CHAPTER VI

HIGHER ORDER DELAY DIFFERENCE EQUATIONS

6.1 INTRODUCTION

In the last four chapters, we studied the asymptotic and oscillatory behavior of solutions of first and higher order neutral delay difference equations. If \( p(n) = 0 \), then the equations considered in those chapters become ordinary delay difference equations. Numerous results exist for first and second order difference equations with or without delays. However, very few results are available for oscillation and asymptotic behavior of higher order difference equations [1-4,45,47].

Therefore, in this chapter, we investigate the oscillatory and asymptotic behavior of solutions of higher order delay difference equations. In Section 6.2, we establish sufficient conditions for the oscillation of all solutions of the delay difference equations of the form

\[ \Delta^m y(n) + q(n) f(y(n-\delta(n))) = 0, \ m \text{ even} \]

and in Section 6.3 we study the oscillatory behavior of solutions of damped delay difference equations of the form

\[ \Delta^m y(n) + q(n) f(y(\sigma(n))) h(\Delta^{m-1}(\delta(n))) = 0, \ m \text{ even}. \]

Finally, in Section 6.4, we obtain sufficient conditions for the oscillatory and asymptotic behavior of solution of the forced equations of the form

\[ \Delta^m y(n) + q(n) f(y(\sigma(n))) h(\Delta^{m-1}(\delta(n))) = e(n), \ m \text{ even.} \]

6.2 OSCILLATION THEOREMS FOR DELAY DIFFERENCE EQUATIONS

In the \( m^{th} \) order difference equation

\[ \Delta^m y(n) + q(n) f(y(n-\delta(n))) = 0, \ n \in \mathbb{Z}, \quad (6.1) \]

where \( m \) is even, \( \{q(n)\} \) is a sequence of non negative real numbers and \( q(n) \) is not identically equal to zero for infinitely many values of \( n \), \( \{\delta(n)\} \) is a given sequence.
of integers with \( \lim_{n \to \infty} (n \cdot \delta(n)) = \infty \) and \( f: \mathbb{R} \to \mathbb{R} \) is continuous such that \( u \cdot f(u) > 0 \) for \( u \neq 0 \).

By a solution of equation (6.1), we mean a real sequence \( \{y(n)\} \) defined for \( n \geq \min_{\geq 0} (i - \delta(i)) \) and satisfies equation (6.1) for all \( n \geq 0 \). As usual, we define the oscillatory and nonoscillatory solution of equation (6.1).

In the sequel, we need the following lemmas given in [1] and [47].

**LEMMA 6.2.1**: Let \( 1 \leq j \leq m-1 \) and \( y(n) \) be defined for all \( n \in \mathbb{Z} \). Then

\[
\lim_{n \to \infty} \Delta^i y(n) > 0, \text{ implies } \lim_{n \to \infty} \Delta^i y(n) = \infty, \quad 0 \leq i \leq j-1,
\]

\[
\limsup_{n \to \infty} \Delta^i y(n) < 0, \text{ implies } \liminf_{n \to \infty} \Delta^i y(n) = -\infty, \quad 0 \leq i \leq j-1.
\]

**LEMMA 6.2.2**: Let \( \{y(n)\} \) be a sequence of real numbers on \( \mathbb{Z} \). Let \( y(n) \) and \( \Delta^m y(n) \) be of constant sign with \( \Delta^m y(n) \) not being identically zero on any subset of the form \( \{n_1, n_1 + 1, \ldots\} \) of \( \mathbb{Z} \). If

\[
y(n) \Delta^m y(n) \leq 0,
\]

then

(i) there is an integer \( n_2 \geq n_1 \), such that the sequence \( \{\Delta^i y(n)\}, 1 \leq j \leq m-1 \) are of constant sign for \( n \geq n_2 \);

(ii) there exists a number \( \ell \) in \( \{0,1,\ldots,(m-1)\} \) with \((-1)^{m-\ell} = 1\) and such that

\[
y(n) \Delta^j y(n) > 0, \text{ for } j = 0, 1, 2, \ldots, \ell, \quad n \geq n_2,
\]

\[
(-1)^{\ell+1} y(n) \Delta^j y(n) > 0, \text{ for } j = \ell + 1, \ldots, (m-1), \quad n \leq n_2.
\]

**LEMMA 6.2.3**: Let \( \{y(n)\} \) be as defined in Lemma 6.2.2 and such that \( y(n) > 0 \) and \( \Delta^m y(n) \leq 0 \) for all \( n \geq n_0 \in \mathbb{Z} \) and not identically equal to zero. Then there exists a sufficiently large integer \( n_i \) such that for all \( n \geq n_i \geq n_0 \)

\[
y(n) \geq \frac{(n-n_i)^{(m-1)}}{(m-1)!} \Delta^{m-1} y(2^{m-\ell} - 1, n),
\]

where \((n-n_i)^{(m-1)}\) is the usual factorial notation. Moreover, if \( \{y(n)\} \) is increasing then
Now we present oscillation results for equation (6.1).

**THEOREM 6.2.4**: Assume

\[ \lim \inf_{n \rightarrow \infty} f(u) > 0, \]  
(6.2)

and

\[ \sum q(n) = \infty. \]  
(6.3)

Then every solution of equation (6.1) oscillates.

**Proof**: Let \( \{y(n)\} \) be a nontrivial solution of equation (6.1). Without loss of generality assume \( y(n) > 0 \) and \( y(n - \delta(n)) > 0 \) for all \( n \in \mathbb{Z} \). From equation (6.1), we have

\[ \Delta^m y(n) = -q(n) f(y(n - \delta(n))) \leq 0, \text{ for } n \geq n_1, \]

and using Lemma 6.2.2 we obtain

\[ \Delta y(n) > 0, \Delta^{m-1} y(n) > 0 \text{ for all } n \geq n_1. \]

As a consequence, \( \lim_{n \rightarrow \infty} y(n - \delta(n)) \) exists and suppose

\[ L = \lim_{n \rightarrow \infty} y(n - \delta(n)). \]

Then \( L > 0 \) and is finite or infinite.

**Case 1**: \( L > 0 \) finite.

From the continuity of \( f \) we have

\[ \lim_{n \rightarrow \infty} f(y(n - \delta(n))) = f(L) > 0. \]

Then we may choose an integer \( n_2 > n_1 \) such that

\[ f(y(n - \delta(n))) > \frac{1}{2} f(L) \text{ for } n \geq n_2. \]

Now from equation (6.1), we have

\[ \Delta^m y(n) + \frac{1}{2} f(L) q(n) < 0 \text{ for } n \geq n_2, \]

and summing the above inequality from \( n_2 \) to \( n-1 \), we obtain

\[ \Delta^{m-1} y(n) - \Delta^{m-1} y(n_2) + \frac{1}{2} f(L) \sum_{s=n_2}^{n-1} q(s) < 0. \]

Letting \( n \rightarrow \infty \) in the last inequality and using condition (6.3) we obtain
\( \Delta^{m-1} y(n) \to -\infty \) as \( n \to \infty \) and in particular \( \Delta^{m-1} y(n) < 0 \) for sufficiently large \( n \). In view of Lemma 6.2.1, we obtain \( y(n) \to -\infty \) as \( n \to \infty \), which is a contradiction.

**Case 2 :** \( L = \infty \).

We have from condition (6.2), \( \lim \inf_{n \to \infty} f(y(n - \delta(n))) > 0 \) and we may choose a positive constant \( L_1 \) and an integer \( n_3 \) such that

\[
 f(y(n - \delta(n))) \geq L_1 \quad \text{for} \quad n \geq n_3.
\]

Therefore for \( n \geq n_3 \), \( \Delta^n(y(n)) + L_1 q(n) < 0 \).

Rest of the proof is same as that of Case 1. This completes the proof of the theorem.

**REMARK :** Theorem 6.2.4 generalizes Theorem 1 given in [48].

The difference equation

\[
\Delta^4 y(n) + q(n) y^4(n-3) = 0, \quad n \geq 3.
\]

where \( q(n) = (3/2)^{2^{n-9}} \) satisfies all conditions of Theorem 6.2.4 and hence all solutions of (E4) are oscillatory. One such solution is \( \{y(n)\} = \left\{ \frac{(-1)^n}{2^n} \right\} \).

Next we establish an oscillation criterion for the equation (6.1) subject to the conditions

\[ \{\delta(n)\} \text{ is a sequence of nonnegative integers}, \quad (6.4) \]

and

\[
\int_0^T \frac{du}{f(u)} < \infty \quad \text{and} \quad \int_T^\infty \frac{du}{f(u)} > -\infty \quad \text{for all} \quad T > 0. \quad (6.5)
\]

**THEOREM 6.2.5 :** In addition to (6.4) and (6.5), assume \( f \) is nondecreasing and

\[
-f(uv) \geq f(uv) \geq k_1 f(u) f(u) \quad \text{for} \quad u, v > 0 \quad (6.6)
\]

where \( k_1 \) is a positive constant. If

\[
\sum_{m=1}^\infty q(n) f((\frac{n - \delta(n)}{2^{m-1}})^{m-1}) = \infty.
\]

(6.7)
Then every solution of equation (6.1) is oscillatory.

**PROOF:** Let \( \{y(n)\} \) be a nonoscillatory solution of equation (6.1) and without loss of generality assume \( y(n) > 0 \) and \( y(n - \delta(n)) > 0 \) for \( n \geq n_1 \in \mathbb{Z} \). From equation (6.1), we have

\[
\Delta^m y(n) = -q(n) f(y(n-\delta(n))) \leq 0 \quad \text{for} \quad n \geq n_1
\]

and by Lemma 6.2.2., we have

\[
\Delta \ y(n) > 0, \ \Delta^{m-1} \ y(n) > 0 \quad \text{for} \quad n \geq n_1.
\]

From (6.8), \( \{\Delta^{m-1} y(n)\} \) is positive and nonincreasing, we have from Lemma 6.2.3.

\[
y(n - \delta(n)) \geq \frac{1}{(m-1)!} \Delta^{m-1} y(n) \left( \frac{n-\delta(n)}{2^{m-1}} \right)^{(m-1)} \quad \text{for all} \quad n \geq n_2 \geq 2n_1.
\]

Now from equation (6.1), we have for \( n \geq n_2 \)

\[
\Delta^m y(n) + q(n) f \left( \frac{1}{(m-1)!} \right) \Delta^{m-1} y(n) \left( \frac{n-\delta(n)}{2^{m-1}} \right)^{(m-1)} \leq 0
\]

and using condition (6.6) to the last inequality, we obtain

\[
\Delta^m y(n) + q(n) k^2 f \left( \frac{1}{(m-1)!} \right) \left( \frac{n-\delta(n)}{2^{m-1}} \right)^{(m-1)} \leq 0
\]

or

\[
k^2 f \left( \frac{1}{(m-1)!} \right) q(n) f \left( \frac{n-\delta(n)}{2^{m-1}} \right)^{(m-1)} \leq \frac{-\Delta^m y(n)}{f(\Delta^{m-1} y(n))}.
\]

Summing the last inequality from \( n_2 \) to \( n \), we obtain

\[
k^2 f \left( \frac{1}{(m-1)!} \right) \sum_{s=n_2}^{n} q(s) f \left( \frac{s-\delta(s)}{2^{m-1}} \right)^{(m-1)} \leq \sum_{s=n_2}^{n} \frac{\Delta^m y(n)}{f(\Delta^{m-1} y(n))}
\]

\[
\leq \int_{0}^{\infty} \frac{du}{f(u)} < \infty
\]

which is a contradiction. The proof for the case \( y(n) < 0 \) eventually is similar and hence omitted.

**COROLLARY 6.2.6:** Let condition (6.5) be replaced by

\[
\frac{f(u)}{u^\beta} \geq \beta > 0 \quad \text{for} \quad u \neq 0
\]
where \( \alpha \) is the ratio of odd positive integers with \( 0 < \alpha < 1 \). Then every solution of equation (6.1) is oscillatory.

**PROOF**: The proof is similar to that of Theorem 6.2.5 and hence is omitted.

**REMARK**: Corollary 6.2.6 generalizes Theorem 3 of [46].

Consider the difference equation

\[
\Delta^4 y(n) + q(n) y^{1/3} (n - \delta(n)) = 0, \quad n \geq 4,
\]

where \( \delta(n) = 3 \) and \( q(n) = \frac{16(n+2)}{(n-3)^{1/3}} \). All conditions of Theorem 6.2.5 are satisfied and hence all solutions of (E2) are oscillatory. One such solution is \( \{y(n)\} = \{-1\}^n n \}. Also equation (E2) illustrates Corollary 6.2.6.

Note that the results of this section can be easily extended to the equation of the form

\[
\Delta^m y(n) + \sum_{i=1}^{k} q_i(n) f_i(y(n-\delta_i(n))) = 0, \quad n = 0, 1, 2, ...
\]

by assuming appropriate conditions on \( q, f, \) and \( \delta \).

**6.3 OSCILLATION THEOREMS FOR DAMPED DELAY DIFFERENCE EQUATIONS**

Consider the higher order damped delay difference equations of the form

\[
\Delta^m y(n) + q(n) f(y(\sigma(n))) h(\Delta^{m-1} y(\delta(n))) = 0, \quad n \in \mathbb{Z},
\]

where \( m \) is even, \( \{q(n)\} \) is a sequence of nonnegative real numbers and \( q(n) \) is not identically equal to zero for infinite many values of \( n \); \( \{\sigma(n)\} \) and \( \{\delta(n)\} \) are given sequences of integers with \( \lim_{n \to \infty} \sigma(n) = \infty = \lim_{n \to \infty} \delta(n) \). We assume throughout that \( f, h : \mathbb{R} \to \mathbb{R} \) are continuous with \( h(u) > 0, u \neq 0 \). Further, \( uf(u) > 0, u \neq 0 \) and \( f \) satisfies condition (6.6) on \( \mathbb{R} - \{0\} \).

By a solution of (6.9), we mean a real sequence \( \{y(n)\} \) which is defined for all \( n \geq \min_{j \in \mathbb{N}} \{ \sigma(j), \delta(j) \} \) and satisfies (6.9) for sufficiently large values of \( n \).
We begin with the following Theorem.

**THEOREM 6.3.1**: Let condition (6.6) holds. Further, assume that

\[ f(u) \text{ and } f(u) h(u) \text{ are nondecreasing for } u > 0, \]

\[ \delta(n) \leq \inf_{i \geq n} \{ i, \sigma(i) \} \quad \text{for } n \geq n_0 \in \mathbb{Z}, \]

\[ \int_0^\infty \frac{du}{f(u) h(u)} < \infty \text{ and } \int_T^\infty \frac{du}{f(u) h(u)} > -\infty, \text{ for any } T > 0. \]

If

\[ \sum q(n) f(\frac{\delta(n)}{2^{m-1}})^{m-1} = \infty \]

then every solution of equation (6.9) is oscillatory.

**PROOF**: Suppose there is a nonoscillatory solution \( \{y(n)\} \) of equation (6.9). We may assume (and we do) that \( y(n) > 0 \) and \( y(\sigma(n)) > 0 \) for all \( n \geq n_0 \in \mathbb{Z} \). By Lemma 6.2.1, there exists a \( n_i \geq n_0 \) such that \( \Delta y(n) > 0 \) and \( \Delta y(n) > 0 \) for all \( n \geq n_i \). Since \( \{y(n)\} \) is increasing, by Lemma 6.2.2 we have

\[ y(n) \geq \frac{1}{(m-1)!} \Delta^{m-1} y(n) \left( \frac{n}{2^{m-1}} \right)^{m-1}, \text{ for all } n \geq 2n_i. \]

Consequently, since \( \delta(n) \to \infty \) as \( n \to \infty \), there exists \( n_2 \geq 2n_i \) such that

\[ y(\delta(n)) \geq \frac{1}{(m-1)!} \Delta^{m-1} y(\delta(n)) \left( \frac{\delta(n)}{2^{m-1}} \right)^{m-1}, \text{ for all } n \geq n_2. \]

Using conditions (6.10), (6.11) and the fact that \( \{y(n)\} \) is increasing we have

\[ f(y(\delta(n))) \geq f(A(n)), \text{ for all } n \geq n_2, \]

where \( A(n) \) stands for the expression on the right hand side of inequality (6.14).

Thus, from (6.9) we have

\[ \Delta^m y(n) + q(n) f(A(n)) h(\Delta^{m-1} y(\delta(n))) \leq 0, \text{ for all } n \geq n_2. \]

From which it follows that

\[ \Delta^m y(n) + q(n) f\left( \frac{\delta(n)}{2^{m-1}} \right)^{m-1} f(\Delta^{m-1} y(n)) h(\Delta^{m-1} y(n)) \leq 0, \]
for \( n \geq n_2 \), where \( c = k_1 f \left( \frac{1}{(m-1)!} \right) \). Summing the above inequality from \( n_2 \) to \( n \) we get

\[
c \sum_{n=n_2}^{n} q(s) f((\delta(s))^{(m-1)}) \leq - \sum_{n=n_2}^{n} \frac{\Delta^n y(s)}{f(\Delta^{n-1} y(s)) h(\Delta^{n-1} y(s))} \leq \int_{0}^{\Delta^n y(n)} \frac{du}{f(u) h(u)} < \infty,
\]

which contradicts condition (6.13).

If \( y(n) < 0 \) for all \( n \geq n_0 \) we can put \( x(n) = -y(n) \) and transform the equation (6.9) to

\[
\Delta^m x(n) + q(n) f^*(x(\sigma(n))) h^*(\Delta^{m-1} x(\delta(n))) = 0 \tag{6.9}^*
\]

where \( f^*(x) = -f(-x) \) and \( h^*(v) = h(-v) \). Thus we can repeat earlier argument on the positive solutions \( \{y(n)\} \) of (6.9)*. This completes the proof of the theorem.

**COROLLARY 6.3.2:** Let conditions (6.10) and (6.12) both be replaced by

\[
f(u) \text{ is non decreasing and } \frac{f(u) h(u)}{u^\alpha} \geq \beta \geq 0 \text{ for } u \neq 0 \tag{6.18}
\]

where \( \alpha \) is the ratio of odd positive integers with \( 0 < \alpha < 1 \). Then every solution of equation (6.9) is oscillatory.

**PROOF:** The proof is similar to that of Theorem 6.3.1 and hence is omitted.

**EXAMPLE:** Consider, the difference equation

\[
\Delta^m y(n) + 3^{(m-2)/5} 2^{(m-5)/15} y^{1/3} (n+1) (\Delta^{m-1} y(n))^{2/5} = 0, \quad n \geq 2, \tag{E_3}
\]

where \( m \) is even. All the hypotheses of Corollary 6.3.2 are satisfied and hence every solution of (E_3) is oscillatory. One such solution is \( \{y(n)\} = \{(-1)^n 2^n\} \).

Also equation (E_3) illustrates Theorem 6.3.1.

In the following two theorems we replace condition (6.11) by the following:

\[
\delta(n) \leq \sigma(n) < n, \quad \text{for } n \geq n_0 \in \mathbb{Z}. \tag{6.19}
\]

**THEOREM 6.3.3:** In addition to conditions (6.6) and (6.9), suppose that
If
\[ f(u) \text{ is nondecreasing and } \frac{f(u)h(u)}{u} \geq r > 0 \text{ for } u \neq 0. \quad (6.20) \]

then every solution of equation (6.9) is oscillatory. 

**PROOF:** Assume to the contrary, and let \( \{y(n)\} \) be a nonoscillatory solution of equation (6.9), which we assume to be eventually positive. From the proof of Theorem 6.3.1, we obtain (6.16). Put
\[ z(n) = \Delta^{n-1} y(n). \]

Then, in view of conditions (6.6) and (6.20), we have
\[ (\Delta z(n) + r k^2 f(1)) q(n) f((\ell(n))^{(m-1)}) z(n) \leq 0, \quad (6.22) \]
for all \( n \geq n_1 \), from which, arguing exactly as in the proof of Theorem 2.5 in [9], we get
\[ \limsup_{s \to \infty} \sum_{s-k}^{n} q(s) f((\ell(s))^{(m-1)}) \leq \frac{1}{k^2 r f((m-1)!)} \]
which contradicts condition (6.20). Thus the proof of the theorem is complete.

**THEOREM 6.3.4:** Let conditions (6.6), (6.19) and (6.20) hold. Further, suppose that \( \delta(n) = n - k \), where \( k \) is a positive integer. If
\[ \liminf_{s \to \infty} \sum_{s-k}^{n} q(s) f((\ell(s))^{(m-1)}) > \frac{1}{k^2 r f((m-1)!)} \]
Then every solution of equation (6.9) is oscillatory. 

**PROOF:** Let \( \{y(n)\} \) be a nonoscillatory solution of equation (6.9) and as before we assume that \( y(n) > 0 \) and \( y(n-k) > 0 \) for \( n \geq n_o \in \mathbb{Z} \). Proceeding as in Theorem 6.3.3 we obtain (6.22), which can be written as
\[ \Delta z(n) + r k_i^2 \left( \frac{1}{(m-1)!} \right) q(n) f \left( \frac{n-k}{2^{m-1}} \right) z(n-k) \leq 0, \]

for all \( n \geq n_2 \). Applying the techniques used in the proof of Theorem 4.1 in [28] we arrive at a contradiction to (6.23). We omit the rest of the details.

The following examples are illustrative.

**EXAMPLES**: The difference equation

\[ \Delta^6 y(n) + 2^{5/3} y^{13}(n-1) \left( \Delta^5 y(n-1) \right)^{2/3} = 0 \quad (n \geq 1) \quad \text{(E_4)} \]

has the oscillatory solution \( \{y(n)\} = \{(-1)^n\} \). We note that all the hypothesis of Theorems 6.3.3 and 6.3.4 are satisfied.

The difference equation

\[ \Delta^m y(n) + \frac{(3/2)^{m-2/3}}{2^{m-3}} y^{13}(n-1) \left( \Delta^{m-1} y(n-2) \right)^{2/3} = 0, \quad n \geq 1, \quad \text{(E_5)} \]

where \( m \) is even, has the oscillatory solution \( \{y(n)\} = \{(-1)^n/2^n\} \). Here all conditions required in Theorems 6.3.3 and 6.3.4 are satisfied.

**REMARKS**:

(a) If \( \sigma(n) = \delta(n) = n \), then Theorem 6.3.1 is the same as Theorem 2 in [45].

(b) If \( h(u) = 1 \) or \( h(u) \geq \alpha \) for \( u \neq 0 \) and \( \sigma(n) = n - \delta(n) \), Theorem 6.3.1 reduces to Theorem 6.2.5 of Section 6.2.

### 6.4 Oscillation Theorem for Forced Delay Difference Equations

In this section we concern with the oscillatory and asymptotic behavior of solutions of the forced equations of the form

\[ \Delta^m y(m) + q(n) f(y(\sigma(n))) h(\Delta^{m-1} y(\sigma(n))) = e(n), \quad m \text{ even}. \quad \text{(6.24)} \]

Here the sequences \( \{q(n)\}, \{\sigma(n)\} \) and the functions \( f \) and \( g \) are as in (6.9), and \( \{e(n)\} \) is a real sequence which is oscillatory.

We prove the following result.

**Theorem 6.4.1**: Let condition (6.6) hold. Further, assume that

\[ h(-u) \geq h(u) > 0, \quad h(u) \text{ and } f(u) \text{ are non-decreasing for } u \neq 0; \quad \text{(6.25)} \]
and there exist an oscillatory sequence \( \{g(n)\} \) such that
\[
\Delta^m g(n) = e(n) \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad g(n)/(n)^{(m-1)} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.26}
\]

If
\[
\limsup_{n \to \infty} \sum_{s=0}^{n-1} q(s) f((\sigma(s))^{(m-1)}) = \infty, \tag{6.27}
\]
then every solution of equation (6.24) is either oscillatory or \([\Delta^{m-1} y(n) - \Delta^{m-1} g(n)] \to 0\) as \(n \to \infty\).

**PROOF:** Let \( \{y(n)\} \) be a nonoscillatory solution of equation (6.24) and assume that \( y(n) > 0 \) and \( y(\sigma(n)) > 0 \) for all \( n \geq n_0 \in \mathbb{Z} \). Put \( y(n) = x(n) + g(n) \). It follows that there exists an integer \( n_1 \) such that \( x(\sigma(n)) + g(\sigma(n)) > 0 \) for all \( n \geq n_1 \). Thus
\[
\Delta^m x(n) + q(n) f(y(\sigma(n)))h(\Delta^{m-1} y(\sigma(n))) \leq 0, \quad \text{for} \quad n \geq n_1. \tag{6.28}
\]

Consequently, all the differences of \( \{x(n)\} \) up to order \( m-1 \) are monotone and are of one sign for all sufficiently large values of \( n \). Now, if \( x(n) < 0 \) for \( n \geq n_1 \), then there is an integer \( n_2 \) such that \( x(\sigma(n)) < 0 \) for \( n \geq n_2 \). In which case, in view of the oscillatory character of \( g(n) \), \( x(\sigma(n)) + g(\sigma(n)) \) assumes negative values for infinitely many \( n \) larger than \( n_2 \), which contradicts the fact that \( y(n) > 0 \) for \( n \geq n_1 \). Thus we have \( x(n) > 0 \) for all \( n \geq n_1 \). Since the hypotheses of Lemma 6.2.2 are satisfied for \( n \geq n_2 \), there exists an integer \( n_3 \) such that
\[
\Delta^m x(n) > 0 \quad \text{and} \quad \Delta^{m-1} x(n) > 0 \quad \text{for} \quad n \geq n_3 \geq n_2 \geq n_1. \tag{6.29}
\]

Since \( \{x(n)\} \) is increasing we use Lemma 6.2.3 to get
\[
x(n) \geq \frac{(n-1)^{m-1}}{(m-1)!} \Delta^{m-1} x(n), \quad \text{for} \quad n \geq 2 n_3.
\]

Since \( \sigma(n) \to \infty \) as \( n \to \infty \), we can choose an integer \( n_4 \) such that
\[
x(\sigma(n)) \geq \frac{1}{(m-1)!} \Delta^{m-1} x(\sigma(n)) \left( \frac{\sigma(n)}{2^{m-1}} \right)^{(m-1)}, \quad \text{for} \quad n \geq n_4 \geq 2n_3. \tag{6.30}
\]

In view of conditions (6.6) and (6.25), equation (6.28) reduces to
\[ \Delta^m x(n) + k_1^2 q(n) f \left( \frac{1}{(m-1)!} \right) f \left( \frac{\sigma(n)}{2^{m-1}} \right) f \left[ \Delta^{m-1} x(\sigma(n)) + \frac{(m-1)! g(\sigma(n))}{\frac{\sigma(n)}{2^{m-1}}} \right] h(\Delta^{m-1} x(\sigma(n)) + \Delta^{m-1} g(\sigma(n))) \leq 0, \quad (6.31) \]

for \( n \geq n_4 \). Put \( \omega(n) = \Delta^{m-1} x(n) \) to obtain

\[ \Delta \omega(n) + c q(n) f \left( \frac{\sigma(n)}{2^{m-1}} \right) f(\omega(\sigma(n))) + \frac{(m-1)! g(\sigma(n))}{\frac{\sigma(n)}{2^{m-1}}} h(\omega(\sigma(n)) + g(\sigma(n))) \leq 0, \quad n \geq n_4 \quad (6.32) \]

where \( c = k_1^2 f \left( \frac{1}{(m-1)!} \right) \). Since \( \Delta \omega(n) < 0 \) for \( n \geq n_4 \) we have \( \lim_{n \to \infty} \omega(n) = a \) for some constant \( a \). If \( a < 0 \) then we get a contradiction to (6.29). For \( a > 0 \) we note that

\[ [\omega(\sigma(n)) + \frac{(m-1)! g(\sigma(n))}{\frac{\sigma(n)}{2^{m-1}}} ] \to a \]

and

\[ [\omega(\sigma(n)) + \Delta^{m-1} g(\sigma(n))] \to a \text{ as } n \to \infty. \]

Thus there is an integer \( n_5 \geq n_4 \) such that

\[ [\omega(\sigma(n)) + \frac{(m-1)! g(\sigma(n))}{\frac{\sigma(n)}{2^{m-1}}} ] \geq \frac{a}{2} \]

and

\[ [\omega(\sigma(n)) + \Delta^{m-1} g(\sigma(n))] \geq a/2, \text{ for } n \geq n_5. \]

Using condition (6.25) we have from (6.32) the following inequality

\[ \Delta \omega(n) + c q(n) f \left( \frac{\sigma(n)}{2^{m-1}} \right) f \left( \frac{a}{2} \right) h \left( \frac{a}{2} \right) \leq 0, \quad (6.33) \]

for all \( n \geq n_5 \). Summing (6.33) from \( n_5 \) to \( n-1 \), we obtain
\[
\begin{align*}
\sum_{i=1}^{n-1} q(s) f \left( \frac{(s)}{2^{m-1}} \right) & \leq \omega(n_2) - \omega(n) \leq \omega(n_1) < \infty,
\end{align*}
\]
a contradiction to the condition (6.27). This completes the proof of the theorem.

**REMARK:** In Theorem 6.4.1 we do not require that the functions \( f \) and \( h \) satisfy conditions of the form

\[
\int_{a}^{b} \frac{du}{f(u)h(u)} < \infty, \text{ or } \int_{a}^{b} \frac{du}{f(u)h(u)} < \infty
\]

or that the function \( fh \) be almost linear. Thus our result holds for strongly superlinear, sublinear and linear cases. Furthermore, we impose no restriction on the sequence \( \{ \sigma(n) \} \) and hence this sequence can be retarded, advanced or of mixed type.

**EXAMPLES:** Each of the following equations

\[
\Delta^m y(n) + \frac{\left[ \frac{3}{2} \right]^{m-1} 2^{m-1} 2^{2m}}{e^{\frac{1}{3} (1+e)^m} \left( y(n+1) \right)^{1/3} \left( 1 + \left| \Delta^{m-1} y(n) \right| \right)^{1/3}} = \frac{(-1)^n}{2} e^n (1 + e)^m
\]

has oscillatory solutions \( \{ y_1(n) \} = \left\{ \frac{(-1)^n}{2^n} \right\} \), \( \{ y_2(n) \} = \left\{ (-1)^n \right\} \) and

\( \{ y_3(n) \} = \{ (-1)^n e^n \} \) respectively. All conditions of Theorem 6.4.1 are satisfied in each case.