CHAPTER IV

On Difference Equations Over Discrete Hardy Fields

4.1 Introduction

This chapter deals with the discrete analogue of certain results presented in chapter I, II and III. There are three sections in this Chapter. In Section 1, we introduce the notion of the canonical valuation of a discrete Hardy field and establish the discrete analogue of Theorem B of chapter I. One can find more properties of the canonical valuations of Hardy fields in M. Rosenlicht [19,20,21]. As far as the author knows no one has discussed so far the canonical valuations of discrete Hardy fields. This is a very significant result with very interesting properties. We use the properties of the canonical valuations of discrete Hardy fields to prove the Theorems of chapters IV and V.

In Section 2, we define the terms such as oscillating and non oscillating sequences, oscillatory and non oscillatory difference equations and the Casoratian of n functions of a discrete Hardy field \( K \). The Theorems 4 to 8 and 2,3 of this section are discrete analogue of Theorems O, L, P of chapter I, 8, 4 of chapter II and I,K of chapter I respectively.

In Section 3, we shall consider \( n^{th} \) order difference equations and generalize some of the Theorems of Section 2. Theorems 11, 12 and 13 are the discrete version of our Theorems of chapter III. Throughout we shall assume that \( K \) stands for a perfect discrete Hardy field unless and otherwise specified.
To start with we shall establish the following Theorem which is the discrete analogue of Theorem B of Chapter I.

**Theorem 1**

Let $K$ be a discrete Hardy field. Then there exists a map $V$ from the set of non zero elements $K'$ of $K$ onto an ordered abelian group such that

1) If $a_n, b_n \in K^*, b_n > 0, \Delta b_n < 0$ and $V(a_n), V(b_n) > 0$ then $V(a_n) \geq V(b_n)$ if and only if $V(\Delta a_n) \geq V(\Delta b_n)$

2) If $a_n, b_n \in K^*, b_n > 0, \Delta b_n > 0$ and $V(a_n) > V(b_n) < 0$ then $V(\Delta a_n) > V(\Delta b_n)$.

**Proof**

Let $K'$ be the set of non zero elements of a discrete Hardy field $K$. Let $a_n, b_n \in K'$. Define $a_n \equiv b_n$ if $\lim_{n \to \infty} a_n/b_n$ is a finite nonzero number. It can be proved easily that $\equiv$ is an equivalence relation on the elements of $K'$. This equivalence relation decomposes $K'$ as a union of mutually disjoint equivalence classes. Let $V(a_n)$ denote the equivalence class of $a_n$. Denote the set of equivalence classes on $K'$ by $\Gamma$. Thus $\Gamma = \{V(a_n) : a_n \in K'\}$.

If $a_n, b_n, c_n, d_n \in K'$ and $a_n \equiv b_n$ and $c_n \equiv d_n$ then $a_n c_n \equiv b_n d_n$, so that multiplication on $K'$ induces a composition of elements of $\Gamma$.

Thus $\Gamma$ becomes an abelian group and the map $V : K' \to \Gamma$ is a homomorphism. Let us follow the convention of writing the
composition law on $\Gamma$ additively as it is done in the continuous case by R. Rosenlicht [19].

If $a_n$, $b_n \in K^\prime$, we write $V(a_n) > V(b_n)$ (or $V(b_n) < V(a_n)$) if $\lim_{n \to \infty} a_n / b_n = 0$. This definition clearly depends only on the equivalence classes $V(a_n)$ and $V(b_n)$ of $a_n$ and $b_n$ and it induces a total ordering on the set $\Gamma$.

If $a_n \in K^\prime$, $V(a_n) > 0$ (or $V(1)$) means simply that $\lim_{n \to \infty} a_n = 0$ and so if $V(a_n)$, $V(b_n) > 0$, then $V(a_n) + V(b_n)$ ($= V(a_n b_n)$) > 0. Thus $\Gamma$ is an ordered abelian group with identity element $V(1)$.

If $a_n \in K^\prime$ then $V(a_n) \geq V(b_n)$ means simply that $\lim_{n \to \infty} a_n / b_n$ is finite. Thus we have associated with the discrete Hardy field $K$ an ordered abelian group $\Gamma$ and an onto map $V : K^\prime \to \Gamma$ such that

i) If $a_n$, $b_n \in K$ then $V(a_n b_n) = V(a_n) + V(b_n)$

ii) If $a_n$, $b_n \in K$, $a_n \neq b_n$ then

$V(a_n + b_n) \geq \min \{ V(a_n), \ V(b_n) \}$.

This map $V$ is called the canonical valuation of $K$ with value group $\Gamma$. In order to extend the applicability of (i) and (ii) for all $a_n$, $b_n \in K$, we shall define $V(0) = \infty$. It should be noted that if $a_n$, $b_n \in K^\prime$ and $V(a_n) \neq V(b_n)$ then $V(a_n + b_n) = \min \{ V(a_n), \ V(b_n) \}$. Thus we have established the existence of a surjective map $V$ from the non zero elements $K^\prime$ of $K$ onto an ordered abelian group $\Gamma$ satisfying the above conditions (i) and (ii). It remains to prove (1) and (2). In order to prove these we shall apply the discrete L'Hospital Rule for sequences [15a].

Thus the proof is complete.
Now we shall define the canonical valuation of a discrete Hardy field.

Definition 4.0

The canonical valuation of a discrete Hardy field \( K \) is a homomorphism from the multiplicative group \( K^* = K\setminus \{0\} \) of \( K \) onto an ordered abelian group (value group) \( V(K) = \Gamma \).

The kernel of \( V \) consists of all \( f_n \in K^* \) such that \( \lim_{n \to \infty} f_n \) is finite and non zero, while \( V(f_n) > 0 \) if and only if \( \lim_{n \to \infty} f_n = 0 \) and \( V(f_n) < 0 \) if and only if \( \lim_{n \to \infty} f_n = \pm \infty \).

Remark 4.1

As it can be seen from the proof that we have also proved two more properties of \( V \) namely (i) if \( a_n, b_n \in K^* \) than \( V(a_n b_n) = V(a_n) + V(b_n) \) and (ii) \( V(a_n + b_n) \geq \min \{V(a_n), V(b_n)\} \).
Section 2

4.3 On Second Order Difference Equations Over K

In this Section we shall establish some properties of second order difference equations over the discrete Hardy field K. Let us start with the definition of non-oscillatory sequences.

Definition 4.1

Let B_s denote the class of real sequences \( \{a_n\} \) defined for all large integers \( n \). A sequence \( \{a_n\} \in B_s \) is said to be non-oscillating if \( a_n a_{n+1} > 0 \) for all \( n \) sufficiently large; otherwise oscillating.

Definition 4.2

The difference equation
\[
\Delta^2 y_n - p_y y_{n+1} = 0
\]
over K is called non-oscillatory if all its solutions are non-oscillating; otherwise called oscillatory.

For example,
\[
\Delta^2 x_n - a_n x_n = 0
\]
is non-oscillatory when
\[
a_n \Delta^2 \left( \frac{\sqrt{n-1}}{\sqrt{n}} \right) \in K
\]

This may be thought of as a discrete analogue of the well known example of non-oscillatory differential equation
This has \( t^8 \) and \( t^8 \log t \) as solutions and the coefficient of \( y \) is

\[
\frac{1}{4t^2} = \frac{(t^{1/2})''}{\sqrt{t}}
\]

The above definition (4.2) is valid for any \( n \)th order linear difference equation over \( K \) also.

It must be noted that a solution means a non trivial solution for large values of \( n \) and

\( \Delta \) stands for the forward operator defined by

\[
\Delta x_\ell = x_{\ell+1} - x_\ell
\]

Definition 4.3

The Casoratian of the functions \( a_n, b_n \) of a discrete Hardy field \( K \) is denoted by \( C(a_n, b_n) \) and is defined by

\[
C(a_n, b_n) = a_n b_{n+1} - b_n a_{n+1}
\]

Definition 4.4

The Casoratian of the \( m \) functions \( f_i(n) \in K \) (\( i = 1, 2, \ldots, m \)) is given by

\[
C(m) = C(f_1, f_2, \ldots, f_m) = \left| \begin{array}{cccc}
    f_1(n) & f_2(n) & \cdots & f_m(n) \\
    f_1(n+1) & f_2(n+1) & \cdots & f_m(n+1) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_1(n+m-1) & f_2(n+m-1) & \cdots & f_m(n+m-1)
  \end{array} \right|
\]
Theorem 2

Let $K$ be any discrete Hardy field and $B_\mathbb{R}$ denote the field of real sequences. Assume that $(Y_n) \in B_\mathbb{R}$ satisfies the difference equation

$$\Delta^2 Y_n - p_n Y_{n+1} = 0.$$  \hspace{1cm} (4.1)

Where $p_n \in K$ be a real rational function. If equation (4.1) is non oscillatory then $Y_n \in E_\mathbb{R}(K)$.

Proof

Define $z_n = Y_{n+1}/Y_n$. Then $z_n \in B_\mathbb{R}$ is positive and satisfies the Riccati difference equation

$$z_{n+1} = P(z_n, k(n))$$  \hspace{1cm} (4.2)

Where $P(z_n, k(n)) = -1/z_n + 2 + p_n$.

Now $P(z_n, k(n))$ satisfies all the conditions of Theorem D of Chapter I and so $z_n \in E_\mathbb{R}(K)$. But $Y_{n+1} = z_n Y_n$ and so

$Y_n = z_1 \ldots z_{n+1}$, $Y_1 \in E_\mathbb{R}(K)$. 
Remark 4.2

Theorem 2 shows that the non-oscillatory sequences satisfying the difference equation (4.1) belong to $E_s(K)$. This is a particular discrete analogue of Theorem I of Chapter I, particular because $p_k \in K$ is a rational function whereas the function $\Phi$ in Theorem I of chapter I need not be so.

Example 4.1

Consider the difference equation

$$\Delta^2 y_n - 4y_{n+4} = 0.$$ 

over the minimal perfect discrete Hardy field $E_s$. Clearly this equation is non-oscillatory with solution basis \{ $(3+2\sqrt{2})^n$, $(3-2\sqrt{2})^n$ \} and they belong to $E_s$.

Now we shall establish the discrete analogue of Theorem K of Chapter I for the equation (4.3) given below.

Theorem 3

Let $K$ be any discrete Hardy field and the equation (4.1) be non-oscillatory. If $Q_n \in K$, then the solutions of

$$\Delta^2 y_n - p_k y_{n+1} = Q_n,$$

belong to $E_s(K)$. 

(4.3)
Proof

Let \( Y_1^{(1)} \) and \( Y_2^{(2)} \) be the linearly independent solutions of the equations \( (4.1) \). They belong to \( E_s(K) \) in view of Theorem 2.

The general solution of the equation \( (4.3) \) is of the form,

\[
Y_n = c_1 Y_1^{(1)} + c_2 Y_2^{(2)} + Y_n^{[p]}
\]

(4.4)

where \( c_1 \) and \( c_2 \) are arbitrary constants and \( Y_n^{[p]} \) is the particular integral. To prove the theorem it is sufficient if we prove \( Y_n^{[p]} \in E_s(K) \). It can be found by variation of constants method that

\[
y_n^{[p]} = y_1^{(1)} \sum_{i=0}^{n-1} Q_i y_1^{(i)} + y_2^{(2)} \sum_{i=0}^{n-1} Q_i y_2^{(i)}
\]

choosing the Casoratian unity. Clearly \( Y_n^{[p]} \in E_s(K) \). Thus the proof is complete.

Corollary 3.1

The equation \( (4.3) \) is non oscillatory if and only if the equation \( (4.1) \) is non oscillatory.

This is the discrete analogue of corollary 2.1 of Theorem 2 of Chapter III for second order linear differential equation over \( K \) (when \( n=2 \) and \( p(x) = 1 \)).

Corollary 3.2

Consider the difference equations

\[
\begin{align*}
ax_{n+1} + bx_{n-1} &= p_n \quad \text{and} \\
A_n x_{n+1} + A_{n-1} x_{n-1} &= B_n x_n &= 0 \quad \text{over} \ K
\end{align*}
\]

with \( a_n, b_n, A_n, B_n > 0 \). If \( A_n \leq a_n \) and \( B_n - A_n - A_{n-1} \leq b_n - a_n - a_{n-1} \) for all sufficiently large \( n \) and if the latter equation is non-oscillatory then the former is non oscillatory and the solutions belongs to \( E_s(K) \). This follows by Theorem (R) of Chapter I and Theorem 3.
Example 4.2

Consider the difference equation
\[ Y_n^2 - 3 Y_{n-1} + 2 Y_n = 4^k + 3 k^2 \]
over the discrete Hardy field \( E_s \). The homogeneous part is non oscillatory with solution basis \( \{2^n, 1\} \in E_s \).

The particular integral is \( 1/6 4^k - n^3 - 3/2 n^2 - 13/2 n \in E_s \). So any solution of the given difference equation belongs to \( E_s \). Thus theorem 3 is verified.

Theorem 4

Suppose that the linear difference equation,
\[ \Delta^2 y_n - P_n y_{n+1} = 0 \tag{4.5} \]
has two linearly independent non oscillatory solutions in \( K \). Then there are linearly independent solutions \( Y_n^{(1)} \) and \( Y_n^{(2)} \) such that \( V(Y_n^{(1)}) > V(Y_n^{(2)}) \). If \( Y_n^{(1)} \) and \( Y_n^{(2)} \) are chosen positive then their Casoratian \( C(Y_n^{(1)}, Y_n^{(2)}) \) is a positive constant \( = C \) and

\[ \Delta \left( \frac{y_n^{(1)}}{y_n^{(2)}} \right) - \frac{-C}{y_n^{(2)}} \Delta \left( \frac{y_n^{(2)}}{y_n^{(1)}} \right) - \frac{C}{y_n^{(1)}} = \frac{1}{y_n^{(1)}} \left[ \begin{array}{c} y_n^{(1)} \\ y_n^{(2)} \end{array} \right] \]

If \( v_i = \frac{\Delta y_n^{(1)}}{y_n^{(1)}} (i = 1, 2) \) then \( v_2 - v_1 = \frac{C}{y_n^{(1)} y_n^{(2)}} > 0 \).

Proof

Any two linearly independent solutions of the difference equation (4.5) with the same \( V \) value have a quotient that approaches a non zero real limit as \( n \to \infty \), therefore have a non zero real linear combination with higher \( V \) value. This shows that
existence of $Y_n^{(1)}$ and $Y_n^{(2)}$ as desired. Thus there are linearly independent solutions such that $V(Y_n^{(1)}) > V(Y_n^{(2)})$.

Now

$$\Delta Y_n = \frac{c(Y_n^{(2)}, Y_n^{(1)})}{Y_n^{(2)}}$$

(4.6)

Now $Y_n^{(1)} / Y_n^{(2)}$ is positive and approaches zero as $n \to \infty$; hence decreasing and so

$$\Delta Y_n < 0$$

Since $Y_n^{(2)}$ is non oscillatory, $(Y_n^{(2)} Y_{n+1}^{(2)}) > 0$, so it follows $C(Y_n^{(1)}, Y_n^{(2)}) > 0$. (4.7)

Further,

$$\Delta Y_n^{(2)} = \frac{c(Y_n^{(1)}, Y_n^{(2)})}{Y_n^{(1)}} > 0$$

(4.8)

$$Y_{n+1} - Y_n = \frac{\Delta Y_n^{(2)}}{Y_n^{(2)}} - \frac{\Delta Y_n^{(1)}}{Y_n^{(1)}} - \frac{c(Y_n^{(1)}, Y_n^{(2)})}{Y_n^{(1)} Y_n^{(2)}} > 0$$

(4.9)

Now we shall prove that $C(Y_n^{(1)}, Y_n^{(2)})$ is a positive constant. An easy computation shows,

$$\Delta (Y_n^{(1)} Y_{n+1}^{(2)} - Y_n^{(2)} Y_{n+1}^{(1)}) = 0$$

(4.10)

From (4.7) and (4.10) $C(Y_n^{(1)}, Y_n^{(2)})$ is a positive constant.

Thus the proof is complete.
Example 4.3

Theorem 4 can be easily verified by considering the difference equation

$$\Delta^2 Y_n = 0 \text{ over } K.$$ 

It should be noted that this equation has a solution basis \(\{1, n\}\) with the Casoratian unity.

Theorem 5

If \(Y_n^{(1)}, Y_n^{(2)}\) are non-oscillatory solutions in \(K\) of equation (4.5) with \(V(Y_n^{(1)}) > V(Y_n^{(2)})\) then

1. \(\sum_{n=1}^{\infty} \frac{1}{y_{n+1}^{(1)} + y_{n+1}^{(2)}}\) is convergent
2. \(\sum_{n=1}^{\infty} \frac{1}{y_{n+1}^{(1)} + y_{n+1}^{(2)}}\) is divergent
3. \(\sum_{n=1}^{\infty} \frac{1}{y_{n+1}^{(1)} + y_{n+1}^{(2)}}\) is convergent.

Proof

We have,

$$\Delta \left( \frac{y_{n}^{(1)}}{y_{n}^{(2)}} \right) = -\frac{c}{y_{n}^{(2)} + y_{n+1}^{(2)}}.$$ 

Therefore,

$$\frac{y_{n+1}^{(1)}}{y_{n}^{(2)} + y_{n+1}^{(2)}} = -\frac{c}{y_{n+1}^{(2)} + y_{n+1}^{(2)}} \quad (4.11)$$
Summing the equation (4.11) from 1 to n and taking the limit as \( n \to \infty \) and using \( \mathcal{V}(Y_n^{[1]}) > \mathcal{V}(Y_n^{[2]}) \) we get (i). Similarly we can prove (ii) and (iii).

**Theorem 6**

Let \( P_n \) and \( Q_n \) be elements of a discrete Hardy field in which each of the difference equations

\[
\Delta^2 Y_n - p_n Y_{n+1} = 0 \\
\text{and} \\
\Delta^2 Z_n - q_n Z_{n+1} = 0
\]

has two non oscillatory linearly independent solutions.

Let \( Y_n^{[1]} \), \( Y_n^{[2]} \) and \( Z_n^{[1]} \), \( Z_n^{[2]} \) respectively be linearly independent solutions of the given difference equations with \( \mathcal{V}(Y_n^{[1]}) > \mathcal{V}(Y_n^{[2]}) \) and \( \mathcal{V}(Z_n^{[1]}) > \mathcal{V}(Z_n^{[2]}) \) and suppose that \( \mathcal{V}(Y_n^{[1]}) > \mathcal{V}(Z_n^{[1]}) \) then

\[
i) \quad \frac{\Delta Y_n^{[1]}}{Y_n^{[1]}} < \frac{\Delta Z_n^{[1]}}{Z_n^{[1]}} \\
ii) \quad p_n > q_n \\
iii) \quad \mathcal{V}(Y_n^{[2]}) < \mathcal{V}(Z_n^{[2]}) \\
iv) \quad \frac{\Delta Y_n^{[2]}}{Y_n^{[2]}} > \frac{\Delta Z_n^{[2]}}{Z_n^{[2]}}
\]

**Proof**

Since \( \mathcal{V}(Y_n^{[1]}) > \mathcal{V}(Z_n^{[1]}) \) and \( \mathcal{V}(Z_n^{[1]}) > \mathcal{V}(Z_n^{[2]}) \) we have \( \mathcal{V}(Y_n^{[1]}/Z_n^{[1]}) > 0 > \mathcal{V}(Z_n^{[2]}/Z_n^{[1]}) \). (4.12)

By Theorem 1, (4.12) implies,
Thus we have
\[
V\left(\Delta\left(\frac{y^{(1)}_n}{z^{(1)}_n}\right)\right) > V\left(\Delta\left(\frac{z^{(2)}_n}{z^{(1)}_n}\right)\right)
\]

\[
V\left(\frac{z^{(1)}_n y^{(1)}_{n+1} - z^{(1)}_n z^{(2)}_{n+1}}{z^{(1)}_n z^{(2)}_{n+1}}\right) > V\left(\frac{z^{(1)}_n z^{(2)}_{n+1} - z^{(2)}_n z^{(1)}_{n+1}}{z^{(2)}_n z^{(1)}_{n+1}}\right)
\]

or
\[
V\left(z^{(1)}_n y^{(1)}_{n+1} - y^{(1)}_n z^{(2)}_{n+1}\right) > V\left(z^{(1)}_n z^{(2)}_{n+1} - z^{(2)}_n z^{(1)}_{n+1}\right) = 0.
\]

Assuming as we may, \(Y_n^{(1)}\), \(z^{(1)}_n > 0\), we have \(Y_n^{(1)}/z^{(1)}_n\) positive and approaching zero, hence with
\[
\Delta Y^{(1)}_n < 0
\]

so that
\[
z^{(1)}_n y^{(1)}_{n+1} - y^{(1)}_n z^{(2)}_{n+1} < 0 \text{ i.e., } z^{(1)}_n \left(\Delta Y^{(1)}_n + y^{(1)}_n\right)
\]

\[
- y^{(1)}_n \left(\Delta z^{(1)}_n + z^{(1)}_n\right) < 0 \text{ so that } \frac{\Delta z^{(1)}_n}{\Delta y^{(1)}_n} < \frac{\Delta y^{(1)}_n}{\Delta z^{(1)}_n}
\]
Since \( z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 1 \leq 0 \) and

\[
V(z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 1) > 0
\]

Thus \( z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 1 \) is negative and approaches zero. Hence

\[
\Delta(z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 1) > 0.
\]

this gives

\[
z^{(1)}_n y^{(1)}_n + 2 - z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 2 + z^{(1)}_n y^{(1)}_n + 1 > 0
\]

i.e.,

\[
z^{(1)}_n y^{(1)}_n + 2 - z^{(1)}_n y^{(1)}_n + 1 - y^{(1)}_n z^{(1)}_n + 2 + z^{(1)}_n y^{(1)}_n + 1 > 0
\]

so that \( (p_n - Q_n) z^{(1)}_n y^{(1)}_n + 1 \) > \( 0 \) so \( p_n > Q_n \).

Similarly we can prove the other parts also.

Theorem 7

Let \( K \) be a discrete Hardy field. Assume that

\[
\Delta^3 y_n - p_n y_{n+1} = 0.
\]

be non oscillatory with solution basis

\[
\{y^{(1)}_n, y^{(2)}_n\} (y^{(1)}_n) > 0 \quad i=1,2
\]

in some discrete Hardy field be such that

\[
V(y^{(1)}_n) > 0 \quad \text{or} \quad V(y^{(2)}_n) > 0 \quad \text{then} \quad p_n > 0.
\]
Proof

By hypothesis, \( V(Y_n^{[1]}) > 0 > V(n) \) \hfill (4.13)

By Theorem 1, equation (4.13) implies

\[ V(\Delta Y_n^{[1]}) > V(\Delta n) - 0. \]

(4.14)

Assume that \( V(Y_n^{[1]}) > 0. \) Thus \( Y_n^{[1]} \) is positive and approaches zero and hence decreasing so that

\[ \Delta Y_n^{[1]} < 0. \]

Similarly,

\[ \Delta Y_n^{[1]} < 0 \text{ with } V(\Delta Y_n^{[1]}) > 0 \text{ gives } \Delta^2 Y_n^{[1]} > 0. \]

Therefore \( p_n Y_n^{[1]} > 0; \) so \( p_n > 0. \)

Similarly we can show that \( p_n > 0 \) when \( V(Y_n^{[2]} > 0. \)

The above Theorem can be easily verified by the followin example.

Example 4.4

Consider the difference equation

\[ \Delta^2 Y_n - 2 Y_{n+1} = 0. \]

over the perfect discrete Hardy field \( K. \) In this case

\[ Y_n^{[1]} = (2 - \sqrt{3})^n \text{ with } V(Y_n^{[1]}) > 0. \]

It should be noted that real constant sequences belong to any perfect discrete Hardy field \( K \) by Section 4 of Chapter 1.
Theorem 8

Let equation (4.5) over $K$ be non-oscillatory with the solution basis \{ $Y_i^{(1)}, Y_i^{(2)}$ \} ($Y_i^{(1)} > 0$, $i = 1, 2$) such that $V(Y_i^{(1)}) > V(Y_i^{(2)}) < 0$.

If $\Delta y_n^{(1)}$ is nondecreasing then $y_n^{(1)}$ is nonincreasing.

Proof

We have,

$$
\Delta y_n^{(1)} = -c \frac{Y_n^{(1)}}{Y_n^{(2)}},
$$

where $C$ is the Casoratian $C(Y_n^{(1)}, Y_n^{(2)})$ which is a positive constant since $V(Y_n^{(1)}) > V(Y_n^{(2)})$. So we have

$$
y_n^{(1)} - y_n^{(2)} = (\sum_{r = n}^{m} \frac{C}{y_r^{(2)} y_r^{(1)}})
$$

Therefore

$$
\Delta y_n^{(1)} = y_n^{(2)} \Delta(\sum_{r = n}^{m} \frac{C}{y_r^{(2)} y_r^{(1)}})
$$

$$
+ \Delta y_n^{(2)} \sum_{r = n}^{m} \frac{C}{y_r^{(2)} y_r^{(1)}}
$$

$$
= -c \Delta y_n^{(2)} \sum_{r = n}^{m} \frac{C}{y_r^{(2)} y_r^{(1)}}
$$

$$
\leq c \sum_{r = n}^{m} \frac{y_r^{(2)} - y_r^{(1)}}{y_r^{(2)}} - \frac{c}{y_n^{(1)}} \leq 0.
$$
Hence $y_{n}^{[1]}$ is non-increasing.

Remark 4.3

Theorems 4 to 8 are the discrete analogues of Theorems 0, 1, 2 of chapter I and 3, 4 of chapter II.

Theorem 9

Consider the difference equation

\[ \Delta y_{n} + p_{n} y_{n+1} = 0. \tag{4.15} \]

over $K$. Let (4.15) be non-oscillatory and

\[ z_{n} = \frac{\Delta y_{n}}{y_{n}}. \]

Then

\[ \sum_{n=1}^{\infty} \frac{z_{n}^{2}}{1 + z_{n}} \quad \text{and} \quad \sum_{n=1}^{\infty} p_{n} \quad \text{are convergent.} \]

Proof

Let $y_{n}$ be a non-oscillatory solution of (4.15). So $y_{n} \neq 0$ for sufficiently large $n$.

Since (4.15) is transformed into

\[ z_{n} = \frac{\Delta y_{n}}{y_{n}}, \quad y_{n+1} = (1 + z_{n}) y_{n} \quad \text{and} \]
\[ y_{n+2} = (1 + z_{n})(1 + z_{n+1}) y_{n} \]

\[ \Delta z_{n} + \frac{z_{n}^{2}}{1 + z_{n}} = p_{n}. \]

Summing from 1 to $n$,

\[ \sum_{r=1}^{n} \frac{z_{r}^{2}}{1 + z_{r}} + \sum_{r=1}^{n} p_{r} = z_{1} - z_{n+1} < z_{1}. \]

As $n \to \infty$, \[ \sum_{r=1}^{\infty} \frac{z_{r}^{2}}{1 + z_{r}} \quad \text{is convergent and} \quad \sum_{r=1}^{\infty} p_{r} \quad \text{is convergent.}\]
Section 3

4.4 On nth Order difference equations Over K

In this Section we investigate the properties of the nth order difference equations over K and generalize certain results of Section 2 of Chapter V.

Theorem 10

Consider the difference equation

\[ L(y) = y_{k+1} + a_1(k)y_{k+1} + \ldots + a_n(k)y_k = 0 \quad (4.16) \]

over K. If (4.16) is non oscillatory with \( y_k^{(i)} \) (i=1,2,...,n) as linearly independent solutions in K, then there exists an ordered solution basis \( \{y_k^{(1)}, y_k^{(2)}, \ldots y_k^{(n)}\} \) in K such that

\[ V(y_k^{(i)}) \geq V(y_k^{(2)}) \geq \ldots \geq V(y_k^{(n)}) \quad (4.17) \]

Further the Casoratian of \( y_k^{(i)} \) (i=1,2,...,n) ;

\[ C(K) = C(y_k^{(1)}, \ldots y_k^{(n)}) \] satisfies the linear difference equation

\[ C(k+1) = (-1)^n a_n(k) C(k) \]

whose solution is

\[ C(k) = (-1)^k C(K_0) \prod_{i=1}^{k-1} a_n(i) \]

where \( K_0 \) is some fixed value of k.

Proof

The first part of the theorem is obvious since K is an ordered field. The second part follows by a simple computation as in [16].

Example 4.5

Consider the mth order difference equation

\[ \Delta^m y_k = 0 \quad \text{over } E_k. \]
The ordered basis in $E_s$ is $\{1, k, k^2, \ldots, k^{m-1}\}$ with $V(k^i) > V(k^j)$ ($j > i$, and $i, j \in \{1, 2, \ldots, m-1\}$). Thus the Theorem is verified.

**Example 4.6**

Let $\mathcal{E}$ denote the shifting operator defined by $\mathcal{E}(y_k) = y_{k+1}$. Consider the $n^{th}$ order linear difference equation with constant coefficients,

$$(\mathcal{E}-\alpha_1)(\mathcal{E}-\alpha_2)\ldots(\mathcal{E}-\alpha_n)y_k = 0$$

over $E_s$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all positive real constants. Then clearly the solution basis is ordered and all the solutions belongs to $E_s$ and satisfies the condition (4.17) of the Theorem.

Next we shall prove the discrete analogue of Theorem 2 of Chapter III (assuming the coefficient of $y_{k+n}$ as unity).

**Theorem 11**

The solutions of the non-homogeneous difference equation

$$L(y) = R_k \quad (R_k \in K)$$

belong to $K$ provided the solutions of the corresponding homogeneous equation $L(y) = 0$ belong to $K$.

**Proof**

Let $\{y_k^{[1]}, y_k^{[2]}, \ldots, y_k^{[n]}\}$ be the ordered solution basis for the equation (4.17). By hypothesis they lie in $K$. To prove the theorem it is enough if we prove that the particular solution $y_k^{[p]} \in K$. By variation of constants method, it can be shown \[16\] that

$$y_k^{[p]} = \sum_{j=1}^{n} c_j(k) y_k^{[j]}$$

where

$$\Delta c_j(k) = \frac{f_i(k)}{c(k+1)}$$

$i = 1, 2, \ldots, n$, $f_i(k)$
are known functions given in terms of $R_k$ and $y^{(i)}_k$. The Casoratian $C(k+1) \neq 0$ (Since $y^{(1)}_k, \ldots, y^{(n)}_k$ are linearly independent). So it follows that $y^{(i)}_k \in K$.

**Remark 4.2**

Theorem 11 is the $n$th order generalization of Theorem 3 of Section 2.

Next we shall generalize Theorem 5 of Section 2 to $n$th order difference equations.

**Theorem 12**

Assume that the equation (4.16) over $K$ be non oscillatory and that the solutions belong to $K$. Let $(y^{(1)}_k, y^{(2)}_k, \ldots, y^{(n)}_k)$ $(y^{(i)}_k > 0 \text{ for } i=1,2,\ldots,n)$ be an ordered solution basis in $K$ such that $V(y^{(i)}_k) > V(y^{(j)}_k)$ $(j > i, i,j \in \{1,2,\ldots,n\})$.

Then

i) $\sum_{k=1}^{n} \frac{c(y^{(i)}_k, y^{(j)}_k)}{y^{(i)}_k y^{(j)}_{k+1}}$ is convergent

ii) $\sum_{k=1}^{n} \frac{c(y^{(i)}_k, y^{(j)}_k)}{y^{(i)}_k y^{(j)}_{k+1}}$ is divergent

iii) $\sum_{k=1}^{n} \frac{c(y^{(i)}_k, y^{(j)}_k)}{y^{(i)}_k y^{(j)}_{k+1} + y^{(i)}_{k+1} y^{(j)}_k}$ is convergent

iv) Further $c(y^{(i)}_k, y^{(j)}_k)$ is positive and if

$$v_i = \frac{\Delta y^{(i)}_k}{y^{(i)}_k}, \quad v_j = \frac{\Delta y^{(j)}_k}{y^{(j)}_k}$$

then

$$v_i - v_j = \frac{c(y^{(i)}_k, y^{(j)}_k)}{y^{(i)}_k y^{(j)}_k} > 0.$$
Proof

The proof is similar to that of Theorem 5 and Theorem 4 Section 2 of Chapter IV.

Theorem 13

Let the equation (4.16) over K be non oscillatory and solutions belong to K.

Let \( \{y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(n)}\} \) \( (y_k^{(i)} > 0 \text{ for } i=1,2,\ldots,n) \) be an order basis for (4.16) in K with \( V(y_k^{(1)}) > V(y_k^{(2)}), \ldots > V(y_k^{(n)}). \)

\[
\begin{align*}
&i) \quad V(y_k^{(i)}) > V(y_k^{(j)}) \quad (j > i, i, j \in \{1,2,\ldots,n\}) \\
&ii) \quad \{\Delta y_k^{(i)}\} \text{ is non decreasing} \\
&iii) \quad V(y_k^{(j)}) < 0 \\
&iv) \quad C(y_k^{(j)}, y_k^{(i)}) \text{ is a positive decreasing function or a constant}
\end{align*}
\]

then \( \{y_k^{(i)}\} \) is non increasing

Proof

The proof is similar to that of Theorem 8 of Section 2.

Remark 4.4

Theorem 8 is a particular case of Theorem 13.

In the case of non-oscillatory linear differential equations over a Hardy field K, the germs of the differential equation belong to \( E(K) \) (Chapter III, Theorem 1).
Now there arise a question will the solutions of the associated difference equation over the discrete Hardy field $K$ belong to $E_s(K)$. This question is still open for more general differential equations with variable coefficients.

However, in the case of non oscillatory linear differential equations with constant coefficients over the field $E$, the germs of the equation may belong to $E$ but the associated linear difference equations over $E_s$ need not have its solutions in $E_s$.

For instance, consider $y''' + 3y'' + 2y = 0$ over $E$. The solution basis is $\{e^{-x}, e^{2x}\}$ and so any solution is non-oscillating and belongs to $E$. But the associated difference equation $y_{k+2} + 3y_{k+1} + 2y_k = 0$ over $E_s$ has the solution basis $\{(-1)^k, (-2)^k\}$. So any solution of the difference equation is not non-oscillating and do not belong to $E_s$.

The next Theorem establishes the criteria for the $n^{th}$ order difference equation with constant coefficients over $E_s$ and the corresponding associated differential equation over $E$ to be non oscillatory so that the former has solutions in $E_s$ and the latter has solutions in the $E$-field.

**Theorem 14**

An $n$th order homogeneous linear difference equation with constant coefficients over $E_s$ is non-oscillatory and the solutions belong to $E_s$ if and only if the associated linear homogeneous differential equation over the $E$-field has non negative real characteristic roots.
The proof is obvious and so we omit it.

It must be noted that the linear difference equation with constant coefficients $y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = 0 \quad (a_n \neq 0)$

(4.19)

has the characteristic equation

$$f(r) = r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$$

(4.20)

and that the solutions of (4.19) belong to $E_\beta$ if the roots of (4.19) are real and positive.
Remark 4.5

In the Proof of the above Theorem we have made use of the well known fact [10a] that every solution of a linear difference equation with constant coefficients oscillates if and only if its characteristic equation has no positive roots.

In the next Chapter we shall study the properties of non oscillatory sequences over discrete Hardy fields.