CHAPTER 3

OSCILLATION RESULTS FOR FORCED NONLINEAR SECOND ORDER DELAY DIFFERENCE EQUATION
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3.1. INTRODUCTION

This chapter deals with the oscillation problem for the solutions of the forced nonlinear difference equation

\[ \Delta^2 y_{n-1} + q_n y^\gamma_{\sigma(n)} = e_n, n \in \mathbb{N} \]  

(3.1)

where \( \gamma \) is a quotient of odd positive integers, \( \{q_n\} \), \( \{e_n\} \) are non-negative real sequences and \( \{\sigma(n)\} \) is an increasing sequence of integers with \( \sigma(n) \leq n \) and \( \sigma(n) \to \infty \) as \( n \to \infty \). When \( \sigma(n) = n \), equation (3.1) takes the form

\[ \Delta^2 y_{n-1} + q_n y^\gamma_n = e_n, \gamma > 0 \]  

(3.2)

which is the forced discrete Emden-Fowler equation. By a solution of equation (3.1), we mean a real sequence \( y = \{y_n\} \) defined for \( n \geq \min_{i \in \mathbb{Z}} \sigma(i) \), satisfying equation (3.1) for \( n \in \mathbb{N} \). Further for \( \gamma > 1 \), the equation (3.1) is called superlinear and for \( 0 < \gamma < 1 \), the equation (3.1) is called sublinear.

We are interested in finding necessary and sufficient conditions for the oscillation of all solutions of equation (3.1). Previously known results [1,20,23,28,35,43,55] of the type obtained here are either for less general equations or assumed conditions different from those we impose. The technique of the proof mainly depends on the assumption that there exists an oscillatory sequence \( \{h_n\} \) such that \( \Delta^2 h_{n-1} = e_n \). Other results on forced
oscillation for nonlinear second order equations can be found in [68,77] and for linear equations in [51].

In the next section, for nonnegative \( \{q_n\} \) and bounded \( \{h_n\} \) \((\Delta^2 h_{n-1} = e_n)\), we give necessary and sufficient conditions for the equation (3.1) to be oscillatory in the superlinear and the sublinear cases.

All our results here could be obtained equally well for the difference equation

\[
\Delta^2 y_{n-1} + q_n \left| y_{(n)} \right|^{\gamma} \text{sgn} y_{(n)} = e_n, \quad \gamma > 0
\]

with no essential change in the proofs given. For simplicity of notation, we instead restrict \( \gamma \) to be a quotient of odd positive integers and discuss equation (3.1).

### 3.2. OSCILLATION RESULTS

We first establish necessary and sufficient conditions for the oscillation of all solutions of equation (3.1) for the super linear case.

Let \( \gamma > 1 \) in equation (3.1) and assume the following conditions:

i) \( q_n \geq 0 \) for all \( n \in \mathbb{N} \), and for every \( N \geq 1 \), \( q_n > 0 \) for some \( n > N \),

ii) there exists a bounded sequence \( \{h_n\} \) such that \( \Delta^2 h_{n-1} = e_n \) and let \( |h_n| \leq M \) for all \( n \).

We begin with the following theorem, which gives necessary condition for the oscillation of equation (3.1).
THEOREM 3.1. Assume conditions (i) and (ii) are satisfied. If
\[
\sum_{n=1}^{\infty} nq_n < \infty, \tag{3.3}
\]
then equation (3.1) has a nonoscillatory solution.

PROOF. Choose \(N \in \mathbb{N}\) sufficiently large so that
\[
\sum_{n=N}^{\infty} nq_n < \frac{1}{2} \min \left\{ \frac{1}{(2M+1)^\gamma}, \frac{1}{\gamma (2M+1)^{\gamma -1}} \right\}. \tag{3.4}
\]
Consider the complete metric space \(S\) consisting of all real sequences \(y = \{y_n\}\), \(n \in \mathbb{N}\) and satisfying the inequalities
\[
\frac{1}{2} \leq y_n \leq 2M + 1 \tag{3.5}
\]
endowed with the metric
\[
\rho(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|.
\]
The operator \(T\) defined by
\[
(Ty)_n = (M+1) + h_n - \sum_{s = n+1}^{\infty} (s - n)q_s y_{\sigma(s)}, \quad n \geq N,
\]
maps \(S\) into itself. In fact, if \(y \in S\), then \((Ty)_n \leq 2M+1\) since \(h_n \leq M\).

Moreover, from (3.4) and (3.5) we see that
\[
(Ty)_n \geq 1 - \sum_{s = N}^{\infty} sq_s y_{\sigma(s)}, \geq 1 - (2M+1)^\gamma \sum_{s = N}^{\infty} sq_s \geq \frac{1}{2}.
\]
Now, we shall show that \(T\) has a fixed point. For this,
\[
|(Ty)_n - (Tz)_n| \leq \sum_{s = n+1}^{\infty} (s - n)q_s |y_{\sigma(s)} - z_{\sigma(s)}| \leq \sum_{s = N}^{\infty} sq_s |y_{\sigma(s)} - z_{\sigma(s)}|.
\]
Using the Mean value theorem applied to the function \( f(x) = x^\gamma \), we see that

\[
\rho(Ty, Tz) \leq \gamma \rho(y, z) \sum_{s=N}^{\infty} \sqrt{q_s x_{\sigma(s)}^\gamma}
\]

where \( x_{\sigma(s)} \) lies between \( y_{\sigma(s)} \) and \( z_{\sigma(s)} \), \( s \geq N \), that is, satisfies the inequalities (3.5). So, we have

\[
\rho(Ty, Tz) \leq \gamma \rho(y, z) (2M + 1)^{\gamma-1} \sum_{s=N}^{\infty} \sqrt{q_s}.
\]

From (3.4), we see that \( \rho(Ty, Tz) \leq \frac{1}{2} \rho(y, z) \). Thus \( T \) is a contraction on \( S \). So by the known Banach contraction mapping theorem, \( T \) has a unique fixed point \( y \in S \), that is,

\[
y_n = M + 1 + h_n - \sum_{s=n+1}^{\infty} (s-n)q_s y_{\sigma(s)}^\gamma, \quad n \geq N.
\]

Taking difference twice, we see that \( \{y_n\} \) is a nonoscillatory solution of equation (3.1). The proof is now complete.

Next we prove that the condition

\[
\sum_{n=1}^{\infty} nq_n = \infty \quad (3.6)
\]

is sufficient for all solutions of equation (3.1) to be oscillatory assuming that \( \{h_n\} \) is oscillatory and satisfies the condition:

(iii) \( \{h_n\} \) is oscillatory and there exist two sequences \( \{n_i\} \), \( \{n_j\} \) tending to infinity such that for all \( j \)

\[
h_{n_j} = \inf\{h_n : n \geq n_j\}
\]

\[
h_{n_j^-} = \sup\{h_n : n \geq n_j^-\}.
\]
THEOREM 3.2. Assume conditions (i), (ii) and (iii) are satisfied. If condition (3.6) holds, then all solutions of equation (3.1) are oscillatory.

PROOF. Suppose \( \{y_n\} \) is a nonoscillatory solution of equation (3.1) and assume without loss of generality that \( y_n > 0, y_{\sigma(n)} > 0 \) for all \( n \geq N \), for some \( N \in \mathbb{N} \). Put \( y_n = z_n + h_n \). Then \( z_n \) satisfies the equation

\[
\Delta^2 z_{n+1} + q_n y_{\sigma(n)} = 0. \tag{3.7}
\]

From this we see that \( \Delta^2 z_{n+1} \leq 0 \). Hence \( z_n \) is of one sign and, definitely, it is positive; otherwise \( \{y_n\} \) will not be oscillatory. Further, if \( \Delta z_n \leq 0 \) for \( n \geq N \), then there exists an integer \( N_1 > N \) such that \( \Delta z_n \leq \Delta z_{N_1} < 0 \). Summing the last inequality from \( N_1 \) to \( n-1 \) and then taking \( n \to \infty \), we see that \( z_n \to -\infty \), a contradiction. Thus \( \Delta z_n > 0 \) and we have

\[
z_n > 0, \Delta z_n > 0, \Delta^2 z_{n+1} \leq 0. \tag{3.8}
\]

From the increase of \( \{z_n\} \) and condition (iii) on \( \{h_n\} \), we easily see that there exists an integer \( N_1 > N \) such that \( z_n + h_n \geq \beta_0 > 0 \) for all \( n > N_1 \), that is,

\[
y_n \geq \beta_0 > 0 \text{ for } n > N_1. \tag{3.9}
\]

This implies that there exists a positive number \( \beta \) such that

\[
y_n \geq \beta z_n. \tag{3.10}
\]

If this is not true, then there exists a sequence \( \{n_j\} \) tending to infinity such that

\[
y_{n_j} = z_{n_j} + h_{n_j} \leq \frac{1}{j} z_{n_j}. \tag{3.11}
\]
So \(1 - \frac{1}{j}\)z_{n_j} + h_{n_j} \leq 0. If \(z_{n_j} \to \infty\), then \(h_{n_j}\) will tend to \(-\infty\) which contradicts the fact \(\{h_n\}\) is bounded. If, on the other hand, \(z_{n_j} \to \text{constant}\), then \(y_{n_j} \to 0\) which contradicts (3.9). Hence (3.10) is true. Now, put

\[
w_n = -\frac{n\Delta z_{n-1}}{Z_{\alpha(n-1)}^\gamma}, n > N_1.
\]

From equation (3.7), we obtain

\[
\Delta w_n = \frac{nq_n y_{\alpha(n)}^\gamma}{Z_{\alpha(n)}^\gamma} - \frac{\Delta Z_{\alpha(n)}^\gamma}{Z_{\alpha(n)}^\gamma} + \frac{n\Delta Z_{n-1} Z_{\alpha(n)}^\gamma}{Z_{\alpha(n-1)}^\gamma Z_{\alpha(n)}^\gamma}.
\]

From (3.8) and (3.10) and the Mean value theorem, we have

\[
\Delta w_n \geq \beta' nq_n - \frac{\Delta Z_{\alpha(n)}^\gamma}{Z_{\alpha(n)}^\gamma} + \frac{cnw_{n+1}^2}{(n+1)^2},
\]

where \(c = \gamma Z_{\alpha(N_1-1)}^\gamma\). Summing the last inequality from \(N_1\) to \(n-1\), we get

\[
w_n \geq w_{N_1} + \beta' \sum_{s=N_1}^{n-1} sq_s + \int_{s=1}^{z_{\alpha(n)}} \frac{ds}{s^\gamma} + c \sum_{s=N_1}^{n-1} \frac{sw_{s+1}^2}{(s+1)^2},
\]

\[
= w_{N_1} + \beta' \sum_{s=N_1}^{n-1} sq_s + \frac{1}{\gamma-1} \left[ Z_{\alpha(n)}^{\gamma+1} - Z_{\alpha(N_1)}^{\gamma+1} \right] + c \sum_{s=N_1}^{n-1} \frac{sw_{s+1}^2}{(s+1)^2}.
\]

From (3.6), we see that there exists an integer \(N_2 > N_1\) such that

\[
w_n \geq c \sum_{s=N_1}^{n-1} \frac{s}{(s+1)^2} w_{s+1}^2, n > N_2.
\]  \hspace{1cm} (3.12)

Letting \(R_n = c \sum_{s=N_1}^{n-1} \frac{s}{(s+1)^2} w_{s+1}^2\), we see that \(\Delta R_n = \left[ \frac{cn}{(n+1)^2} \right] w_{n+1}^2\).
From (3.12), we get $\Delta R_n = \left[ \frac{cn}{(n+1)^2} \right] R_{n+1}^2$. Dividing by $R_{n+1}^2$ and summing from $N_2$ to $n-1$, we get

$$\int_{R_{N_2}}^{R_n} \frac{ds}{s^2} \geq \sum_{s=N_2}^{n-1} \frac{\Delta R_s}{R_{s+1}^2} > c \sum_{s=N_2}^{n-1} \frac{s}{(s+1)^2}.$$ 

Thus

$$c \sum_{s=N_2}^{n-1} \frac{s}{(s+1)^2} \leq \frac{1}{R_{N_2}}$$

which gives a contradiction as $n \to \infty$. This completes the proof of the theorem.

Next we establish necessary and sufficient conditions for the oscillation of all solutions of equation (3.1) for the sublinear case. That is, assume $0 < \gamma < 1$ in equation (3.1). The following theorem gives necessary condition for the oscillation of equation (3.1).

**THEOREM 3.3.** Assume conditions (i) and (ii) are satisfied. If

$$\sum_{n=1}^{\infty} n^\gamma q_n < \infty$$

then equation (3.1) has a nonoscillatory solution.

**PROOF.** Choose $N \in \mathbb{N}$ large enough so that

$$\sum_{n=N}^{\infty} n^\gamma q_n < \frac{1}{8}, \quad N > 8M.$$  \hspace{1cm} (3.13)

Let $B_N$ be the Banach space of all real sequences which satisfies $\sup_{n \in \mathbb{N}} \frac{|y_n|}{n} < \infty$. 

with the norm $\|y\| = \sup_{n \in \mathbb{N}} \frac{|y_n|}{n}$. We define a partial ordering on $B_N$ as follows:

for given $x, y \in B_N$, $x \leq y$ means $x_n \leq y_n$ for $n \geq N$. Let $S = \{y \in B_N : \frac{1}{2} \leq \frac{y_n}{n} \leq 1, n \geq N\}$. Define the operator $T$ acting in $S$ by

$$(Ty)_n = \frac{n}{2} + M + h_n + \sum_{s=0}^{n} s^{+} q_{s} y_{\sigma(s)}^\gamma + n \sum_{s=n+1}^{\infty} q_{s} y_{\sigma(s)}^\gamma, \quad n \geq N.$$

Now we will show that $T$ maps $S$ into itself. In fact, for $y \in S$, $(Ty)_n \geq \frac{n}{2}$ and from (3.13), we have

$$(Ty)_n \leq \frac{n}{2} + 2M + \frac{n}{2} \sum_{s=0}^{n} s^{+} q_{s} + \frac{n}{2} \sum_{s=0}^{\infty} s^{+} q_{s} \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \frac{n}{8} = n.$$

From the hypotheses, one can easily see that $T$ is an increasing mapping. Hence by the Knaster-Tarski fixed point theorem [49], there exists $y \in S$ such that $Ty = y$, that is,

$$y_n = \frac{n}{2} + M + h_n + \sum_{s=0}^{n} s^{+} q_{s} y_{\sigma(s)}^\gamma + n \sum_{s=n+1}^{\infty} q_{s} y_{\sigma(s)}^\gamma.$$

Taking the difference twice, we see that $\{y_n\}$ is a nonoscillatory solution of equation (3.1). This completes the proof.

Finally, we give sufficient conditions for the oscillation of all solutions of equation (3.1).

**THEOREM 3.4.** Assume conditions (i), (ii) and (iii) are satisfied. If

$$\sum_{n=1}^{\infty} n^{+} q_{n} = \infty,$$  \hspace{1cm} \text{(3.14)}

then all solutions of equation (3.1) are oscillatory.
PROOF. Suppose \( \{y_n\} \) is a nonoscillatory solution of equation (3.1) and assume without loss of generality that \( y_n > 0, y_{\sigma(n)} > 0 \) for all \( n \geq N \), for some \( N \in \mathbb{N} \). Put \( y_n = z_n + h_n \). Then \( z_n \) satisfies equation (3.7). Further from the proof of Theorem 3.2, we obtain

\[ \Delta^2 z_n \leq 0, \Delta z_n > 0, z_n > 0, y_{\sigma(n)} > \beta z_{\sigma(n)} \text{ for } n \geq N \]  
(3.15)

and from the Lemma 4.1 of Hooker and Patula [30], there exists some constant \( b > 0 \) such that

\[ z_n < bn. \]  
(3.16)

Now define \( w_n = -\frac{n^\gamma \Delta z_{n-1}}{Z_{\sigma(n-1)}} \) and we obtain

\[ \Delta w_n = \frac{n^\gamma q_n y_{\sigma(n)}^{\gamma}}{Z_{\sigma(n)}} \Delta n^\gamma \Delta z_n + \frac{n^\gamma \Delta z_{n-1} \Delta z_{\sigma(n-1)}^{\gamma}}{Z_{\sigma(n-1)} Z_{\sigma(n)}}. \]

From (3.15) and (3.16) and the Mean value theorem, we have

\[ \Delta w_n \geq \beta^\gamma n^\gamma q_n - \gamma(n+1)^{\gamma-1} \frac{\Delta z_{\sigma(n)}}{Z_{\sigma(n)}} + \frac{\gamma n^{2\gamma-1}}{(n+1)^{2\gamma}} b^{\gamma-1} w_{n+1}. \]

Summing from \( N \) to \( n-1 \), using the assumption (3.14) of the theorem and that \( 0 < \gamma < 1 \), we obtain for sufficiently large \( n \)

\[ w_n \geq \gamma b^{\gamma-1} \sum_{s=N}^{n-1} \frac{s^{2\gamma-1}}{(s+1)^{2\gamma}} w_{s+1}^2. \]

Rest of the proof is similar to that of Theorem 3.2 and hence the details are omitted.

We conclude this chapter with the following remark.
REMARK 3.1. When $e_n = 0$, the theorems of Hooker and Patula [30] for the oscillation of unforced equation (3.2) follow as consequences of Theorems 3.1-3.4. Further, the proof given here for the Theorems 3.2 and 3.4 are different from that of Hooker and Patula [30].