CHAPTER 6

SOME OSCILLATION RESULTS FOR FORCED NONLINEAR NEUTRAL EQUATIONS
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6.1. INTRODUCTION

In the previous chapters, we have studied the oscillatory and asymptotic behavior of solutions of different type of second order difference equation with or without delay. Similar results are available in plenty for homogenous neutral type difference equations [2,3,14,21,24,41,44,46,52,89,90,96]. However, the work on forced equations is scanty. In this chapter, we investigate the oscillatory and nonoscillatory behavior of solutions of neutral type forced difference equation

$$\Delta(a_n \Delta(y_{n} + p_n y_{n-k})) + q_n f(y_{n+1}) = e_n, \quad n \in \mathbb{N} (n_0)$$

(6.1)

where $\{a_n\}, \{p_n\}, \{q_n\}$ and $\{e_n\}$ are real sequences such that $a_n > 0$ and $q_n \geq 0$ for all $n \in \mathbb{N}(n_0)$, $f: \mathbb{R} \to \mathbb{R}$ is continuous with $u f(u) > 0$ for $u \neq 0$, $k$ and $\ell \in \mathbb{N}_0$ and there exists a real sequence $\{E_n\}$ such that $\Delta(a_n \Delta E_n) = e_n$.

Let $M = \max \{k, \ell\}$ and $n_0 \in \mathbb{N}_0$. By a solution of equation (6.1), we mean a real sequence $\{y_n\}$ defined for $n \geq N_0 - M$ and satisfies (6.1) for $n \geq n_0$.

Now for each result pertinent to the difference equation (6.1) that we shall prove, we require some of the following conditions:

$(c_1)$ $0 \leq p_n \leq A < 1$, where $A$ is a constant;

$(c_2)$ $p_n p_{n-k} \geq 0$ and $-1 < -B \leq p_n \leq A \leq 1$; where $A$ and $B$ are positive constants;

$(c_3)$ $f(u)$ is bounded away from zero if $u$ is bounded away from zero, that is,
\[ |u| > \delta \text{ implies that } |f(u)| > \eta, \text{ where } \eta > 0 \text{ and } \delta > 0; \]

\[ (c_4) \sum_{n=n_0}^{\infty} (n+1)q_n = \infty; \]

\[ (c_5) \sum_{n=n_0}^{\infty} (n+1)^\alpha q_n = \infty \text{ where } \alpha < 1; \]

\[ (c_6) \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty; \]

\[ (c_7) \sum_{n=n_0}^{\infty} \frac{1}{a_{n-\ell}} \left( \sum_{s=n}^{\infty} q_s \right) = \infty; \]

\[ (c_8) f \text{ is nondecreasing and superlinear, that is,} \]

\[ \int_{c}^{\infty} \frac{du}{f(u)} < \infty \text{ and } \int_{-\infty}^{c} \frac{du}{f(u)} < \infty \text{ for every } c > 0; \]

\[ (c_9) f(u) \text{ is sublinear, that is,} \]

\[ \int_{0}^{c} \frac{du}{f(u)} < \infty, \int_{0}^{c} \frac{du}{f(u)} < \infty \text{ for every } c > 0; \]

\[ (c_{10}) f(uv) \geq f(u) f(v) \text{ for all } u > 0 \text{ and large } v \text{ and} \]

\[ \sum_{n=n_0}^{\infty} q_n f(R_{n+1-\ell}) = \infty, \text{ where } R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}; \]

\[ (c_{11}) \lim_{n \to \infty} E_n = 0; \]

\[ (c_{12}) \{E_n\} \text{ is } k\text{-periodic.} \]

We may note that \( p_n p_{n+k} \geq 0 \) if \( \{p_n\} \) is \( k\)-periodic. In Section 6.2, we establish sufficient conditions for the oscillation of all / bounded solutions of

\[ (6.1) \text{ when } \lim_{n \to \infty} E_n = 0 \text{ and in Section 6.3, we obtain similar results when } \{E_n\} \text{ is periodic.} \]
6.2. OSCILLATION OF EQUATION (6.1) WHEN \( \lim_{n \to \infty} E_n = 0 \).

We begin with the following lemma which is needed in the sequel.

**Lemma 6.1.** Let \((c_2)\) holds and \(\{E_n\}\) and \(\{y_n\}\) are real sequences such that \(\{E_n\}\) is bounded and \(\{y_n\}\) is eventually positive. Then \(z_n = y_n + p_n y_{n-k} - E_n\) is bounded, if and only if, \(\{y_n\}\) is bounded. Further, if \(\{z_n\}\) is of one sign for large \(n\), then \(\{y_n\}\) is unbounded implies that \(\{z_n\}\) is unbounded and \(z_n > 0\) for large \(n\).

**Proof.** Let \(y_n > 0\) and \(y_{n+k} > 0\) for \(n \geq n_1 > n_0 \in \mathbb{N}\). Clearly, \(\{y_n\}\) bounded implies that \(\{z_n\}\) is bounded. Next suppose that \(\{z_n\}\) is bounded.

Assume \(\{y_n\}\) is unbounded. So there exists a sequence \(\{n_j\}\) such that

\[
\lim_{j \to \infty} n_j = \infty, \quad \lim_{j \to \infty} y_{n_j} \to \infty \quad \text{and} \quad y_{n_j} = \max \{y_n : n_1 \leq n \leq n_j\}.
\]

Since \(n-k \leq n\),

\[
y_{n-j-k} \leq \max \{y_n : n_1 \leq n \leq n_{j-k}\} \leq \max \{y_n : n_1 \leq n \leq n_j\} = y_{n_j}.
\]

Thus \(z_{n_j} + E_{n_j} \geq y_{n_j} - By_{n_j-k} \geq (1-B)y_{n_j}\) leads to a contradiction as \(j \to \infty\).

Hence \(\{y_n\}\) is bounded.

Next suppose that \(z_n > 0\) or \(< 0\) for \(n \geq n_1^* > n_1\). Clearly \(\{y_n\}\) is unbounded implies that \(\{z_n\}\) is unbounded. If \(z_n < 0\) for \(n \geq n_1^*\), then arguing as above, we obtain \(E_{n_j} > z_{n_j} + E_{n_j} \geq (1-B)y_{n_j}\) which leads to a contradiction as \(j \to \infty\). Thus \(z_n > 0\) for \(n \geq n_1^*\). This completes the proof of the lemma.

**Theorem 6.2.** In addition to conditions \((c_2), (c_3), (c_4), (c_6)\) and \((c_11)\) assume that the sequence \(\{a_n\}\) is bounded. Then all bounded solutions of equation (6.1) are either oscillatory or tend to zero as \(n \to \infty\).
PROOF. Let \( \{y_n\} \) be a bounded nonoscillatory solution of equation (6.1) for \( n \in \mathbb{N}(n_0) \). So there exists a \( n_1 \in \mathbb{N}(n_0) \) such that \( y_n > 0 \) or \( < 0 \) for \( n \geq n_1 \). Let \( y_n > 0 \) for \( n \geq n_1 \). Hence there exists a \( n_2 > n_1 \) such that \( y_{n-k} > 0 \) and \( y_{n+c} > 0 \) for \( n \geq n_2 \). Letting \( z_n = y_n + p_n y_{n-k} - E_n \) for \( n \geq n_2 \), we obtain from (6.1)

\[
\Delta (a_n \Delta z_n) = - q_n f(y_{n+1}) \leq 0. \tag{6.2}
\]

Then \( \{z_n\} \) is bounded and \( z_n > 0 \) or \( < 0 \) for \( n \geq n_3 \geq n_2 \). Let \( z_3 > 0 \) for \( n \geq n_3 \). From (6.2) and (c_6), it follows that \( a_n \Delta z_n > 0 \) for \( n \geq n_4 > n_3 \) and hence \( \Delta z_n > 0 \) for \( n \geq n_4 \). Let \( \lim_{n \to \infty} z_n = \lambda, \ 0 < \lambda < \infty \). Clearly there exists an integer \( n_5 > n_4 \) such that \( n-k > n_2 \) for \( n \geq n_5 \) and hence, for \( n \geq n_5 \),

\[
(1-A) z_n < y_n + |E_n| + |E_{n-k}|. \tag{6.3}
\]

For \( 0 < \varepsilon < (1-A) \lambda \), there exists \( n_6 > n_5 \) such that \( (1-A) z_n < y_n + \varepsilon \) for \( n \geq n_6 \).

Thus, \( \liminf_{n \to \infty} y_n > 0 \). On the other hand, multiplying (6.2) by \((n + 1)\) and summing the resulting equality, we obtain

\[
\sum_{s=n_5}^{n-1} (s+1)q_s f(y_{s+1}) \leq n_5 a_{n_5} \Delta z_{n_5} + Kz_{n_5} < \infty, \text{ because } a_n \leq K \text{ for all } n \geq n_5. \]

This, in turn, implies, in view of (c_3) and \( \liminf_{n \to \infty} y_n > 0 \), a contradiction to (c_4).

Thus \( z_n < 0 \) for \( n \geq n_3 \). Consequently, from the definition of \( z_n \), we get

\[
0 \leq (1-B) \limsup_{n \to \infty} y_n \leq 0. \quad \text{Hence } \lim_{n \to \infty} y_n = 0. \quad \text{The case } y_n < 0 \text{ for } n \geq n_1 \text{ may be proved similarly. This completes the proof of the theorem.}
\]

**THEOREM 6.3.** In addition to conditions (c_1), (c_2), (c_4), (c_6), (c_8) and (c_11), assume that \( \{a_n\} \) is bounded. Then every solution of equation (6.1) is either oscillatory or tends to zero as \( n \to \infty \).
PROOF. In view of Theorem 6.2, it is enough to prove that no nonoscillatory solution of equation (6.1) is unbounded. Let \( \{y_n\} \) be an unbounded nonoscillatory solution of equation (6.1). Proceeding as in the proof of Theorem 6.2 and setting \( x_n = (1 - A) z_n - \varepsilon \), we obtain \( 0 < x_n < y_n, \Delta x_n > 0 \), \( \Delta(a_n \Delta x_n) \leq 0 \) and
\[
\Delta(a_n \Delta x_n) + (1 - A) q_n f(x_{n+1}, \varepsilon) \leq 0.
\] (6.4)
Rest of the proof is similar to that of Theorem 6.2 and hence the details are omitted.

THEOREM 6.4: Let \((c_2), (c_6), (c_7)\) and \((c_{11})\) hold. Suppose that \( f \) is monotonically increasing and
\[
\lim_{u \to \infty} \inf \frac{|f(u)|}{|u|^{1+\beta}} > 0, \beta > 0.
\]
Then every solution of equation (6.1) is either oscillatory or tends to zero as \( n \to \infty \).

PROOF. Let \( \{y_n\} \) be a nonoscillatory solution of (6.1) such that \( y_n > 0 \) for \( n \geq n_1 \in \mathbb{N}(n_0) \). Proceeding as in Theorem 6.2, we obtain \( z_n > 0 \) and
\[
\lim_{n \to \infty} z_n = \infty \text{ if } \{y_n\} \text{ is unbounded.}
\]
Letting \( x_n \) as in Theorem 6.3, we obtain (6.4) and \( x_n \to \infty \) as \( n \to \infty \). Summing (6.4) twice and using \( |f(u)| > K |u|^{1+\beta} \) for \( |u| \) large and \( x_n \to \infty \) as \( n \to \infty \), we have
\[
(1 - A) K \sum_{j=n_5}^{n-1} \frac{1}{a_{j-\ell}} \left( \sum_{i=j}^{\infty} q_i \right) < \int_{x_{n_5-\ell}}^{x_2-\ell} \frac{dt}{t^{1+\beta}} < \frac{1}{\beta x_{n_5-\ell}^\beta},
\]
a contradiction to \((c_7)\). Hence \( \{y_n\} \) is bounded. Rest of the proof is similar to that of Theorem 6.2 and hence the details are omitted.
In the following, we obtain a result which holds for sublinear \( f \).

**THEOREM 6.5.** Let \((c_2), (c_5) (c_{10})\) and \((c_{11})\) hold. If \( \ell \geq 1 \) and \( f \) is monotonically increasing, then every solution of equation (6.1) is either oscillatory or tends to zero as \( n \to \infty \).

**PROOF.** Proceeding as in Theorem 6.4 and defining \( x_n \) as in Theorem 6.3, we have

\[
x_n > \sum_{s=n_4}^{n-1} \Delta x_s > a_n \Delta x_n R_n, \quad \text{where} \quad R_n = \sum_{s=n_4}^{n-1} \frac{1}{a_s} \quad \text{for} \quad n \geq n_5 > n_4.
\]

Now summing from \( n_6 \) to \( n-1 \), we obtain

\[
(1 - A) \sum_{s=n_6}^{n-1} q_s f(R_{s+1-\ell}) \leq - \sum_{s=n_6}^{n-1} \frac{\Delta(a_s \Delta x_s)}{f(a_s \Delta x_s)} \leq \int_{a_s \Delta x_s}^{a_{s+1} \Delta x_{s+1}} \frac{\Delta x_s}{f(t)} \leq \int_0^\infty \frac{dt}{f(t)}.
\]

This in view of \((c_0)\) contradicts \((c_{10})\). Thus \( \{y_n\} \) is bounded. Rest of the proof is similar to that of Theorem 6.2 and hence the details are omitted.

Finally, in this section we obtain a result subject to the condition \( a_n = 1 \) for all \( n \in \mathbb{N} (n_0) \).

**THEOREM 6.6.** Let conditions \((c_2), (c_5)\) and \((c_{11})\) hold. If \( f \) is monotonically increasing and \( \lim \inf_{|u| \to \infty} \frac{f(u)}{u} > 0 \), then every solution of equation (6.1) is either oscillatory or tends to zero as \( n \to \infty \).

**PROOF.** Proceeding as in Theorem 6.4 and setting \( v_n = \frac{\Delta x_n}{x_{n-\ell}} \), we have

\[
\Delta v_n + v_n^2 + (1 - A) q_n \frac{f(x_{n+1-\ell})}{x_{n+1-\ell}} \leq 0.
\]
Multiplying the above inequality by \((n+1)^\alpha\) and then summing from \(n_5\) to \(n-1\), we obtain

\[
0 > -n_5^\alpha v_{n_5} - \alpha \sum_{s=n_5}^{n-1} \xi^{\alpha-1} v_s + \sum_{s=n_5}^{n-1} (s+1)^\alpha v_s^2 + K(1-A) \sum_{s=n_5}^{n-1} (s+1)^\alpha q_s, \quad s < \xi < s + 1,
\]

because, for large \(|u|, f(u) > Ku\) and \(x_n \to \infty\) as \(n \to \infty\). By the method of completing the squares, we obtain

\[
K(1-A) \sum_{s=n_5}^{n-1} (s+1)^\alpha q_s < n_5^\alpha v_{n_5} + \frac{\alpha^2 (n_5 + 1)^{\alpha-1}}{4 (1-\alpha)}
\]

a contradiction to \((c_3)\). Hence \(\{y_n\}\) is bounded. The rest of the proof is similar to that of Theorem 6.2 and hence the details are omitted.

**REMARK 6.1** We may note that in Theorems 6.2-6.6, we obtain

\[
\lim_{n \to \infty} y_n = 0\quad \text{for a nonoscillatory solution } \{y_n\}\quad \text{of equation (6.1) in the case } z_n < 0
\]

for large \(n\). If \(0 \leq p_n \leq A < 1\) and \(\{E_n\}\) is oscillatory or \(= 0, z_n < 0\) implies that \(0 < y_n < y_{n+1} y_{n+2} < E_n\), a contradiction. Hence in Theorems 6.3-6.6, (Theorem 6.2) conclusion would read as every (every bounded) solution is oscillatory if \((c_2)\) is replaced by \((c_1)\) and \(\{E_n\}\) is either oscillatory or \(= 0\).

In the following, we give some examples to illustrate the above results.

**EXAMPLE 6.1.** Consider the difference equation

\[
\Delta^2 \left( y_n + \frac{2(-1)^n + 1}{4} y_{n-2} \right) + \frac{12(n-1)^2}{(n-2)n(n+1)(n+2)} y_{n-1}^3 = \frac{2}{n(n+1)(n+2)} + \frac{2(-1)^n (2n^2 - 4n + 1) + 1}{2n(n-1)(n-2)} + \frac{12}{(n-2)(n-1)n(n+1)(n+2)}, \quad n \geq 3.
\]
Here \( a_n = 1 \), \( E_n = \frac{1}{n} + \frac{2(-1)^n + 1}{4(n-2)} + \frac{1}{n(n-1)(n-2)} \) and \(-1 < -\frac{1}{4} \leq p_n \leq \frac{3}{4} < 1\).

From Theorem 6.2, it follows that all bounded solutions of equation (6.5) are either oscillatory or tend to zero as \( n \to \infty \). In particular, \( \{y_n\} = \left\{ \frac{1}{n} \right\} \) is one such solution of equation (6.5).

**EXAMPLE 6.2.** Consider the difference equation

\[
\Delta \left[ \frac{2n+2}{2n+1} \Delta \left( y_n + \frac{(-1)^n}{n(n-2)} y_{n-2} \right) \right] + \frac{2(2n+3)}{(n-1)^2} y_{n-1}^3 = \frac{2}{n(2n+1)} - \frac{2}{(n+1)(2n+3)}, \quad n \geq 4.
\]

Here \( a_n = \frac{2n+2}{2n+1}, \ E_n = \frac{1}{n} \) and \(-1 < -\frac{1}{15} \leq p_n \leq \frac{1}{8} < 1\).

From Theorem 6.3, it follows that all solutions of equation (6.6) are either oscillatory or tend to zero as \( n \to \infty \). In particular \( \{y_n\} = \{(-1)^n n\} \) is an oscillatory solution of equation (6.6).

### 6.3. OSCILLATION OF EQUATION (6.1) WHEN \( \{E_n\} \) IS PERIODIC

In this section, we obtain conditions for the oscillation of all solutions of equation (6.1) when \( \{E_n\} \) is \( k \)-periodic.

**THEOREM 6.7.** Let conditions \((c_1), (c_3), (c_4), (c_5)\) and \((c_{12})\) hold. If \( \{a_n\} \) is bounded, then every bounded solution of equation (6.1) is oscillatory.

**PROOF.** Let \( \{y_n\} \) be a bounded nonoscillatory solution of (6.1) for \( n \geq n_1 \in \mathbb{N} (n_0) \). Proceeding as in Theorem 6.2 we obtain \((1 - A)(z_n + E_n) \leq y_n\) for \( n \geq n_2 \). Since \( \{E_n\} \) is \( k \)-periodic, there exist real constants \( b_1 \) and \( b_2 \) and sequences \( \{n'_1\} \) and \( \{n''_1\} \) such that
\[ \lim_{j \to \infty} n_j' = \lim_{j \to \infty} n_j'' = \infty, E_{n_j} = b_1, E_{n_j} = b_2 \text{ and } b_1 \leq E_n \leq b_2. \]

For \( n > n_3 > \max \{ n_2, n_j' \} \) where \( n_j' > n_1 \), we have

\[ 0 < (1-A)(z_{n_j} + E_{n_j}) \leq (1-A)(z_n + b_1) \leq (1-A)(z_n + E_n) \leq y_n. \]

Setting \( x_n = (1-A)(z_n + b_1) \), we obtain \( 0 < x_n \leq y_n \), \( \{x_n\} \) bounded, \( \Delta x_n > 0 \) and \( \Delta(a_n \Delta x_n) \leq 0 \) for \( n \geq n_3 \). If \( \lim_{n \to \infty} x_n = \lambda, \) \( 0 < \lambda < \infty \), then, for \( 0 < \varepsilon < \lambda \), there exists an integer \( n_4 > n_3 \) such that \( 0 < \lambda - \varepsilon < x_{n+1-\varepsilon} \leq y_{n+1-\varepsilon} \), for \( n \geq n_4 \). Hence

\[ f(y_{n+1-\varepsilon}) > \lambda > 0 \text{ for } n \geq n_4. \]

From (6.2), we get

\[ 0 \geq \Delta(a_n \Delta x_n) + (1-A) q_n f(y_{n+1-\varepsilon}) \geq \Delta(a_n \Delta x_n) + (1-A) \lambda^* q_n \text{ for } n \geq n_4. \]

Rest of the proof is similar to that of Theorem 6.2 and hence the details are omitted.

**THEOREM 6.8.** Let conditions (c1), (c4), (c6), (c8) and (c12) hold. If \( \{a_n\} \) is bounded, then all solutions of equation (6.1) are oscillatory.

**PROOF.** The proof is similar to that of Theorems 6.7 and 6.3 and therefore the details are omitted.

**REMARK 6.2.** One can obtain theorems similar to Theorems 6.4-6.6 and hence the details are omitted.

We conclude this chapter with the following example.

**EXAMPLE 6.3.** Consider the difference equation

\[ \Delta^2 \left[ y_n + \frac{2}{3} y_{n-4} \right] + y_{n+1}^3 = -\frac{10}{3} \cos \frac{n\pi}{2}, n \geq 4. \]  

(6.7)

Here \( E_n = \frac{5}{3} \sin \frac{n\pi}{2} + \frac{1}{2} \sin \frac{3n\pi}{2} \) is 4-periodic. From Theorem 6.8, it follows that all solutions of equation (6.7) are oscillatory.
In particular $\{y_n\} = \left\{\sin \frac{n\pi}{2}\right\}$ is an oscillatory solution of equation (6.7).

**REMARK 6.3.** If $p_n=0$, $a_n=1$ and $f(u)=u^7$, then the equation (6.1) is similar to that of equation (3.1), but the conditions imposed on the equations (3.1) and (6.1) are different and therefore the direct comparison between the results are not possible.