CHAPTER I
INTRODUCTION

Difference equations arise themselves as mathematical models describing real life situations not only in science and technology but also in such diverse fields as economics, psychology, sociology etc. Therefore, difference equations are not the discrete analogues of differential equations, in fact they paved the way for the development of the latter. In [1,5,6,12,26,37] several examples from the diverse fields have been illustrated; these are sufficient to convey the importance of the qualitative as well as quantitative study of the difference equations. A detailed study of difference equations with many references can be found in [1,27,32] as well as in Chapter 7 in [22].

The qualitative theory of difference equations is in a process of continuous development, as it is apparent from the large number of research papers concerned with it. It provides methods and criteria which describe the asymptotic behavior of solutions of given difference equations without actually constructing or approximating them. In contrast with differential equations, the existence and uniqueness of solutions of discrete initial value problems can be established recursively.

In the theory of difference equations, oscillatory and asymptotic behavior of solutions play an important role. A nontrivial solution of a difference equation is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Although several results in the discrete case are similar to those already known in the continuous case, the adaptation from the
continuous to the discrete case is not direct but requires some special devices. Further, it was shown in [21,23] that there exist some properties of differential equations which do not carry over directly to the corresponding difference equations. Therefore it is useful to study the oscillatory and asymptotic behavior of solutions of difference equations. Some interesting investigations of such problems can be found in the monographs by Agarwal [1] and Gyori and Ladas [22].

For the last few years, the oscillation and asymptotic behavior of solutions of delay difference equations are being extensively studied. Erbe and Zhang [9], Chen and Zhang [8], Ladas [28], Ladas, Philos and Sticas [29], Gyori [20], Gyori and Ladas [22] Lalli and Zhang [34,35] and Szmanda [39] have done extensive work on this topic. In contrast neutral delay difference equations, that is equations in which the highest order difference of the unknown function appears both with and without delays has received very little attention, see for example, [10,11,33-36,44,47].

From the point of view of applications, it is important to study neutral difference equations because these equations are discrete analogues of neutral delay differential equations, which appear in problems dealing with networks containing lossless transmission lines. Such networks arise in high speed computers, where transmission lines are used to interconnect circuits. They also arise as Euler equations for the minimizations of functionals involving a time delay. For further applications of neutral type equations see [7,20,40].
Keeping in view the importance of the subject, we have obtained some important results on the following topics.

1) Asymptotic and oscillatory behavior of solutions of first order nonlinear neutral delay difference equations.

2) Oscillatory properties of first order nonlinear neutral difference equations with variable delays.

3) Oscillatory and asymptotic behavior of solutions of second order nonlinear neutral delay difference equations.

4) Classification of nonoscillatory solutions of neutral delay difference equations of arbitrary order.

5) Oscillatory and asymptotic behavior of solutions of neutral delay difference equations of arbitrary order.

6) Oscillation of solutions of delay difference equations of higher order with and without forcing term.

The plan of the thesis is as follows:

In the second and third sections of Chapter II of the thesis we consider first order neutral delay difference equations of the form

$$\Delta (y(n)+p(n)y(n-k)) + \delta \sum_{j=1}^{m} q_j(n)f_j(y(n-\sigma_j)) = 0, \ n \in \mathbb{Z}, \quad (1.1)$$

where $\mathbb{Z} = \{0,1,2,\ldots\}, \delta = \pm 1, \ {p(n)}, \ {q_j(n)} (1 \leq j \leq m)$ are real sequences, $f_j: \mathbb{R} \to \mathbb{R}$ are continuous and $\sigma_j (1 \leq j \leq m)$ are nonnegative integers. We have obtained several sufficient conditions for the asymptotic
behavior of nonoscillatory solutions and the oscillation of all and/or bounded solutions of equation (1.1).

In Section 4 of Chapter II we consider nonlinear neutral difference equations with variable delays of the form

$$\Delta \left( y(n) + p(n) y(n-k) \right) + \delta f \left( n, y(g_1(n)), \ldots, y(g_m(n)) \right) = 0, \; n \in \mathbb{Z}, \tag{1.2}$$

where $\delta = \pm 1$, $(p(n))$ is a given real sequence, $f : \mathbb{Z} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous and $(g_j(n))$, $(1 \leq j \leq m)$ are sequences of nonnegative integers. We have established necessary as well as sufficient conditions for the existence of oscillatory and nonoscillatory solutions of equation (1.2) using Knaster - Tarski fixed point theorem [38].

Nonlinear second order neutral delay difference equations of the form

$$\Delta^2 \left( y(n-1) + p(n-1) y(n-1-k) \right) + q(n) f(y(n-\sigma)) = 0, \; n \in \mathbb{Z}, \tag{1.3}$$

where $(p(n))$, $(q(n))$ are real sequences, $k$ and $\sigma$ are nonnegative integers and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and nondecreasing, are considered in Sections 2 and 3 of Chapter III. Several sufficient conditions for the oscillation and asymptotic behavior of solutions of equation (1.3) have been obtained.

In Section 4 of Chapter III we consider second order nonlinear neutral delay difference equations of the form

$$\Delta^2 \left( y(n) + p(n) y(n-k) \right) - F \left( n, y(n-\sigma_1), \ldots, y(n-\sigma_m) \right) = 0, \; n \in \mathbb{Z}, \tag{1.4}$$
where \( \{p(n)\} \) is a real sequence, \( k, \sigma_1, ... \sigma_m \) are nonnegative integers and \( F: \mathbb{Z} \times \mathbb{R}^m \rightarrow \mathbb{R} \) is continuous. We have obtained sufficient conditions for the growth of nonoscillatory solutions and oscillation of all bounded solutions of equation (1.4) when \( F \) satisfies different type of conditions.

Chapter IV deals with higher order neutral delay difference equations of the form

\[
\Delta^m (y(n) - p(n) y(n-k)) + \delta q(n) y(\sigma (n+m-1)) = 0, \quad n \in \mathbb{Z}
\]  

(1.5)

where \( \delta = \pm 1 \), \( \{p(n)\} \), \( \{q(n)\} \) and \( \{\sigma(n)\} \) are real sequences, and \( k \) is a nonnegative integer. We have classified the possible nonoscillatory solutions of equation (1.5) according to their asymptotic behavior as \( n \rightarrow \infty \) and constructed, with the aid of fixed point techniques, nonoscillatory solutions having certain types of asymptotic behaviors.

In Chapter V we consider higher order nonlinear neutral delay difference equations of the form

\[
\Delta^m (y(n-m+1)+p(n-m+1) y(n-m+1-k))+\delta F(n, y(n-\sigma)) = 0, \quad n \in \mathbb{Z}
\]  

(1.6)

where \( \delta = \pm 1 \), \( \{p(n)\} \), \( \{q(n)\} \) are real sequences, and \( F: \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous. Several sufficient conditions for the asymptotic behavior of nonoscillatory solutions and the oscillation of all solutions of equation (1.6) have been established.

Several sufficient conditions were established in [2-4, 24] for the oscillation and asymptotic behavior of solutions of higher order ordinary difference equations.
In the last Chapter VI of the thesis we consider higher order delay difference equations of the form

\[ \Delta^m y(n) + q(n) f(y(n-5(n))) = 0, \quad n \in \mathbb{Z}, \quad (1.7) \]
\[ \Delta^m y(n) + q(n) f(y(\sigma(n))) h(\Delta^{m-1}y(\delta(n))) = 0, \quad n \in \mathbb{Z}, \quad (1.8) \]
and
\[ \Delta^m y(n) + q(n) f(y(\sigma(n))) h(\Delta^{m-1}y(\sigma(n))) = e(n), \quad n \in \mathbb{Z} \quad (1.9) \]

where \( m \) is even, \( \{q(n)\}, \{\sigma(n)\}, \{\delta(n)\} \) are real sequences and \( f, h : \mathbb{R} \rightarrow \mathbb{R} \) are continuous with \( h(u) > 0 \) for \( u \neq 0 \). We have established sufficient conditions for the oscillation of all solutions of equations (1.7), (1.8) and (1.9).

The results obtained in the thesis are partially motivated by results given in [2-4, 9-11, 13, 14, 16-19, 23, 25, 28, 34, 36, 44, 46-48]. Examples have been inserted then and there to illustrate the results obtained in the thesis.