iv) \int_{c}^{d} \frac{w(f, g)}{g^2} \cos(f/g) \, dt\)

\(v) \int_{c}^{d} \frac{w(f, g)}{g^2} \, dt \text{ and the integral}\)

\(vi) \int_{c}^{d} \frac{w(f, g)}{f^2} \, dt \text{ is divergent}\)

**Proof**

The proof is similar to that of Theorem 4 of Chapter II.

**Remark 6.3**

If the non-oscillatory functions pair \([g(x), f(x)]\) are the germs of the second order differential equation \(y'' + p(x) y' + Q(x) y = 0\) over \(K\), then

\[w(f, g) = c e^{\int p_{1x}}\]

In this case Theorem 4 of Chapter II can be deduced from the above Theorem 5. Also if \(P(x) = 0\), second part of Boshernitzan Theorem 1 of Chapter I can be deduced from Theorem 5. It should be noted that the formula for \(w(f, g)\) holds generally also.
Theorem 4

Let \([u, v]\) be a non-oscillatory functions pair in \(K\) such that

i) \(V(u) < 0.\)

ii) \(W(v, u)\) is a positive decreasing function or a constant and iii) \(u'(x)\) is non-decreasing.

Then \(v(x)\) is non-increasing.

Proof

The proof is similar to that of Theorem 4 of Chapter II.

Remark 6.4

Theorem 4 of Chapter II can be deduced from the above theorem 4.

Theorem 5

If \([u_1, u_2, \ldots, u_n]\) be \(n\) non-oscillatory functions in \(K\) such that they are non-oscillatory functions pairs when taken two by two. If \(V(u_1) < V(u_2) < \ldots V(u_n)\) then

\[
\sum_{k(j)} \frac{W(u_k, u_j)}{u_k^2} < 0
\]

Proof

The proof is obvious.
Theorem 6

If \([u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n]\) are \(n\) non-oscillatory functions pairs with Wronskians \(\phi_1, \phi_2, \ldots, \phi_n\) respectively then

\[
\sum_{r=1}^{n} \frac{v_r}{u_r} - \int_{x}^{n} \left( \sum_{r=1}^{n} \frac{\phi_r^2}{u_r^2} \right) dx
\]

Proof

The Proof follows from the definition.

Theorem 7

Let \([u, v]\) be a non-oscillatory function pair with Wronskian \(W(v, u)\). If \(V(W) > 0\) then \((u''/u) < (v''/v)\).

Proof

By hypothesis, \(W(v, u)\) is a positive germ which decreases to zero and so \(W'(v, u) < 0\). This gives the desired result.

Example 6.2

The above theorem can be easily verified by considering the non-oscillatory function pair \([x, 1/2x]\) with Wronskian \(1/x\).

Corollary 7.1

Let \((u, u_1)\) and \((v, v_1)\) be the linearly independent germs of the differential equations

\[
y'' + P(x) y = 0 \quad \text{and} \quad z'' + Q(x) z = 0 \quad (P, Q \in K).
\]

If \([u, v]\) is a non-oscillatory functions pair in \(K\) with Wronskian \(W(v, u)\) and if \(V(W) > 0\) then \(P(x) > Q(x)\).
Theorem 8

If $u$ and $v$ are both infinitely increasing
(i.e., $\lim_{x \to \infty} u(x) = \infty$ and $\lim_{x \to \infty} v(x) = \infty$) non-oscillatory functions in $K$ and $u \geq v$. Then $V(v' / v) \geq V(u' / u)$. If the inequality is strict and $u' / u, v' / v > 0$ then $(u' / v') < (u'' / v'')$.

Proof

The first part follows by Theorem C of Chapter I. To prove the second part, consider $V(v' / v) > V(u' / u)$.

By hypothesis $(v'u / u'v)' < 0$. This gives the desired result.

Theorem 9

Let $f_1, f_2, \ldots, f_n$ be $n$ linearly independent non-oscillatory functions in $K$. If $f \in C' \left( a, \infty \right)$ be any other real valued function such that the Wronskian $W(f_1, f_2, \ldots, f_n, f) = \phi(x) \in K$.

Then

(i) $f$ is given by

$$f(x) = \sum_{k=1}^{n} f_k(x) \int_{x_0}^{x} \frac{w_k(t) \phi(t) dt}{w^2(f_1, f_2, \ldots, f_n)(t)}$$

(6.1)

Where $W(f_1, f_2, \ldots, f_n)$ is the Wronskian of $f_1, f_2, \ldots, f_n$ and $W_k$ is the determinant obtained from $W(f_1, f_2, \ldots, f_n)$ by replacing the $k_{th}$ column $(f_k, f_k', \ldots, f_k^{(n-1)})$ by $(0, 0, \ldots, 0, 1)$

(iii) $f \in K$

(iii) $f$ is non-oscillatory.
Proof

Expanding the Wronskian we get an nth order non-homogeneous linear differential equation with $\phi(x)/W(f_1,\ldots,f_n)$ on the right side and $f_1,f_2,\ldots,f_n$, the solutions of the corresponding homogeneous equation. Using the method of Variation of Constants equation (6.1) follows easily. $f(x) \in K$ since all the terms on the right of (6.1) belong to K. $f(x)$ is non-oscillatory follows by proof by contradiction.

Example 6.3

When $n=2$, if $f_1(x) = x$, $f_2(x) = x^2$, $\phi(x) = 2x^3$, $I = (0,\infty)$ then $f(x) = x^3 \in K$ and is non-oscillatory. Thus the theorem is verified.

Definition 3

The indicator of the non-oscillatory function pair $[f,g]$ in $K$ is defined as

$$I(f,g,x) = \frac{2W(2W - W'') - 3(W')^2}{4W^2}$$

Where $W$ and $W_1$ denote the Wronskian's $W(g,f)$ and $W(g',f')$ respectively.

If $W$ is a constant, $I(f,g,x) = W_1/W$. It must be noted that $I(f,g,x) \in K$ since $g,f \in K$.

Theorem 10

Let $[f_2,f_1]$ and $[g_2,g_1]$ be any two non-oscillatory functions pairs in $K$. If they have constant Wronskians and if $V(f_1) > V(g_1)$ then

$$I(f_1,f_1,x) < I(g_1,g_1,x).$$
Proof

\[ V(g_1) > V(g_2) \text{ and } V(f_1) > V(f_2) \text{ gives } \]
\[ V(f_1/g_1) > 0 > V(g_2/g_1) \quad (6.2) \]

On differentiating, we get

\[ V(g'_1 f'_1 - f'_1 g'_1) > V(g'_2 g'_2 - g'_2 g'_1) = 0 \]

Thus \( V(g'_1 f'_1 - f'_1 g'_1) > 0 \). Since \( V(f_1) > V(g_1) \) we have

\[ \frac{f'_1}{f_1} < \frac{g'_1}{g_1} \quad (6.3) \]

Since \( (g'_1 f'_1 - f'_1 g'_1) \) is negative and approaches zero on the positive half line, it has a positive derivative so that

\[ \frac{g''_1}{g_1} < \frac{f''_1}{f_1} \quad (6.4) \]

From (6.3) and (6.4)

\[ \frac{f'_1}{g'_1} < \frac{f'_1}{g'_1} < \frac{f''_1}{g''_1} \quad (6.5) \]

Similarly it can be shown that

\[ \frac{f'_2}{g'_2} > \frac{f'_2}{g'_2} > \frac{f''_2}{g''_2} \quad (6.6) \]
From (6.5) and (6.6) and using the definition of indicators the required result follows.

Remark 6.5

Chapters V and VI are of independent interest. It is for the future researchers to find applications of the non-oscillatory sequences pair and non-oscillatory functions pair.