CHAPTER III

On Higher Order Differential Equations Over Hardy Fields

3.1 Introduction

In this chapter, we shall generalize certain results of chapter II to $n^{th}$ order differential equations over Hardy fields.

Having this aim in mind, we consider the linear differential equations.

\begin{align*}
L(y) &= 0 \quad (3.1) \\
L(y) &= f(x) \quad (f(x) \neq 0 \text{ for } x \in [a, \infty)) \quad (3.2) \\
M(y) &= 0 \quad (3.3) \\
M(y) &= g(x) \quad (g(x) \neq 0 \text{ for } x \in [a, \infty)) \quad (3.4)
\end{align*}

where

\[ L(y) = (py^{(a-1)})^{\sum_{i=1}^{a} b_i(x)} y^{(i-1)} , \quad (p(x) > 0) \]

and

\[ M(y) = (py^{[a]})^{\sum_{i=1}^{a} b_i(x)} y^{(i-1)} , \quad (p(x) > 0) \]

with $p(x), q(x), f(x), g(x), b_i(x)$ ($i=1, 2, \ldots, n$) belong to the Hardy field $K$.

We study the behaviour of the germs of the equations (3.1) to (3.4) in $E(K)$ and obtain $n^{th}$ order generalization of the Theorems 1 to 5 of chapter II so that not only Boshernitzan's Theorems I, J, K of Chapter I but also our theorems 1 to 5 of Chapter II become particular cases of our most general results.

---

This Chapter is accepted for publication as 'On Higher Order Differential Equations Over Hardy Fields' in Archivum Mathematicum (BRNO), Czechoslovakia.
Further we deduce a set of necessary and sufficient conditions for the equations (3.2) and (3.4) to be non-oscillatory.

3.2 The regular growth of non-oscillating germs of Higher order linear differential equations.

The following is the $n^{th}$ order generalization of Theorem 1 of Chapter II.

Theorem 1

If the equation (3.1) over $K$ is non-oscillatory then every germ of (3.1) belong to $E(K)$.

To prove the Theorem we need the following Lemma.

Lemma 1*.

Every $n^{th}$ order non-oscillatory linear differential equation

$$y^{(n)} + a_1(x) \frac{y^{(n-1)}}{Y(x)} + a_2(x) \frac{y^{(n-2)}}{Y(x)} + \ldots + a_n Y = 0$$

over a Hardy field $K$ has at least one germ in $E(K)$.

Proof of the Lemma 1*.

If $Y = Y(x) = 0$, the claim of the lemma 1* is obvious. Otherwise, $\frac{Y'}{Y(x)}$ is a non-oscillating germ and therefore $z_j = \frac{Y'(x)}{Y(x)}$ is well defined for sufficiently large $x$ and solves the equation

$$[z_1^{(n-1)} + a_1(x) z_1^{(n-2)} + a_2(x) z_1^{(n-3)} + \ldots + a_n(x)] = 0.$$ 

That is, $z_1$ satisfies an equation of the form

$$z_1^{(n-1)} = f_1(x, z_1^{(n-2)}, z_1^{(n-3)}, \ldots, z_1).$$

Substituting, $z_2 = z_1'$, $z_3 = z_2'$, $z_4 = z_3'$, $z_5 = z_4'$, $\ldots$, $z_{n-1} = z_{n-2}'$, in succession, we obtain

$$z_{n-1}' = f_{n-1}(x, z_{n-1}).$$

This equation satisfies Theorem G of Chapter I. Hence $z_{n-1} \in E(K)$. 

Since \( z_{n-1} = \frac{z_{n-2}}{z_{n-2}} \), we have
\[ z_{n-2} = e^{\int_{a_1}^{x} dx} E(K). \]

Continuing like this we get
\[ Y(X) = e^{\int_{a_1}^{x} dx} E(K). \]

Proof of the Theorem

We shall prove it by induction. When \( n = 1 \), the theorem is true trivially. When \( n = 2 \), the proof is similar to that of Theorem 1 of Chapter II. Thus, the theorem is true for \( n=1 \) and \( n=2 \).

Assume the theorem to be true for all differential equations of type (3.1) of order \((n-1)\) over \( K \).

Consider (3.1) over \( K \) for \( n \geq 2 \).

By Lemma 1* there exists at least one germ of equation (3.1) in \( E(K) \). Let it be \( Y_1(x) \). It must be a non-oscillating germ since the equation is non-oscillatory. So there exists \( x_0 \in I = (a, \infty) \) such that \( Y_1(x) \neq 0 \) for all \( x \geq x_0 \). Without loss of generality we can assume \( Y_1(x) > 0 \). If \( z(x) = u(x) Y_1(x) \) is to be a germ of (3.1) then the coefficient of \( u \) is just \( L(Y_1) = 0 \). Setting \( v = u \), the reduced equation in \( v \) is

\[
\sum_{k=1}^{n} \left[ p^{n-k} C_k Y_1^{(n-k)} + p^{n-k} C_k Y_2^{(n-k)} \right] v^{k-1} = 0.
\]

This is an equation over the Hardy field \( K \) and is of order \((n-1)\) in \( v \) with germs \((Y_2/Y_1)',(Y_3/Y_1)',...,(Y_n/Y_1)'\) where \( Y_k(x)(k = 1,2,3,...,n) \) are the linearly independent non-oscillating germs of (3.1). We claim that the above reduced equation in \( v \) is non-oscillatory. For, if it is oscillatory, it must have at least one oscillating germ. Let \( v_k = \frac{Y_k}{Y_1} \) be a
oscillating germ. By definition, we have $v_k(x_n) = 0$ such that $\lim_{x \to -\infty} x_n = 0$ as $m \to \infty$. That is $w(Y_1, Y_k)(x_n) = 0$ such that $\lim_{x \to -\infty} x_n = 0$ as $m \to \infty$. Since $Y_1(x)$ and $Y_k(x)$ are solutions of the differential equation (3.1), they are analytic. Hence it follows by [0, p.34] that $Y_1(x)$ and $Y_k(x)$ are linearly dependent. This is a contradiction since $Y_1(x)$ and $Y_k(x)$ are linearly independent non-oscillating germs of the equation (3.1).

By induction hypothesis, this reduced equation has a solution basis $\{ v_2, v_3, ..., v_n \}$ in $E(K)$. So $v_i = (Y_j(x)/Y_1(x)) \in E(K)$ ($i = 2, 3, ..., n$). By Theorem A of Chapter I, $Y_1(x)/Y_1(x) \in E(K)$ and $Y_i(x) \in E(K)$ ($i = 1, 2, 3, ..., n$). Hence the theorem.

Example 3.1

Consider $(x^3 Y')' - 3x Y' = 0$ over the Hardy field $K$. This equation is non-oscillatory with the solution basis \{ $1/x^2$, 1, $x^2$ \} $\in E(K)$.

Remark 3.1

Theorem 1 fails if the equation (3.1) is not taken over a Hardy field. For instance consider the following example.

Example 3.2

$e^{slnx}$ is a non-oscillating germ of the differential equation

$$Y''' - Y'' \cos x + 2Y' \sin x + Y \cos x = 0 \quad (3.5)$$

But $e^{slnx}$ does not belong to $E(K)$ for any Hardy field $K$. 
Remark 3.2

In general in the theory of linear differential equations with variable coefficients a non homogeneous linear differential equation need not be non-oscillatory even if all the solutions of the corresponding homogeneous equation is non-oscillatory. For instance consider

\[(x^3 y')' - 3x y' = \sin (2 \log x) \quad (3.6)\]

The homogeneous part has a solution basis \(1, x^2, x^3\) but the non-homogeneous equations can have a solution \(\cos(2 \log x)/16\) which is oscillatory.

In contrast to this situation in the classical theory of linear differential equations, we have the following theorem in the Hardy fields.

Theorem 2

If the equation (3.1) over \(K\) is non-oscillatory then the germs of (3.2) belong to \(E(K)\) and equation (3.2) is non-oscillatory.

Proof

Let \((z_1(x), z_2(x), \ldots, z_n(x))\) be the solution basis of the equation (3.1). In view of theorem 1, all the germs and their linear combinations belong to \(E(K)\).
Let \( y(x) = \sum_{i=1}^{n} \gamma_i(x)z_i(x) \)  
(3.7)

and we impose \((n-1)\) conditions on the unknown functions

\[ \gamma_i(x) (i=1,2,\ldots,n) \]

\[ \sum_{i=1}^{n} \gamma_j'(x)z_i^{(j-1)}(x) = 0, \quad (j=1,2,\ldots,(n-1)) \]

(3.8)

Then using (3.8) repeatedly, we have

\[ p y^{(n-1)} = p \left( \sum_{i=1}^{n} \gamma_i z^{(n-1)}_i \right) \]

This implies

\[ (p y^{(n-1)})' = p \left( \sum_{i=1}^{n} \gamma_i' z^{(n-1)}_i \right) - \sum_{i=1}^{n-1} b_i(x) y^{(i-1)} \]

Therefore,

\[ (p y^{(n-1)})' + \sum_{i=1}^{n-1} b_i(x) y^{(i-1)} = p \left( \sum_{i=1}^{n} \gamma_i' z^{(n-1)}_i \right) \]

Thus
Solving (3.8) and (3.9) and having the Wronskian

\[ w(z_1, z_2, \ldots, z_n) = \frac{w(o)}{p(x)} \exp \left\{ \int_0^\infty \frac{b_n(s)}{p(s)} ds \right\} \]

We have

\[ \gamma_k(x) = W_k(x) \frac{f(x)}{w(o)} \exp \left\{ \int_0^\infty \frac{b_n(s)}{p(s)} ds \right\} (k = 1, 2, \ldots, n) \]

(3.10)

where \( W_k(x) \) denotes a determinant obtained from the Wronsky's determinant \( W(z_1, z_2, \ldots, z_n) \), replacing its \( k \)th column for \((0, 0, \ldots, 0, 1)\). Integrating (3.10) on the interval \([a, x]\) \((x > a)\)

\[ \gamma_k(x) = \gamma_k(a) + \int_a^x W_k(t) \frac{f(t)}{w(o)} \exp \left\{ \int_0^\infty \frac{b_n(s)}{p(s)} ds \right\} dt. \]

\[ (k = 1, 2, \ldots, n) \]

(3.11)

Since \( E(k) \) is perfect, all the terms on the right side of (3.11) belong to \( E(K) \) and so

\[ \gamma_k(x) \in E(K) \]

From (3.7) and (3.11) we infer that any germ \( Y(x) \) of (3.2) belongs to \( E(K) \). Further \( Y(x) \) cannot be oscillatory, for if so, \( Y(x) \) will have arbitrarily large number of zeroes which contradicts the property of the Hardy field (section 2.1 of Chapter I). Hence the theorem.
Corollary 2.1

The equation (3.2) over K is non-oscillatory if and only if equation (3.1) over K is non-oscillatory.

Example 3.3

The equation \((x^{y''} - 3xy') = \log x\) over the rational field of functions is non-oscillatory.

Theorem 3

If equation (3.3) over K is non-oscillatory then all its germs belong to \(E(K)\).

Equation (3.3) can be transformed to (3.1) where \(n = m + 1\), \(b_n(x) = (1-1)p'(x)\) etc.

Theorem 4

If equation (3.3) over K is non-oscillatory then all the germs of equation (3.4) belong to \(E(K)\) and equation (3.4) is non-oscillatory. Theorem 4 is a consequence of Theorem 2.

Corollary 4.1

The equation (3.4) over K is non-oscillatory if and only if equation (3.3) over K is non-oscillatory. This is a consequence of Corollary 2.1.

Example 3.4

The equation \((x^{y'})^{(2)} = \log x\) over K is non-oscillatory.

Remark 3.3

Theorems 1 and 3 and Theorems 2 and 4 are generalization of Theorems 1 and 2 of chapter II respectively.

Thus we see that all non-oscillating germs of homogeneous or non-homogeneous linear differential equations over K belong to \(E(K)\).
3.3 Some Properties of the germs of the Higher Order linear differential equations over $K$.

In this section we shall prove some interesting properties of the Higher order linear differential equations over $K$ using Canonical valuation $V$.

To start with, we shall show that there exists an ordered basis for the solution space of non-oscillatory homogeneous equations over Hardy fields and that the basis lie in a Hardy field of finite rank.

Theorem 5

If equation (3.1) or (3.3) over $K$ is non-oscillatory, there exists an ordered basis $\{y_1, \ldots, y_n\}$ in $E(K)$ such that

$$V(y_1) \geq V(y_2) \geq \ldots \geq V(y_n) \quad (3.12)$$

If $K = \mathbb{R}$ the basis lie on $R(y_1, \ldots, y_n) \subset E(\mathbb{R})$ and it is of finite rank utmost $n$.

Proof

Since any Hardy field is ordered, the first part follows trivially. By adjoining $y_1, y_2, \ldots, y_n$ to $\mathbb{R}$ we get the Hardy field $R(y_1, \ldots, y_n)$ which has utmost $n$ comparability classes. Hence the result follows.

Example 3.5

If equation (3.1) over $K(=\mathbb{R})$ has the solution basis $\{\log x, x\}$ in $E(\mathbb{R})$, clearly $V(\log x) > V(x)$ and the solution basis lie in a Hardy field $R(x, \log x)$ which has rank 2. Thus the theorem is verified.

Remark 3.4

Equality in (3.12) cannot be dropped. For instance if $\{\log x, x, 1+x\}$ is a solution basis for the equation (3.1) or
(3.3) then \((x/1+x)\) belongs to the kernel of \(V\) and so the
equality holds there, while for the other two elements \(V(\log x) > V(x)\).

In the next two theorems we find some important properties
of the germs in the solution space.

**Theorem 6**

Let \(\{y_1, y_2, \ldots, y_n\}\) be an ordered basis for the non-oscillatory
equation (3.1) or (3.3) over \(K\) be such that
\[
V(y_1) > V(y_2) > \cdots > V(y_n)
\]
then for all
\[
V(y_k) > V(y_j) (j > k, j, k \in \{1, 2, \ldots, n\})
\]
(i) \(W(y_k, y_j) > 0\)

(ii) \(\int_0^\infty W(y_k, y_j) \phi(y_k, y_j) \, dt\) is convergent when \(\phi(y_k, y_j)\)
belongs to the set
\[
\left\{ \frac{1}{y_j^2}, \frac{1}{y_k^2 + y_j^2}, \frac{\exp(y_k/y_j)}{y_j^2}, \frac{1}{y_j^2} \sin(y_k/y_j), \frac{1}{y_j^2} \cos(y_k/y_j) \right\}
\]
(iii) \(\int_0^\infty \frac{w(y_k, y_j)}{y_k^2} \, dt\) is divergent.

**Proof**

Since \(V(y_k) > V(y_j)\), \(y_k/y_j\) is a positive germ approaching
zero as \(x \to \infty\) and hence is decreasing. So \((y_k/y_j)' < 0\);
This gives (i).

\[
\frac{y_k'}{y_j} - \frac{w(y_j, y_k)}{y_j^2}
\]
Now

Integrate the previous equality with respect to "t" from c to \( t \) and make \( t \rightarrow \omega \). Then using \( V(Y_k) > V(Y_j) \) we have

\[
\int_c^t \frac{w(y_k, y_j)}{y_j^2} \, dt \text{ is finite.}
\]

This proves the first part of (ii). Similarly we can prove the other parts.

Theorem 7

Let \( \{y_1, y_2, \ldots, y_n\} \) be the ordered solution basis in \( \mathbb{R}(K) \) for the non-oscillatory equation (3.1) (or (3.3)) over \( K \) be such that \( V(Y_1) > V(Y_2) > \ldots > V(Y_n) \).

If

i) \( V(Y_k) > V(Y_j) < 0 \) \( (j, k = 1, 2, \ldots, n, j > k) \)

ii) \( W(y_k, y_j) \) is positive decreasing function or a constant

iii) \( y_j'(x) \) is non-decreasing, then \( y_k(x) \) is non-increasing.

Proof.

By hypothesis

\[
\left( \frac{y_k}{y_j} \right)' = \frac{w(y_j, y_k)}{y_j^2}
\]

Integrate the previous equality with respect to "t" from \( x \) to \( t \) and make \( t \rightarrow \omega \). Then using \( V(Y_k) > V(Y_j) \) we have

\[
y_k(x) - y_j(x) \int_x^t \frac{w(y_k, y_j)}{y_j^2} \, dt.
\]
Differentiating and using $y_j'(x)$ is non-decreasing, 

$$y_k'(x) \leq \int_{y}^{y_j(t)} w(y_k, y_j)(t) \frac{y_j^2(t)}{y_j'(t)} dt - \frac{w(y_k, y_j)(x)}{y_j'(x)}.$$ 

If $w(y_k, y_j) = \text{constant}$, $y_k'(x) \leq 0$.

If $w(y_k, y_j)$ is a positive decreasing function, using Bonnet's form of Second Mean Value Theorem, $y_k'(x) \leq 0$. Hence the theorem.

Remark 3.5

Theorems 3, 4 and 5 of Chapter II are particular cases of our most general theorems 5, 6, 7 of chapter III.

As it follows from the proof, under the stated conditions, the Theorem 7 is true in the general case also.

In Chapter IV, we concentrate on the discrete analogue of certain results of Chapters I, II and III.
4.2 Canonical Valuations of a Discrete Hardy Field

Let $K^i$ be the set of non zero elements of a discrete Hardy field $K$. If $a_n, b_n \in K^i$, we write $a_n \approx b_n$ if $\lim_{n \to \infty} a_n/b_n$ is a finite nonzero number. It is clear that $\approx$ is an equivalence relation on the elements of $K^i$. For $a_n \in K^i$, the equivalence class of $a_n$ is denoted by $\mathcal{V}(a_n)$. Let $\Gamma = \{ \mathcal{V}(a_n) : a_n \in K^i \}$ be the set of all equivalence classes on $K^i$. If $a_n, b_n, c_n, d_n \in K^i$ and $a_n \approx b_n$ and $c_n \approx d_n$ then clearly $a_n c_n \approx b_n d_n$, so that multiplication on $K^i$ induces a composition of elements of $\Gamma$.

Thus $\Gamma$ becomes an abelian group and the map $\mathcal{V} : K^i \to \Gamma$ is a homomorphism. We follow the convention of writing the composition law on $\Gamma$ additively.

If $a_n, b_n \in K^i$, we write $\mathcal{V}(a_n) > \mathcal{V}(b_n)$ (or $\mathcal{V}(b_n) < \mathcal{V}(a_n)$) if $\lim_{n \to \infty} a_n/b_n = 0$, this definition clearly depends only on the equivalence classes $\mathcal{V}(a_n)$ and $\mathcal{V}(b_n)$ of $a_n$ and $b_n$ and it induces a total ordering on the set $\Gamma$.

For $a_n \in K^i$, $\mathcal{V}(a_n) > 0$ (or $\mathcal{V}(1)$) means simply that $\lim_{n \to \infty} a_n = 0$ and it follows that if $a_n, b_n \in K^i$ and $\mathcal{V}(a_n), \mathcal{V}(b_n) > 0$, then also $\mathcal{V}(a_n) + \mathcal{V}(b_n) = \mathcal{V}(a_n b_n) > 0$.

That is $\Gamma$ is an ordered abelian group.