CHAPTER VI
PERTURBED DIFFERENCE EQUATIONS
6.1 INTRODUCTION

For the second order nonlinear difference equations with perturbed term relatively few oscillation criteria are investigated, see for instance, Szmanda [47] and Thandapani [52]. In this chapter, we establish several sufficient conditions which ensure all solutions of more general nonlinear second order perturbed difference equations of the forms

\[ \Delta(a_n h(y_{n+1}) \Delta y_n) + Q(n, y_{n+1}) = P(n, y_{n+1}, \Delta y_n), \quad n \in \mathbb{Z} \quad (6.1) \]

and

\[ \Delta(a_n \psi(n, y_{n+1}) \Delta y_n) + Q(n, y_{n+1}) = P(n, y_{n+1}, \Delta y_n), \quad n \in \mathbb{Z} \quad (6.2) \]

where \( \{a_n\} \) is a positive real sequence, \( h: \mathbb{R} \to \mathbb{R} \) is continuous such that \( h(u) > 0 \) for all \( u \), \( Q: \mathbb{N} \times \mathbb{R} \to \mathbb{R} \) and \( P: \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R} \) are continuous.

By a solution of equation (6.1) or (6.2), we mean a nontrivial real sequence \( \{y_n\} \) satisfying (6.1) or (6.2) for all \( n \in \mathbb{Z} \).

Throughout this chapter, we assume that there exist real sequences \( \{q_n\} \) and \( \{p_n\} \) with \( p_n \geq 0 \) and continuous functions \( g, f_1, f_2: \mathbb{R} \to \mathbb{R} \) such that

\[ \frac{Q(n, u)}{f_1(u)} \geq q_n \quad \text{and} \quad \frac{P(n, u, v)}{f_2(u) g(v)} \leq p_n, \quad \text{for } u, v \neq 0, \quad (6.3) \]
where

\[ u f_i(u) > 0 \text{ for all } u \neq 0, \ i = 1, 2, \quad (6.4) \]

\[ \frac{f_2(u)}{f_1(u)} \leq k \text{ for } u \neq 0, \text{ for some constant } k > 0, \quad (6.5) \]

\[ f_1(u) - f_1(v) = g_1(u,v) (u-v), \text{ for } u \neq v, \ g_1 \text{ is a nonnegative function} \quad (6.6) \]

\[ o < g(v) \leq c, \text{ for some constant } c, \quad (6.7) \]

\[ o < c_1 \leq h(u) \text{ for some constant } c_1, \quad (6.8) \]

\[ \psi (n, u, v) > 0, \text{ for all } u, v \neq 0, \ n \in \mathbb{Z}, \quad (6.9) \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty. \quad (6.10) \]

In Section 6.2 we provide some oscillation criteria which ensure that all solutions of equation (6.1) are oscillatory. Similar results have been established for the equation (6.2) in Section 6.3.

The results obtained here generalize and improve some of the previous known results.

6.2 OSCILLATION CRITERIA

We begin with the following theorem.

**THEOREM 6.1.** In addition to conditions (6.3) - (6.8) and (6.10), if

\[ \sum_{n=1}^{\infty} (q_n - \mu p_n) = \infty \quad (6.11) \]

where \( \mu = ck \), then all solutions of equation (6.1) are oscillatory.
PROOF: Suppose \( \{y_n\} \) is a nonoscillatory solution of equation (6.1) say \( y_n \neq 0 \) for \( n \geq N_0 \in \mathbb{Z} \). Then

\[
\Delta \left( \frac{a_n h(y_{n+1}) \Delta y_n}{f_1(y_n)} \right) \leq \frac{-Q(n,y_{n+1})}{f_1(y_{n+1})} + \frac{P(n,y_{n+1},\Delta y_n)}{f_1(y_{n+1})} + \frac{a_n h(y_{n+1}) g_1(y_n, y_{n+1}) (\Delta y_n)^2}{f_1(y_n) f_1(y_{n+1})} - 2(q_n - \mu_p_n).
\]

Summing the above inequality from \( N_0 \) to \( n-1 \), we obtain

\[
\begin{align*}
\sum_{s=N_0}^{n-1} & \left( \frac{a_n h(y_{s+1}) \Delta y_s}{f_1(y_s)} \right) \\
& \leq \frac{a_{N_0} h(y_{N_0+1}) \Delta y_{N_0}}{f_1(y_{N_0})} - \sum_{s=N_0}^{n-1} (q_s - \mu p_s). \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
\end{align*}
\]

We assume that \( y_n > 0 \) for \( n \geq N_1 \geq N_0 \in \mathbb{Z} \); the proof for the case \( y_n < 0 \), \( n \geq N_1 \) is similar and will be omitted. In view of condition (6.11) it follows from (6.12) that there exists \( N_2 \geq N_1 \) such that \( \Delta y_n < 0 \) for \( n \geq N_2 \). It also follows from condition (6.11) that there exists an integer \( N_3 \geq N_2 \) such that

\[
\sum_{s=N_3}^{n-1} (q_s - \mu p_s) > 0 \quad \text{for} \quad n \geq N_3.
\]

Now summing equation (6.1) and using condition (6.3), we have

\[
\begin{align*}
a_n h(y_{n+1}) \Delta y_n & \leq a_{N_3} h(y_{N_3+1}) \Delta y_{N_3} - \sum_{s=N_3}^{n-1} f_1(y_{s+1}) (q_s - \mu p_s) \\
& = a_{N_3} h(y_{N_3+1}) \Delta y_{N_3} - \sum_{s=N_3}^{n-1} f_1(y_{n+1}) (q_s - \mu p_s)
\end{align*}
\]

\[
\sum_{s=N_3}^{n-1} (q_s - \mu p_s) > 0 \quad \text{for} \quad n \geq N_3.
\]
Since \( \{y_n\} \) is positive decreasing and \( h \) is positive and continuous, there exists a positive constant \( K \) and an integer \( N_4 \geq N_3 \) such that \( 0 < h(y_{n+1}) \leq K, \ n \geq N_4. \)

Hence,

\[
\Delta y_n \leq \frac{a_{N_4} h(y_{N_4+1}) \Delta y_{N_4}}{K a_n}.
\]

From (6.10) it follows that \( y_n \rightarrow -\infty \) as \( n \rightarrow \infty \) which is a contradiction. This completes the proof of the theorem.

**REMARK 6.1**: Theorem 6.1 generalizes Theorem 1 in [52]. The conditions imposed on the functions \( Q \) and \( P \) are more general than the corresponding conditions required, for example, in Theorems 5-7 in [47]. Also Theorem 6.1 includes a result of [53] as a special case.

**EXAMPLE 6.1**: Consider the difference equation

\[
\Delta (n(1+y_{n+1}^2) \Delta y_n) + (10n+4) y_{n+1}^3 = 4n \frac{y_{n+1}^3}{1+y_{n+1}^2}, \ n \geq 1.
\]  

(6.13)

Choose \( f_1(u) = u^3, f_2(u) = u^3/(1+u^2) \), then all hypotheses of Theorem 6.1 are satisfied. That this equation (6.13) is oscillatory does not appear to be deducible from other known oscillation criteria.
In the following theorem we study the oscillation criteria for equation (6.1) subject to the conditions:

\[ f_1 \quad \text{is a nondecreasing function on } \mathbb{R}, \quad (6.14) \]

and

\[ \int_{-\infty}^{\alpha} \frac{h(x)}{f_1(x)} \, dx < \infty \quad \text{and} \quad \int_{-\infty}^{\alpha} \frac{-\alpha h(x)}{f_1(x)} \, dx > -\infty, \quad \text{for all } \alpha > 0. \quad (6.15) \]

**THEOREM 6.2**: Suppose that conditions (6.3) - (6.8), (6.10), (6.14) and (6.15) hold, and in addition, there exists \( B > 0 \) such that

\[ |\sum (q_n - \mu p_n)| \leq B, \quad \text{for all } n, \quad (6.16) \]

\[ \liminf_{n \to \infty} \sum_{s=N_0}^{n-1} (q_s - \mu p_s) \geq 0 \quad \text{for large } N_0 \in \mathbb{Z}, \quad (6.17) \]

and

\[ \lim_{n \to \infty} \sum_{s=N_0}^{n} \frac{1}{a_s} \left[ \sum_{t=s}^{\infty} (q_t - \mu p_t) \right] = \infty \quad (6.18) \]

are satisfied. Then all solutions of equation (6.1) are oscillatory.

**PROOF**: Let \( \{y_n\} \) be a nonoscillatory solution of equation (6.1), say \( y_n > 0 \) for all \( n \geq N_0 \in \mathbb{Z} \). For any \( N_1 \geq N_0 \), summation of (6.1) yields

\[ \frac{a_n h(y_{n+1}) \Delta y_n}{f_1 (y_{n+1})} \leq \frac{a_{N_1} h(N_{1+1}) \Delta y_{N_1}}{f_1 (y_{N_{1+1}})} - \sum_{s=N_1}^{n-1} (q_s - \mu p_s). \quad (6.19) \]
Now, if $\Delta y_n \geq 0$ for all $n \geq N_1 \geq N_0 \in \mathbb{Z}$, we have from (6.19)
\[
\sum_{s=N_1}^{\infty} a_N h(y_{N_1+1}) \Delta y_{N_1} \leq \sum_{s=N_1}^{\infty} (q_s \mu p_s).
\]

Hence, for all $n \geq N_1$ we have
\[
\sum_{s=n}^{\infty} (q_s - \mu p_s) \leq \frac{a_n h(y_{n+1}) \Delta y_n}{f_1(y_{n+1})}.
\]

or
\[
\frac{1}{a_n} \sum_{s=n}^{\infty} (q_s - \mu p_s) \leq \frac{h(y_{n+1})}{f_1(y_{n+1})} \Delta y_n. \tag{6.20}
\]

Observe, for $y_n \leq x \leq y_{n+1}$, we have $h(x) \geq \frac{h(y_{n+1})}{f_1(y_{n+1})}$

and it follows that
\[
\int_{y_n}^{y_{n+1}} \frac{h(x)}{f_1(x)} dx \geq \frac{h(y_{n+1})}{f_1(y_{n+1})} \Delta y_n.
\]

Using the last inequality in (6.20) and summing the resulting inequality from $N_1$ to $n$,
we obtain
\[
\sum_{s=N_1}^{n} a_s \left[ \sum_{t=s}^{\infty} (q_t - \mu p_t) \right] \leq \int_{y_{N_1}}^{y_{n+1}} \frac{h(x)}{f_1(x)} dx. \tag{6.21}
\]

This contradicts (6.19) since the left sum diverges.
If \( \{ \Delta y_n \} \) changes sign, there exists a sequence \( \{ N_k \} \) such that \( \Delta y_{N_k} < 0 \). Choose \( k \) large enough so that (6.17) holds. We then have,

\[
\frac{a_n h(y_{n+1}) \Delta y_n}{f_1 (y_{n+1})} \leq \frac{a_{N_k} h(y_{N_k+1}) \Delta y_{N_k}}{f_1 (y_{N_k+1})} - \sum_{s=N_k}^{n-1} (q_s - \mu p_s)
\]

so

\[
\limsup_{n \to \infty} \frac{a_n h(y_{n+1}) \Delta y_n}{f_1 (y_{n+1})} \leq \frac{a_{N_k} h(y_{N_k+1}) \Delta y_{N_k}}{f_1 (y_{N_k+1})} + \limsup_{n \to \infty} \left[ - \sum_{s=N_k}^{n-1} (q_s - \mu p_s) \right] < 0
\]

which contradicts the fact that \( \{ \Delta y_n \} \) oscillates. Hence there exists an integer \( N_2 \geq N_1 \) such that \( \Delta y_n < 0 \) for all \( n \geq N_2 \). Now condition (6.17) implies that for any integer \( N_3 \geq N_0 \) there exists \( N_4 \geq N_3 \) such that

\[
\sum_{s=N_4}^{n-1} (q_s - \mu p_s) > 0
\]

for all \( n \geq N_4 \). Choosing \( N_4 \geq N_2 \) as indicated and then summing equation (6.1), we have


\[
a_n h(y_{n+1}) \Delta y_n \leq a_{N_4} h(y_{N_4+1}) \Delta y_{N_4} \sum_{s=N_4}^{n-1} f_1(y_{s+1}) (q_s - \mu_p) \\
= a_{N_4} h(y_{N_4+1}) \Delta y_{N_4} \cdot \sum_{s=N_4}^{n-1} f_1(y_{s+1}) (q_s - \mu_p) \\
+ \sum_{s=N_4}^{n-1} \Delta f_1(y_{s+1}) \left( \sum_{t=N_4}^s \mu_p \right) \\
\leq a_{N_4} h(y_{N_4+1}) \Delta y_{N_4}
\]

or

\[
h(y_{n+1}) \Delta y_n \leq a_{N_4} h(y_{N_4+1}) \Delta y_{N_4} \frac{1}{a_n}.
\tag{6.23}
\]

Since \(\{y_n\}\) is positive decreasing and \(h\) is positive and continuous, there exists a positive constant \(K\) and an integrate \(N_5 \geq N_4\) such that \(0 < h(y_{n+1}) \leq K, n \geq N_5\).

Hence,

\[
\Delta y_n \leq \frac{a_{N_5} h(y_{N_5+1}) \Delta y_{N_5}}{K a_n}.
\]

Summing the above inequality from \(N_5\) to \(n-1\), we obtain

\[
y_n - y_{N_5} \leq \frac{a_{N_5} h(y_{N_5+1}) \Delta y_{N_5}}{K} \sum_{s=N_5}^{n-1} \frac{1}{a_s}
\]

which in turn implies from (6.10) an immediate contradiction \(y_n \to -\infty\) as \(n \to \infty\).

A similar proof holds when \(\{y_n\}\) is eventually negative.

**Remark 6.2**: If \(h(u) \equiv 1\) and \(f_1(u) = f_2(u)\), then Theorem 6.2 reduces to Theorem 2 given in [52].
COROLLARY 6.3: If conditions (6.3) - (6.8), (6.10), (6.14) and (6.16) - (6.18) hold, then all bounded solutions of equation (6.1) are oscillatory.

PROOF: Condition (6.15) has been used only in the first part of the proof of Theorem 6.2, we have $y_n > 0$ and $\Delta y_n \geq 0$ for $n \geq N_1 \geq N_0 \in \mathbb{Z}$, so by (6.6), we have $f_1 (y_n) \geq f_1 (y_{N_1})$ for $n \geq N_1$. From (6.18) and (6.21), we then obtain a contradiction to the boundedness of $\{y_n\}$.

Next, we discuss the oscillatory behavior of equation (6.1) subject to the condition

$$\frac{g_1 (u,v)}{h(u)} \geq \lambda > 0 \quad \text{for} \quad u,v \neq 0 \quad (6.24)$$

THEOREM 6.4: Suppose that conditions (6.3) - (6.8), (6.10), (6.17) and (6.24) hold. Assume there exists a positive nondecreasing sequence $\{\beta_n\}$ such that

$$\limsup_{n \to \infty} \frac{1}{(n)^{(a)}} \sum_{s=N_0}^{n-1} (n-s)^{(a)} \beta_s = \infty, N_0 \in \mathbb{Z}, (6.25)$$

for some positive integer $a \geq 1$, where $(n)^{(a)}$ is the usual factorial notation. Then all solutions of equation (6.1) are oscillatory.

PROOF: Let $\{y_n\}$ be a nonoscillatory solution of equation (6.1), say $y_n > 0$ for all $n \geq N_1 \geq N_0 \in \mathbb{Z}$; the proof for the case $y_n < 0$, $n \geq N_1$ is similar and will be omitted. Since (6.17) holds, as before we see from the proof of Theorem 6.2 that $\{\Delta y_n\}$ cannot
change sign for arbitrarily large $n$. Let $\Delta y_n \geq 0$ for $n \geq N_2$ for some integer $N_2 \geq N_1$.

Define

$$z_n = \frac{v_n \beta_n}{f_1(y_n)}$$

where $v_n = a_n h(y_{n+1}) \Delta y_n$.

Then for $n \geq N_2$, we have, using inequalities $v_{n+1} \leq v_n$ and $f_1(y_n) \leq f_1(y_{n+1})$,

$$\Delta z_n \leq -\beta_n (q_n - \mu p_n) + \frac{\Delta \beta_n z_{n+1}}{\beta_{n+1}} - \frac{\lambda \beta_n}{a_n \beta_{n+1}^2} z_{n+1}^2.$$

Since $\sum_{s=N_2}^{n-1} (n-s)z_s = -z_{N_2} + \alpha \sum_{s=N_2}^{n-1} (n-s)(\alpha - 1)z_{s+1}$

we get

$$\sum_{s=N_2}^{n-1} (n-s)(\alpha - 1)z_s + \alpha \sum_{s=N_2}^{n-1} (n-s)(\alpha - 1)z_{s+1}$$

$$\leq \frac{(n-N_2)(\alpha) z_{N_2}}{(n)(\alpha)} - \frac{\alpha}{(n)(\alpha)} \sum_{s=N_2}^{n-1} (n-s)(\alpha - 1)z_{s+1}$$

$$+ \frac{1}{(n)(\alpha)} \sum_{s=N_2}^{n-1} \frac{\Delta \beta_s}{\beta_{s+1}} z_{s+1} (n-s)(\alpha) - \frac{1}{(n)(\alpha)}$$

$$x \sum_{s=N_2}^{n-1} (n-s)(\alpha) \lambda \beta_s a_s \beta_{s+1}^2 z_{s+1}^2$$

$$\leq \frac{(n-N_2)(\alpha) z_{N_2}}{(n)(\alpha)} - \frac{1}{(n)(\alpha)} \sum_{s=N_2}^{n-1} (n-s)(\alpha) \lambda \beta_s a_s \beta_{s+1}^2 z_{s+1}^2$$

$$x \left[ \frac{z_{s+1}^2 + \frac{a_s \beta_{s+1}}{\lambda}}{\frac{\alpha}{(n-s+\alpha-1)} - \frac{\Delta \beta_s}{\beta_s}} \right] z_{s+1}$$
\[
\leq \frac{(n-N_2)^{(a)}}{(n)^{(a)}} \cdot \frac{z_{N_2}}{1} + \frac{1}{(n)^{(a)}}
\times \sum_{s=N_2}^{n-1} \frac{(n-s)^{(a)} a_{\beta_s}}{4\lambda} \left[ \frac{\alpha}{(n-s+\alpha-1)} - \frac{\Delta \beta_s}{\beta_s} \right]^2
\]

or

\[
\frac{1}{(n)^{(a)}} \sum_{s=N_2}^{n-1} (n-s)^{(a)} \beta_s
\]

\[
\times \left[ (q_s - \mu p_s) - \frac{a_s}{4\lambda} \left( \frac{\alpha}{(n-s+\alpha-1)} - \frac{\Delta \beta_s}{\beta_s} \right)^2 \right]
\]

\[
\leq \frac{(n-N_2)^{(a)}}{(n)^{(a)}} \cdot \frac{z_{N_2}}{1} \rightarrow z_{N_2} \text{ as } n \to \infty,
\]

which contradicts (6.25). If \(\Delta y_n < 0\) for large \(n\), we proceed as in the proof of Theorem 6.2. This completes the proof of the theorem.

**COROLLARY 6.5**: If the condition (6.25) is replaced by

\[
\limsup_{n \to \infty} \frac{1}{(n)^{(a)}} \sum_{s=N_0}^{n-1} (n-s)^{(a)} \beta_s (q_s - \mu p_s) = \infty,
\]

and

\[
\limsup_{n \to \infty} \frac{1}{(n)^{(a)}} \sum_{s=N_0}^{n-1} \frac{(n-s)^{(a)} \beta_s a_s}{(n-s+\alpha-1)^2}
\times \left[ \alpha \cdot (n-s+\alpha-1) \left( \frac{\Delta \beta_s}{\beta_s} \right)^2 \right] < \infty
\]

for a positive integer \(\alpha \geq 1\), then all solutions of equation (6.1) are oscillatory.
REMARK 6.4: Theorem 6.6 and Corollary 6.7 extend Theorem 5 and Corollary 6 of [52] respectively.

We conclude this section with the following example.

EXAMPLE 6.3: Consider the difference equation

\[ \Delta(n(1+y_{n+1}^2)\Delta y_n) + (5n+2)(y_{n+1}+y_{n+1}^3) = 4n \frac{y_{n+1}^3}{1+y_{n+1}^2}, \quad n \geq 1, \quad (6.26) \]

satisfies all the conditions of Corollary 6.7. Hence all solutions of equation (6.26) are oscillatory. Here, we let \( f_1(u) = u + u^3 \), \( f_2(u) = u^3 \) and observe that

\[ \frac{Q(n,y_{n+1})}{f_1(y_{n+1})} = 5n+2 \quad \text{and} \quad \frac{P(n, y_{n+1}, \Delta y_n)}{f_2(y_{n+1})} \leq 4n. \]

6.3 OSCILLATION CRITERIA FOR GENERAL PERTURBED EQUATION

In this section, we establish some oscillatory criteria for the more general second order nonlinear perturbed difference equation of the form

\[ \Delta(a_n \psi (n, y_{n+1}, \Delta y_n) \Delta y_n) + Q(n, y_{n+1}) = P(n, y_{n+1}, \Delta y_n) \]

subject to the condition

\[ a_n \psi (n,u,v) \leq 1, \quad \text{for } u,v \neq 0. \quad (6.27) \]

THEOREM 6.8: Suppose conditions (6.3) - (6.7), (6.9), (6.11) and (6.27) are satisfied. Then all solutions of equation (6.2) are oscillatory.
PROOF: Suppose \( \{y_n\} \) is a nonoscillatory solution of equation (6.2) say \( y_n \neq 0 \) for \( n \geq N_0 \in \mathbb{Z} \). Then

\[
\Delta \left( \frac{a_n \psi(n, y_{n+1}, \Delta y_n) \Delta y_n}{f_1(y_n)} \right) \leq -\frac{Q(n, y_{n+1})}{f_1(y_{n+1})} + \frac{P(n, y_{n+1}, \Delta y_n)}{f_1(y_{n+1})}
\]

\[
-\frac{a_n \psi(n, y_{n+1}, \Delta y_n) \Delta y_n g(y_n, y_{n+1}) (\Delta y_n)^2}{f_1(y_n) f_1(y_{n+1})}
\]

\[
\leq -(q_n - \mu p_n).
\]

Summing the last inequality from \( N_0 \) to \( n-1 \), we get

\[
\frac{a_n \psi(n, y_{n+1}, \Delta y_n) \Delta y_n}{f_1(y_n)} \leq \frac{a_{N_0} \psi(N_0, y_{N_0+1}, \Delta y_{N_0}) \Delta y_{N_0}}{f_1(y_{N_0})}
\]

\[
- \sum_{s=N_0}^{n-1} (q_s - \mu p_s).
\]  

(6.28)

We assume that \( y_n > 0 \) for \( n \geq N_1 \geq N_0 \in \mathbb{Z} \); the proof for the case \( y_n < 0, n \geq N_1 \) is similar and will be omitted. In view of condition (6.11), it follows from (6.28) that there exists an integer \( N_2 \geq N_1 \) such that

\[\Delta y_n < 0 \text{ for } n \geq N_2.\]

It also follows from condition (6.11) that there exists an integer \( N_3 \geq N_2 \) such that

\[
\sum_{s=N_3}^{n-1} (q_s - \mu p_s) \geq 0 \text{ for } n \geq N_3.
\]
Now, summing the equation (6.2) and using condition (6.3), we obtain

\[ a_n \, \psi (n, y_{n+1}, \Delta y_n) \Delta y_n \leq a_{N3} \, \psi (N3, y_{N3+1}, \Delta y_{N3}) \Delta y_{N3} \]

\[ \sum_{s=N3}^{n-1} f_1 (y_{s+1}) (q_s - \mu p_s) \]

\[ = a_{N3} \, \psi (N3, y_{N3+1}, \Delta y_{N3}) \Delta y_{N3} - f_1 (y_{n+1}) \sum_{s=N3}^{n-1} (q_s - \mu p_s) \]

\[ + \sum_{s=N3}^{n-1} \Delta f_1 (y_{s+1}) \left[ \sum_{t=N3}^{s} (q_t - \mu p_t) \right] \]

\[ \leq a_{N3} \, \psi (N3, y_{N3+1}, \Delta y_{N3}) \Delta y_{N3}. \]

In view of condition (6.31), we have

\[ \Delta y_n \leq a_n \, \psi (n, y_{n+1}, \Delta y_n) \Delta y_n \leq a_{N3} \, \psi (N3, y_{N3+1}, \Delta y_{N3}) \Delta y_{N3}. \]

From the last inequality we derive the desired contradiction \( y_n \to -\infty \) as \( n \to \infty \). This completes the proof of the theorem.

**REMARK 6.5:** Theorem 6.8 generalizes Theorem 1 in [52].

**EXAMPLE 6.4:** Consider the difference equation

\[ \Delta \left( \frac{1}{(4+n)} \left( \frac{y^{2}_{n+1}}{1 + y^{2}_{n+1} + (\Delta y_n)^2} \right) \Delta y_n \right) + n(y^{3}_{n+1} + y^{n+1}_{n+1}) \]

\[ = \frac{(n^3 + 9n^2 + 18n - 9)}{(4+n)(5+n)} \quad y_{n+1}, \quad n \geq 1. \]  

(6.29)

If we choose \( a_n = \frac{1}{4+n} \), \( \psi (n, u, v) = \frac{u^2}{1 + u^2 + v^2} \), \( f_1(u) = u + u^3 \), \( f_2(u) = u \),
then all hypotheses of Theorem 6.8 are satisfied. Hence the equation (6.29) is oscillatory, does not appear to be deducible from other known oscillation criteria.

**EXAMPLE 6.5:** Consider the difference equation

\[
\Delta \left[ \frac{1}{n+2} \left( \frac{y_{n+1}^2}{1+y_{n+1}^2 + (\Delta y_n)^2} \right) \Delta y_n \right] \\
+ n (y_{n+1}^3 + y_{n+1}) = n^2 e^{-n} y_{n+1}^2 n \geq 1.
\]  

(6.30)

If we take \( f_1(u) = u^3 + u \), \( f_2(u) = u^2 \sgn u \), \( g(u) = 1 \) and \( g_1(u,v) = u^2 + uv + v^2 + 1 \), then it is easy to see that all the hypotheses of Theorem 6.6 are satisfied. Hence equation (6.30) is oscillatory. The oscillation of this equation is not deducible from any previously known oscillation criteria.

In the following theorem, we give the oscillation criteria for equation (6.2) subject to the condition

\[
\int_{-\alpha}^{\alpha} \frac{dx}{f_1(x)} < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{-\alpha \, dx}{f_1(x)} > -\infty, \text{ for all } \alpha > 0
\]  

(6.31)

**THEOREM 6.7:** Suppose that conditions (6.3) - (6.7) (6.9) and (6.11) hold and in addition, there exists \( B > 0 \) such that

\[
\left| \sum (q_n - \mu p_n) \right| < B, \text{ for all } n, \\
\liminf_{n \to \infty} \sum_{s=0}^{n-1} (q_s - \mu p_s) \geq 0, \text{ for large } N_0 \in \mathbb{Z}.
\]  

(6.32)
and
\[
\liminf_{n \to \infty} n \sum_{s=N_0}^{\infty} \left[ \sum_{t=s}^{\infty} (\alpha_i - \mu p_i) \right] = \infty, \quad N_0 \in \mathbb{Z},
\]  
(6.34)

are satisfied. Then all solutions of equation (6.2) are oscillatory.

**Proof**: The proof is similar to that for Theorem 6.2 and hence details are omitted.

**Example 6.6**: Consider the difference equation

\[
\Delta \left( (n+1)^2 \left( \frac{y_{n+1}^2}{1+(n+1)^2 y_{n+1}^2} \right) \Delta y_n \right) + \frac{y_{n+1} \exp(y_{n+1}^2)}{n^{3/2}} = \frac{y_{n+1}}{n^2 \ln (2+y_{n+1}^2 + |\Delta y_n|)}, \quad n \geq 1.
\]  
(6.35)

Here, we let \( f_1(u) = u \exp(u^2) \), \( f_2(u) = u \), \( g(v) = 1 \),

\[
g_1(u,v) = \frac{u \exp(u^2) - v \exp(v^2)}{u - v}, \quad q_n = \frac{1}{n^{3/2}}, \quad p_n = \frac{1}{n^2 \ln 2},
\]

and \( \mu = 1 \). The hypotheses of Theorem 6.7 are satisfied, so all solutions of equation (6.35) are oscillatory.

We conclude this chapter with the following corollary.

**Corollary 6.8**: If conditions (6.3) - (6.7), (6.9), (6.27) and (6.32) - (6.34) hold, then all bounded solutions of equation (6.2) are oscillatory.

**Proof**: The proof is similar to that of Corollary 6.3 and hence the details are not repeated.