CHAPTER V
NONLINEAR NEUTRAL DIFFERENCE EQUATIONS
OF ARBITRARY ORDER

5.1 INTRODUCTION

In this chapter we continue our study on the asymptotic and oscillatory behavior of solutions of higher order nonlinear neutral delay difference equation of the type

\[ \Delta^m (y(n-m+1) + p(n-m+1) y(n-m+1-k)) + \delta F(n, y(n-\sigma)) = 0 \quad (5.1) \]

where \( m \geq 1, \delta = \pm 1, \Delta \) denotes the forward difference operator \( \Delta y(n) = y(n+1) - y(n) \), \( \Delta^i y(n) = \Delta (\Delta^{i-1} y(n)) \), \( 1 \leq i \leq m, k, \sigma \in \mathbb{Z} \), \( \{p(n)\} \) and \( \{q(n)\} \) are sequences of real numbers, \( F: \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) is continuous with \( n F(n, u) \leq 0 \) for \( u \neq 0 \) and \( n \geq N_0 \), and \( F(n, u) \neq 0 \) for \( u \in \mathbb{R} \setminus \{0\} \) and \( n \in N_i \) for every \( N_i \geq N_0 \in \mathbb{Z} \). By a solution of (5.1), we mean a sequence \( \{y(n)\} \) of real numbers which is defined for \( n \geq N_0 - M \) where \( M = \max \{k, \sigma\} + m-1 \) and which satisfies (5.1) for \( n \geq N_0 \). As usual, we defined the oscillatory and nonoscillatory solutions of equation (5.1).

5.2 OSCILLATORY AND ASYMPTOTIC BEHAVIOUR

Many of our results will require the condition that if \( \{u(n)\} \) is a sequence with \( u(n) > 0 \) (\( u(n) < 0 \)) and \( \lim \inf_{n \to \infty} |u(n)| > 0 \), then

\[ \sum_{i \in N_i} F(i, u(i)) = -\infty \quad (5.2) \]

We will begin with a lemma that will facilitate proving the main results in this chapter. In some parts of the lemma we assume that there exist constants \( P_1 \) and \( P_2 \) such that either

\[ P_1 \leq p(n) \leq 0, \quad (5.3) \]

\[ -1 \leq p(n) \leq 0, \quad (5.4) \]

or

\[ P_1 \leq p(n) \leq -1. \quad (5.5) \]

For notational purposes, we let

\[ z(n) = y(n) + p(n) y(n-k). \]
We give proofs only for the case when a nonoscillatory solution is eventually positive since the proofs for an eventually negative solution are similar. In addition, when the proof for $\delta = -1$ is completely analogous to the proof for $\delta = +1$, only the latter will be given.

**LEMMA 5.2.1**: Suppose that condition (5.2) holds and \{y(n)\} is an eventually positive \{negative\} solution of (5.1) with $\delta = +1 \{\delta = -1\}$. Then:

(a) \{\Delta^{m-1} z(n)\} is an eventually nonincreasing \{nondecreasing\} \{nondecreasing \{nonincreasing\}\} sequence and

\[ \Delta^{m-1} z(n) \to \delta L < \infty \ (\ > - \infty) \ \text{as} \ n \to \infty. \]

(b) If $\delta L > - \infty \ (< \infty)$, then \[ \lim \inf_{n \to \infty} |y(n)| = 0. \]

(c) If $z(n) \to 0$ as $n \to \infty$, then \{\Delta' z(n)\} is monotonic and

\[ \Delta' z(n) \to 0 \ \text{as} \ n \to \infty \ \text{and} \ \Delta' z(n) \Delta' z(n) < 0 \]

for $i = 0, 1, \ldots, m - 1$.

(d) Let $z(n) \to 0$ as $n \to \infty$. If $m$ is even \{odd\}, then $z(n) < 0 \ (z(n) > 0)$. If $m$ is odd \{even\}, then $z(n) > 0 \ (z(n) < 0)$.

(e) If (5.3) holds, then either \{\Delta' z(n)\} is nonincreasing \{nondecreasing\} \{nondecreasing \{nonincreasing\}\} with

\[ \Delta' z(n) \to - \infty \ (\infty) \ \{\infty(- \infty)\} \ \text{as} \ n \to \infty \]

for $i = 0, 1, \ldots, m - 1$, or \{\Delta' z(n)\} is monotonic and (5.6) holds.

(f) If (5.3) holds and $m$ is even, then $z(n) < 0 \ (z(n) > 0)$ \{> 0 \ (< 0)\}. If (5.6) holds \ \text{and} \ m \ \text{is odd, then} \ z(n) > 0 \ (z(n) < 0) \ \{< 0 \ (> 0)\}.

(g) If (5.4) \{(5.3)\} holds, then (5.6) holds \{either (5.6) holds or \ $|y(n)| \to \infty$ as $n \to \infty$\}.

(h) If (5.5) holds and $m$ is odd \{even\}, then (5.7) holds.

**PROOF**: Suppose that \{y(n)\} is an eventually positive solution of equation (5.1). Then there exists an integer $N_1 \geq N_0 \in \mathbb{Z}$ such that $y(n-m+1-k) > 0$ and $y(n-o) > 0$ for $n \geq N_i$. From equation (5.1) we have $\delta \Delta^m z(n-m+1) = -F(n, y(n-o)) \leq 0$, so clearly part (a) holds. Summing equation (5.1) from $N_i$ to $n$ and then letting $n \to \infty$, we have
From condition (5.2), we see that (b) holds.

In order to prove (c), suppose that \( \delta = +1 \) and \( z(n) \to 0 \) as \( n \to \infty \). By (a), \( \{\Delta^{m-1} z(n)\} \) is eventually nonincreasing. If \( \Delta^{m-1} z(n) \to L < 0 \) as \( n \to \infty \), then there exists \( L_i < 0 \) and an integer \( N_i \geq N_1 \) such that \( \Delta^{m-1} z(n) \leq L_i \) for \( n \geq N_2 \). This contradicts \( z(n) \to 0 \) as \( n \to \infty \). If \( \Delta^{m-1} z(n) \to L > 0 \) as \( n \to \infty \), then \( \Delta^{m-1} z(n) \geq L \) for \( n \geq N_1 \) again contradicting \( z(n) \to 0 \) as \( n \to \infty \). Therefore, \( \Delta^{m-1} z(n) \to 0 \) as \( n \to \infty \). Since \( \{\Delta^{m-1} z(n)\} \) is nonincreasing and \( F(n, y(n-a)) \neq 0 \), we have \( \Delta^{m-1} z(n) > 0 \) for \( n \geq N_1 \). Hence, if \( m \geq 2 \), then \( \{\Delta^{m-2} z(n)\} \) is nondecreasing and so \( \Delta^{m-2} z(n) \to L > -\infty \) as \( n \to \infty \). If \( L_2 < 0 \), then \( \Delta^{m-2} z(n) \leq L_2 \) for \( n \geq N_1 \) which contradicts \( z(n) \to 0 \) as \( n \to \infty \). Now assume \( L_2 > 0 \); then there is an \( L_3 > 0 \) and an integer \( N_3 \) such that \( \Delta^{m-2} z(n) \geq L_3 \) for \( n \geq N_3 \). Again this contradicts \( z(n) \to 0 \) as \( n \to \infty \). Thus, \( \Delta^{m-2} z(n) \to 0 \) as \( n \to \infty \), and since \( \{\Delta^{m-2} z(n)\} \) is nondecreasing, we have \( \Delta^{m-2} z(n) < 0 \) for \( n \geq n_1 \). Continuing in this fashion we see that (5.6) holds.

Part (d) follows immediately from (5.6) since \( \delta \Delta^m z(n-m+1) \leq 0 \). To prove (e), for \( \delta = +1 \), first note that from (a) and (b) we have \( \{\Delta^{m-1} z(n)\} \) is nonincreasing, \( \Delta^{m-1} z(n) \to L \geq -\infty \) as \( n \to \infty \), and \( \lim \inf_{n \to \infty} y(n) = 0 \) if \( L > -\infty \). If \( L = -\infty \), then clearly (5.7) holds.

If \( -\infty < L < 0 \), then eventually \( z(n) \leq L \) for some \( L_i < 0 \) and so \( p_i y(n-k) \leq p(n) y(n-k) < z(n) \) contradicting \( \lim \inf_{n \to \infty} y(n) = 0 \). Hence, \( L \geq 0 \). If \( L > 0 \), then eventually \( y(n) \geq z(n) \geq L_2 > 0 \) which contradicts \( \lim \inf_{n \to \infty} y(n) = 0 \). Thus, we have \( \Delta^{m-1} z(n) \to 0 \) as \( n \to \infty \). Moreover, \( \Delta^{m-1} z(n) > 0 \) since \( \{\Delta^{m-1} z(n)\} \) is nonincreasing and \( F(n, y(n-a)) \neq 0 \). Hence \( \{\Delta^{m-2} z(n)\} \) is nondecreasing. In addition, \( \Delta^{m-2} z(n) < 0 \) for otherwise \( \{\Delta^{m-2} z(n)\} \) is eventually positive and nondecreasing, which in turn implies \( z(n) \) has a positive lower bound contradicting \( \lim \inf_{n \to \infty} y(n) = 0 \). Now if \( \Delta^{m-2} z(n) \to L_1 < 0 \) as \( n \to \infty \), then it is easy to see that \( z(n) \leq L_3 < 0 \) eventually. This again contradicts \( \lim \inf_{n \to \infty} y(n) = 0 \). Thus, \( \{\Delta^{m-1} z(n)\} \) is nondecreasing and tends to zero as \( n \to \infty \).
Continuing in this way we see that (5.7) holds.

The proof of (f) follows from the fact that either (5.6) or (5.7) implies $z(n) < 0$ if $m$ is even, and (5.7) implies $z(n) > 0$ if $m$ is odd.

To prove (g) when $\delta = +1$, suppose (5.6) does not hold. Then by part (e), (5.7) holds so $z(n) < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Since $p(n) \geq -1$, we have

$$y(n) \leq -p(n) y(n-k) \leq y(n-k).$$

This implies that $\{y(n)\}$ is bounded contradicting (5.7). If $\delta = -1$, and (5.6) does not hold, then part (e) implies that (5.7) holds and so $z(n) \to \infty$ as $n \to \infty$. By (5.3), we have $z(n) \leq y(n) \to \infty$ as $n \to \infty$.

Finally, to prove (h), if (5.7) does not hold, then (5.6) holds. This implies that $\liminf_{n \to \infty} y(n) = 0$. Part (f) implies $z(n) > 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Hence, $y(n) > -p(n) y(n-k) \geq y(n-k)$ which contradicts $\liminf_{n \to \infty} y(n) = 0$.

Our first theorem places very mild restrictions on the sequence $\{p(n)\}$, and as a consequence, the conclusions in the theorem are not very strong. However, it does give us the flavour of the results to be obtained in the subsequent theorems.

**THEOREM 5.2.2**: Suppose that condition (5.2) holds, $m$ is either even or odd, and $\{y(n)\}$ is a nonoscillatory solution of equation (5.1).

(i) If $\delta = +1$ and there exists a constant $P_3$ such that

$$P_3 \leq p(n),$$

then either $|y(n)| \to \infty$ as $n \to \infty$ or $\liminf_{n \to \infty} |y(n)| = 0$. Moreover, if $-1 \leq P_3$, then the second conclusion holds.

(ii) If $\delta = -1$ and there exists $P_4$ such that

$$p(n) \leq P_4,$$

then either $\limsup_{n \to \infty} |y(n)| = \infty$ or $\liminf_{n \to \infty} |y(n)| = 0$. In addition, if $P_4 \leq 0$, then either $|y(n)| \to \infty$ as $n \to \infty$ or $\liminf_{n \to \infty} y(n) = 0$. 

PROOF: Let \( \{y(n)\} \) be an eventually positive solution of equation (5.1), say, \( y(n-m+1-k) > 0 \) and \( y(n-o) > 0 \) for \( n \geq N_1 \) for some \( N_1 \geq N_0 \in \mathbb{Z} \). Part (a) of Lemma 5.2.1 implies \( \Delta^{m-1} z(n) \to \delta L < \infty \) as \( n \to \infty \) and part (b) of Lemma 5.2.1 implies \( \lim \inf_{n \to \infty} y(n) = 0 \) if \( \delta L > -\infty \). If \( \delta L = -\infty \), then \( \delta z(n) \to -\infty \) as \( n \to \infty \). If (i) holds, \( z(n) \to -\infty \) as \( n \to \infty \), and so

\[
P_3 y(n-k) \leq y(n) + p(n) y(n-k) = z(n) \to -\infty
\]
as \( n \to \infty \). Hence, \( p(n) \leq 0 \) eventually and \( y(n) \to \infty \) as \( n \to \infty \). If \( P_3 \geq -1 \), then either \( \lim \inf_{n \to \infty} y(n) = 0 \) or \( y(n) + p(n) y(n-k) = z(n) < 0 \) for all large \( n \). Thus, \( y(n) < -p(n) y(n-k) \leq y(n-k) \) which implies \( \{y(n)\} \) is bounded and this contradicts \( L = -\infty \). If (ii) holds, \( z(n) \to \infty \) as \( n \to \infty \), so we have \( z(n) \leq y(n) + P_4 y(n-k) \to \infty \) as \( n \to \infty \). This implies \( \lim \sup_{n \to \infty} y(n) = \infty \). If \( P_4 = 0 \), then \( z(n) \leq y(n) \to \infty \) as \( n \to \infty \).

REMARK: Theorem 5.2.2 generalizes Theorem 2.3 in [34]

For our next theorem we ask that there exist a positive constant \( P_5 \) such that

\[
0 \leq p(n) \leq P_5 < 1.
\]  

(5.8)

THEOREM 5.2.3: Suppose that conditions (5.2) and (5.8) hold.

(i) If \( m \) is even and \( \delta = +1 \), then all solutions of equation (5.1) are oscillatory, while if \( \delta = -1 \), any solution \( \{y(n)\} \) of equation (5.1) is either oscillatory, \( y(n) \to 0 \) as \( n \to \infty \), or \( |y(n)| \to \infty \) as \( n \to \infty \).

(ii) If \( m \) is odd and \( \delta = +1 \), then either \( \{y(n)\} \) is oscillatory or \( y(n) \to 0 \) as \( n \to \infty \), while if \( \delta = -1 \), then either \( \{y(n)\} \) is oscillatory or \( |y(n)| \to \infty \) as \( n \to \infty \).

PROOF: Let \( \{y(n)\} \) be an eventually positive solution of equation (5.1), say \( y(n-m+1-k) > 0 \) and \( y(n-o) > 0 \) for \( n \geq N_1 \geq N_0 \in \mathbb{Z} \). By part (a) of Lemma 5.2.1, we have \( \{\delta \Delta^{m-1} z(n)\} \) is nonincreasing and \( \{\delta \Delta^{m-1} z(n)\} \) converges to \( \delta L \geq -\infty \) as \( n \to \infty \). If \( \delta L = -\infty \), then \( z(n) \) is eventually negative if \( \delta = +1 \) and \( z(n) \to \infty \) if \( \delta = -1 \). Moreover, since \( p(n) \geq 0 \), the first possibility is excluded. If \( z(n) \to \infty \), then \( \{z(n)\} \) is nondecreasing since \( \{\delta \Delta^{m-1} z(n)\} \) has fixed sign. Hence, we have \( z(n) = y(n) + p(n) y(n-k) \leq y(n) + p(n) z(n-k) \leq y(n) + P_5 z(n) \), so \( z(n) \leq y(n) \to \infty \) as \( n \to \infty \).
If $\delta L > -\infty$, $\lim_{n \to \infty} \inf y(n) = 0$. Since $\{z(n)\}$ is monotonic, $z(n) \to \ell$ as $n \to \infty$. Observe that $\ell \geq 0$ since $\ell < 0$ implies $y(n) < 0$. Assume $\ell > 0$. If $\{z(n)\}$ is nonincreasing, we again obtain $z(n) [1-P_3] \leq y(n)$ which contradicts $\lim_{n \to \infty} \inf y(n) = 0$. If $\{z(n)\}$ is nondecreasing, let $1-P_3 = \epsilon > 0$. Then $z(n) \leq y(n) + P_3 y(n-k)$, and since $\ell$ is finite.

$$\frac{z(n)}{z(n-k)} \leq \frac{y(n)}{z(n-k)} + P_3 \leq \frac{y(n)}{\ell} + P_3.$$

Since $P_3 + \frac{\epsilon}{2} < 1$, there exists $N_2 > N_1$ such that $z(n)/z(n-k) \geq P_3 + \frac{\epsilon}{2}$ for $n \geq N_2 \in Z$. Hence, $y(n) \geq \ell\epsilon/2$ for $n \geq N_2$ contradicting $\lim_{n \to \infty} \inf y(n) = 0$. Thus, $z(n) \to 0$ as $n \to \infty$.

To complete the proof, just observe that part (d) of Lemma 5.2.1 implies that for $m$ even $z(n) < 0$ if $\delta = +1$ and $z(n) > 0$ if $\delta = -1$. But $z(n) < 0$ contradicts $y(n) > 0$, and $z(n) > 0$ implies $y(n) \leq z(n) \to 0$ as $n \to \infty$. Hence (i) holds. If $m$ is odd, Lemma 5.2.1 (d) implies $z(n) > 0$ if $\delta = +1$ and $z(n) < 0$ if $\delta = -1$; part (ii) then follows.

**EXAMPLES:** The equation

$$\Delta^n (y(n-m+1)+py(n-m)) + \frac{(-1)^m (1 + \epsilon y(n-m))}{\epsilon (\gamma+1)} e^{(\gamma+1)m} y^{(n-1)-0}, n \geq 1 \quad (E_1)$$

where $0 \leq \epsilon < 1$ and $\gamma \geq 1$ is the quotient of odd positive integers, satisfies the hypotheses of part (i) of Theorem 5.2.3 with $\delta = -1$ and part (ii) with $\delta = +1$. Here, $\{y(n)\} = \{e^{n}\}$ is a nonoscillatory solution which converges to 0 as $n \to \infty$. Equation $(E_1)$ also satisfies the hypotheses of part (i) of Theorem 5.3.2 provided $p > -1/e$ and $m$ is odd or $p < -1/e$ and $m$ is even. The equation

$$\Delta^n (y(n-m+1)+p y(n-m)) = \frac{(P+1)(e-1)^m e^{(\gamma+1)m} e^{ym1-m}}{\epsilon (\gamma+1)} y^{(n-1)-0}, n \geq 1, \quad (E_2)$$

with $m$ odd, $0 \leq \epsilon < 1$ and $\gamma \leq 1$, the ratio of odd positive integers, satisfies the hypotheses of Theorem 5.2.3 (ii) for $\delta = -1$, and has the nonoscillatory solution.
\[ \{y(n)\} = \{e^n\} \text{ satisfying } e^n \to \infty \text{ as } n \to \infty. \] If \( p < -e \), then Theorem 5.2.2 (i) holds, and if \(-e < p \leq 0\), then Theorem 5.2.2 (ii) holds. In each case \(\{y(n)\} = \{e^n\}\) is an unbounded nonoscillatory solution. As an example of an equation satisfying the hypotheses of Theorem 5.2.3 and having an oscillatory solution, consider

\[ \Delta^n(y(n-m+1) + p y(n-m-1)) + \delta (1 + p) 2^n y(n-\alpha) = 0, \quad n \geq 1, \quad (E_3) \]

where \(0 \leq p < 1\), if \(\delta = +1\) and \(\alpha\) is even, or \(\delta = -1\) and \(\alpha\) is odd, then \(\{y(n)\} = \{(-1)^n\}\) is an oscillatory solution of \((E_3)\). If \(p \leq 0\), then equation \((E_3)\) can also be used to construct examples of equations satisfying Theorem 5.2.2 and having oscillatory solutions.

REMARK: Theorem 5.2.3 (i) generalizes Theorem 5 in [44] and Theorem 3.3.5 in Chapter 3, and Theorem 5.2.3 (ii) generalizes part of Corollary 1(b) in [11].

For our next result we will need a stronger version of condition (5.4), namely, that there exists a constant \(P_6 < 0\) such that

\[ -1 < P_6 \leq p(n) \leq 0. \quad (5.9) \]

THEOREM 5.2.4: Suppose that conditions (5.2) and (5.9) hold and \(m\) is either even or odd. If \(\delta = +1\), then any solution \(\{y(n)\}\) of equation (5.1) is either oscillatory or satisfies \(y(n) \to 0\) as \(n \to \infty\), while if \(\delta = -1\), then either \(\{y(n)\}\) is oscillatory, \(y(n) \to 0\), or \(|y(n)| \to \infty\) as \(n \to \infty\).

PROOF: Suppose that \(\{y(n)\}\) is a nonoscillatory solution of equation (5.1) such that \(y(n-m+1-k) > 0\) and \(y(n-\sigma) > 0\) for \(n \geq N_1 \geq N_0 \in \mathbb{Z}\). Lemma 5.2.1 (g) implies that (5.6) holds if \(\delta = +1\) and either (5.6) holds or \(|y(n)| \to \infty\) as \(n \to \infty\) if \(\delta = -1\). Suppose (5.6) holds. If either \(m\) is even and \(\delta = +1\) or \(m\) is odd and \(\delta = -1\), (5.9) and Lemma 5.2.1 (d) imply that \(z(n) < 0\) eventually. It then follows that \(y(n) \leq -P_6 y(n-k)\) for \(n \geq N_2 \) for some \(N_2 \geq N_1\). Hence, \(y(n+k) \leq (-P_6)^2 y(n-k)\) and by induction we have that \(y(n+jk) \leq (-P_6)^{j+1} y(n-k)\) for every positive integer \(j\). Since \(0 < -P_6 < 1\), this implies that \(y(n) \to 0\) as \(n \to \infty\).

If \(m\) is even and \(\delta = -1\) or \(m\) is odd and \(\delta = +1\), then (5.9) and Lemma
5.2.1 (d) imply $0 < z(n) < -A_1$ for some constant $A_1 > 0$ and sufficiently large $n$, and so $0 < y(n) < -P_6 y(n-k) + A_1$. If $\{y(n)\}$ is unbounded, then there exists an nondecreasing sequence $\{\alpha_i\}$ such that $y(\alpha_i) \to \infty$ as $i \to \infty$, and $y(\alpha_i) = \max \{y(n) : N_i \leq n \leq \alpha_i\}$. For each $i$, $y(\alpha_i) < -P_6 y(\alpha_i-k) + A_1 \leq -P_6 y(\alpha_i) + A_1$, or $(P_6 + 1) y(\alpha_i) \leq A_1$. In view of (5.9) this is impossible. Therefore, $\{y(n)\}$ is bounded and there exists a constant $A_2 > 0$ such that $\limsup_{n \to \infty} y(n) = A_2$. Thus, there is an increasing sequence $\{\beta_j\}$ such that $y(\beta_j) \to A_2$ as $j \to \infty$. From (5.9) we have

$$-P_6 y(\beta_j-k) \geq y(\beta_j) - z(\beta_j).$$

Since $A_2 > 0$, there exists $\epsilon > 0$ such that $(1-P_6) \epsilon < (1+P_6) A_2$ and so

$$0 < -P_6 (A_2+\epsilon) < A_2 - \epsilon.$$

But for all sufficiently large $j$, $y(\beta_j-k) < A_2 + \epsilon$, so we have

$$A_2 - \epsilon > -P_6 y(\beta_j-k) \geq y(\beta_j) - z(\beta_j)$$

for all such $j$. As $j \to \infty$, this contradicts $y(\beta_j) \to A_2$ as $j \to \infty$ since $z(\beta_j) \to 0$ as $j \to \infty$.

**REMARK:** Notice that if $\delta = -1$, Theorem 5.2.4 implies that unbounded solutions must be oscillatory. Theorem 5.2.4 generalizes Corollary 2.1 (v) in [34], Theorem 3.4 in [36], Theorem 4 in [44], Theorems 3.2.2 and 3.3.1 in Chapter 3, and a part of Corollary 1(b) in [11].

**EXAMPLE:** Equation (E1) provides examples of all the different cases in Theorem 5.2.4 depending on whether $-1/e < p \leq 0$ or $-1 < p < -1/e$. For $-1 < p \leq 0$, equation (E2) satisfies the hypotheses of Theorem 5.2.4 with $\delta = -1$ and has an unbounded nonoscillatory solution. Similarly, for $-1 < p \leq 0$, if $\delta = +1$ and $\alpha$ is even, or $\delta = -1$ and $\alpha$ is odd, (E3) yields equations satisfying Theorem 5.2.4 and having oscillatory solutions.

**THEOREM 5.2.5:** Suppose that (5.2) and (5.5) hold. If (i) $m$ is even and $\delta = -1$ or (ii) $m$ is odd and $\delta = +1$, then any solution $\{y(n)\}$ of equation (5.1) is either oscillatory or $|y(n)| \to \infty$ as $n \to \infty$.

**PROOF:** Let $\{y(n)\}$ be a nonoscillatory solution of equation (5.1) such that $y(n-m+1-k) > 0$ and $y(n-o) > 0$ for $n \geq N_i \geq N_0 \in \mathbb{Z}$. Part (h) of Lemma 5.2.1 implies (5.7) holds so $\delta z(n) \to -\infty$ as $n \to \infty$. Now for sufficiently large $n$, (5.5)
implies that $P_2 y(n-k) \leq z(n) \leq y(n)$, and hence $y(n) \to \infty$ as $n \to \infty$.

**EXAMPLE**: If $m$ is even and $-e \leq p \leq -1$ or $m$ is odd and $p \leq -e$, then equation $(E_2)$ satisfies the hypotheses of Theorem 5.2.5 and has the unbounded nonoscillatory solution $\{y(n)\} = \{e^n\}$ with $e^n \to \infty$ as $n \to \infty$.

**REMARK**: Theorem 5.2.5 generalizes Corollary 1(a) in [11], Theorem 4.3 in [36], and Theorems 2 and 7 in [44].

Next, we obtain a result on the behaviour of the bounded solutions of equation (5.1) for the case when $p(n)$ is bounded above away from -1. Assume that there exists a constant $P_1$ such that

$$P_2 \leq p(n) \leq P_1 < -1.$$  \hfill (5.10)

**THEOREM 5.2.6**: Suppose conditions (5.2) and (5.10) hold. If $m$ is even and $\delta = +1$ or if $m$ is odd and $\delta = -1$, then any bounded solution $\{y(n)\}$ of equation (5.1) is either oscillatory or satisfies $y(n) \to 0$ as $n \to \infty$.

**PROOF**: Assume that $\{y(n)\}$ is a bounded nonoscillatory solution of equation (5.1) with $y(n-m+1-k) > 0$ and $y(n-\sigma) > 0$ for $n \geq N_1 \geq N_0 \in \mathbb{Z}$. Lemma 5.2.1 (e) implies that either (5.6) or (5.7) holds. If (5.7) holds, then the argument used in the proof of Theorem 5.2.4 shows that $y(n) \to \infty$ as $n \to \infty$ contradicting $\{y(n)\}$ being bounded. Therefore (5.6) holds. Now Lemma 5.2.1 (c) implies that if $m$ is even and $\delta = +1$ or if $m$ is odd and $\delta = -1$, then $\delta z(n) < 0$ and $|\delta z(n)|$ is nondecreasing and tends to zero as $n \to \infty$. Since $\{y(n)\}$ is bounded, $\limsup_{n \to \infty} y(n) = \ell$ is nonnegative and finite. If $\ell > 0$, then there exists an increasing sequence $\{n_j\}$ such that $n_j > N_1$, and $n_j \to \infty$ and $y(n_j-k) \to \ell$ as $j \to \infty$. Let $c = P_2 + 1 < 0$, $c = -c\ell/8 > 0$, $d = c\ell/8P_1 > 0$ and $\lambda = -3c\ell/4 > 0$. Then, there exists $N_2 \geq N_1$ such that $\delta z(n_j) > -\varepsilon$ and $y(n_j-k) > \ell - d > 0$ for $j \geq N_2$. Hence, for $j \geq N_2$ we have

$$-\varepsilon < \delta z(n_j) < y(n_j) + P_1 (\ell - d).$$

It follows that

$$-y(n_j) < P_1 (\ell - P_2) d + \varepsilon - (c-1)\ell - \frac{c\ell}{4} = -\lambda - \ell,$$

so $\ell + \lambda < y(n_j)$ for $j \geq N_2$. This contradicts $\limsup_{n \to \infty} y(n) = \ell > 0$. Thus,
lim sup_{n \to \infty} y(n) = 0 and so y(n) \to 0 as n \to \infty.

**REMARK:** Theorem 5.2.6 generalizes Theorem 2.3 in [34] and Theorem 3.2.4 in Chapter 3.

**EXAMPLE:** If \( p < -1 \) and \( m \) is either even or odd, then equation (E_1) satisfies the hypotheses of Theorem 5.2.6 and has the solution \( \{y(n)\} = \{e^n\} \). Also, if \(-e < p < -1\) and \( m \) is odd, then equation (E_2) shows that under the hypotheses of Theorem 5.2.6, it is possible for equation (5.1) to have unbounded solutions.

**REMARK:** Equation (E_3) provides examples of equations satisfying the hypotheses of Theorems 5.2.5 and 5.2.6 and having oscillatory solutions. That is, under the conditions given here, it is not possible to obtain results on the limiting behaviour of all solutions of equation (5.1).

Our next two results require a stronger condition on the function \( F \), namely, that there exists a constant \( B > 0 \) such that

\[
|F(n,u)| \geq B |u| \quad \text{for all } n \geq N_0 \text{ and all } u.
\]  

(5.11)

In addition, we ask that there exists \( P_4 > 0 \) such that

\[
0 \leq p(n) \leq P_4.
\]  

(5.12)

**THEOREM 5.2.7:** Let conditions (5.11) and (5.12) hold, \( m \) be even, and \( \{y(n)\} \) be a solution of equation (5.1). If (i) \( \delta = +1 \), then \( \{y(n)\} \) is oscillatory, while if (ii) \( \delta = -1 \) and \( \{y(n)\} \) is bounded, then either \( \{y(n)\} \) is oscillatory or \( y(n) \to 0 \) as \( n \to \infty \).

**PROOF:** Suppose \( \{y(n)\} \) is a solution of equation (5.1) such that \( y(n-m+1-k) > 0 \) and \( y(n-\sigma) > 0 \) for \( n \geq N_1 \geq N_0 \in \mathbb{Z} \). By part (a) of Lemma 5.2.1, \( \{\Delta^{m-1} z(n)\} \) is nonincreasing and satisfies \( \Delta^{m-1} z(n) \to \delta L \geq -\infty \) as \( n \to \infty \). If \( \delta L < 0 \), then \( \{z(n)\} \) is eventually negative which contradicts (5.12). Hence, \( \delta L \geq 0 \) and by (5.11) we have

\[
|\Delta^{m-1} z(N_1-m+1)| \geq \delta \Delta^{m-1} z(N_1-m+1) = \delta L + \sum_{i=N_1}^{\infty} F(i,y(i-\sigma)) \\
\geq \delta L + B \sum_{i=N_1}^{\infty} y(i-\sigma).
\]

If \( \delta = +1 \), \( \Delta^{m-1} z(n) \) is bounded above; if \( \delta = -1 \), the boundness assumption on
y(n) implies that \( |\Delta^{m-1}z(n)| \) is bounded. In either case, the series on the right hand side of the above inequality converges, and so \( y(n) \to 0 \) as \( n \to \infty \). This in turn implies \( z(n) \to 0 \) as \( n \to \infty \). By Lemma 5.2.1 (d), \( z(n) < 0 \) if \( m \) is even and \( \delta = +1 \), so we get a contradiction in this case.

**EXAMPLE**: If \( m \) is even and \( p \geq 0 \), then \( \alpha \) in equation (E2) can be chosen so that the hypotheses of Theorem 5.2.7 are satisfied and (E2) has the oscillatory solution \( \{y(n)\} = \{-1\}^n \). In addition, for \( m \) even, \( \gamma = 1 \), and \( p \geq 0 \), equation (E1) satisfies Theorem 5.2.7 (ii) and has the bounded nonoscillatory solution \( \{y(n)\} = \{e^n\} \) which converges to zero. Equation (E2) with \( m \) even, \( \gamma = 1 \), and \( p \geq 0 \) satisfies part (ii) of Theorem 5.2.7 and has an unbounded nonoscillatory solution.

**THEOREM 5.2.8**: Let conditions (5.11) and (5.12) hold, \( m \) be odd, and \( \{y(n)\} \) be a solution of equation (5.1). If \( \delta = +1 \), then either \( \{y(n)\} \) is oscillatory or \( y(n) \to 0 \) as \( n \to \infty \), while if \( \delta = -1 \) and \( \{y(n)\} \) is bounded, then \( \{y(n)\} \) is oscillatory.

**PROOF**: As in the proof of Theorem 5.2.7, for any nonoscillatory solution \( \{y(n)\} \) we have \( y(n) \to 0 \) and \( z(n) \to 0 \) as \( n \to \infty \). But since \( m \) is odd, if \( \delta = -1 \), Lemma 5.2.1 (d) contradicts \( z(n) > 0 \).

**EXAMPLE**: If \( m \) is odd, \( \gamma = 1 \), and \( p \geq 0 \), equation (E5) satisfies the hypotheses of Theorem 5.2.8. Here \( \{y(n)\} = \{e^n\} \) is a solution. With \( m \) odd, \( \delta = -1 \), \( \alpha \) odd and \( p \geq 0 \), equation (E1) satisfies Theorem 5.2.8 and has the bounded oscillatory solution \( \{y(n)\} = \{-1\}^n \). This also shows that the hypotheses of Theorem 5.2.8 are not sufficient to ensure that oscillatory solutions of equation (5.1) tends to zero as \( n \to \infty \).

**REMARK**: Theorem 5.2.8 generalizes Theorem 2.2.3 in Chapter 2 and part of Corollary 1(b) in [11].

**THEOREM 5.2.9**: Suppose that (5.2) holds and there is a constant \( p > 1 \) such that

\[
p(n) \to p \quad \text{as} \quad n \to \infty
\]

and \( \{y(n)\} \) is a bounded solution of equation (5.1).

(i) If \( m \) is even and \( \delta = +1 \), then \( \{y(n)\} \) is oscillatory, while if \( \delta = -1 \), then \( \{y(n)\} \)
is either oscillatory or satisfies $y(n) \to 0$ as $n \to \infty$.

(ii) If $m$ is odd and $\delta = +1$, then \{y(n)\} is either oscillatory or satisfies $y(n) \to 0$ as $n \to \infty$, while if $\delta = -1$, then \{y(n)\} is oscillatory.

**PROOF:** Suppose \{y(n)\} is a bounded nonoscillatory solution of equation (5.1) with $y(n-m+1-k) > 0$ and $y(n-\sigma) > 0$ for $n \geq N_1 \geq N_0 \in \mathbb{Z}$. By (5.13), \{p(n)\} is bounded and eventually positive, so $z(n) > 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Part (a) of Lemma 5.2.1 implies that $\Delta^{(m)} z(n) \to \delta L < \infty$ as $n \to \infty$. If $\delta L = \infty$, then \{z(n)\} is eventually negative. Thus, $L$ is finite and by Lemma 5.2.1 (b) $\liminf_{n \to \infty} y(n) = 0$.

Since \{z(n)\} is positive, monotone, and bounded, \{z(n)\} converges to a non-negative number as $n \to \infty$. Suppose $\limsup_{n \to \infty} y(n) = \ell > 0$; then there exists an increasing sequence of positive integers $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and $y(\alpha_j - k) \to \ell$ as $j \to \infty$. Now $\liminf_{n \to \infty} y(n) = 0$ implies there is an increasing sequence of positive integers $\{\beta_j\}$ such that $\beta_j \to \infty$ and $y(\beta_j - k) \to 0$ as $j \to \infty$.

Let $\epsilon = \ell (p-1)/2 > 0$. Since $\limsup_{n \to \infty} y(n) = \ell$, there exists $N_2 \geq N_1$ such that $y(n) < \ell + \epsilon$ for $n \geq N_2$. Choose $N_3 \geq N_2$ such that $\beta_{N_3} > N_2$.

Then for $j \geq N_3$, we have

$$z(\alpha_j) = y(\alpha_j) + p(\alpha_j) y(\alpha_j - k) \quad \text{and} \quad z(\beta_j) = y(\beta_j) + p(\beta_j) y(\beta_j - k).$$

Thus, we have

$$y(\alpha_j - k) - \frac{p(\beta_j)}{p(\alpha_j)} y(\beta_j - k) < \frac{z(\alpha_j) - z(\beta_j)}{p(\alpha_j)} + \frac{y(\beta_j)}{p(\alpha_j)} < \frac{z(\alpha_j) - z(\beta_j)}{p(\alpha_j)} + \frac{\ell + \epsilon}{p(\alpha_j)}.$$ 

Letting $j \to \infty$, we see that

$$\ell \leq \frac{\ell + \epsilon}{p} = \frac{\ell (p+1)}{2p} < \ell$$

which is a contradiction. Hence, $\limsup_{n \to \infty} y(n) = 0$ and so $y(n) \to 0$ as $n \to \infty$. Thus $z(n) \to 0$ as $n \to \infty$.  

Therefore, we have $z(n) > 0$, $z(n) \to 0$ as $n \to \infty$, and $\{z(n)\}$ is monotonic, so $\Delta z(n) < 0$. Since $z(n) > 0$ we must have $\Delta' z(n) > 0$, and an argument similar to the one used in part (i) of Lemma 5.2.1, shows that $(-1)^i \Delta' z(n) \geq 0$ for $i = 1, 2, \ldots, m$. This yields a contradiction if $m$ is even and $\delta = +1$ or $m$ is odd and $\delta = -1$.

**EXAMPLE:** Equation (E$_1$) with $p > 1$ satisfies the hypotheses of Theorem 5.2.9 (i) if $m$ is even ($\delta = -1$) and Theorem 5.2.9 (ii) if $m$ is odd ($\delta = +1$); $\{y(n)\} = \{e^n\}$ is a nonoscillatory solution of (E$_1$) that tends to zero as $n \to \infty$. Equation (E$_3$) also satisfies the hypotheses of Theorem 5.2.9 when $p > 1$ and has the bounded oscillatory solution $\{y(n)\} = \{(-1)^n\}$. It is also interesting to observe that (E$_3$) with $p > 1$ is an example of an equation which satisfies the hypotheses of Theorem 5.2.9 for $m$ even or odd and $\delta = -1$, and has the unbounded nonoscillatory solution $\{y(n)\} = \{e^n\}$.

**EXAMPLE:** As a final example, consider the equation

$$\Delta^m (y(n-m+1)+p y(n-m)) + (-1)^q (e+1)^m \left[ 1 - \frac{p}{e} \right] e^{\delta-1-m} y(n-\beta) = 0, n \geq 1+\beta. \quad (E_4)$$

For any value of the nonnegative integer $\beta$, equation (E$_4$) has the unbounded oscillatory solution $\{y(n)\} = \{(-1)^n e^n\}$. Hence, by appropriately choosing the parity of $\beta$, it is possible to obtain examples of equation (5.1) which have unbounded oscillatory solutions for any values of $m$, $\delta$ and $p$.

### 5.3 CONCLUDING REMARKS

We conclude this chapter with a few suggestions for further research.

First, by examining Theorems 5.2.4 - 5.2.6 we see that $p(n) = -1$ behaves as a bifurcation point for the behaviour of nonoscillatory solutions of equation (5.1). Moreover, if $p(n) = -1$ and either (a) $\delta = +1$ and $m$ is even or (b) $\delta = -1$ and $m$ is odd, the behaviour of nonoscillatory solutions, if any, is not fully understood.

If (a) holds, then Theorem 5.2.2 (i) tells us that $\lim \inf_{n \to \infty} |y(n)| = 0$, and if (b) holds, Theorem 5.2.2 (ii) says that either $|y(n)| \to \infty$ as $n \to \infty$ or $\lim \inf_{n \to \infty}$
In fact, when (5.10) \((p_2 \leq p(n) \leq P, < 1)\) and either (a) or (b) holds, we are unable to rule out the possibility of equation (5.1) having a solution \(\{y(n)\}\) with \(\limsup_{n \to \infty} |y(n)| = \infty\) and \(\liminf_{n \to \infty} |y(n)| = 0\) (see Theorem 5.2.6). Further study of this situation is needed.

Secondly, when \(p(n) \geq 1\) the results here require additional hypotheses such as (5.11) or (5.13). Without these added conditions some, albeit minimal, information about the behaviour of solutions is obtainable from Theorem 5.2.2. It would be interesting to see the conclusions of Theorems 5.2.7 - 5.2.9 reached without these added assumptions.