Magnetic Systems

2.1 Introduction

Magnetism is a class of physical phenomena that are mediated by magnetic fields. Electric currents and the fundamental magnetic moments of elementary particles give rise to a magnetic field, which acts on other currents and magnetic moments. All materials are influenced to some extent by a magnetic field. The most familiar effect is on permanent magnets which have persistent magnetic moments caused by ferromagnetism. Most materials do not have permanent moments. Some are attracted to a magnetic field, some are repelled by a magnetic field and others have a much more complex relationship with an applied magnetic field. Substances that are negligibly affected by magnetic fields are known as non-magnetic substances. They include copper, aluminium, gases and plastic. Pure oxygen exhibits magnetic properties when cooled to a liquid state. The magnetic state of
a material depends on temperature so that a material may exhibit more than one form of magnetism. Magnetism at its root arises from two sources such as electric current and nuclear magnetic moments. Ordinarily, the enormous number of electrons in a material are arranged such that their magnetic moments cancel out. This is due to the combination of electrons into pairs with opposite intrinsic magnetic moments as a result of the Paulis Exclusion principle or combining into filled subshells with zero net orbital motion. In both cases the electron arrangement is so as to exactly cancel the magnetic moments from each electron. Magnetism are of five types. It includes paramagnetism, diamagnetism, ferromagnetism, antiferromagnetism and ferrimagnetism. A ferromagnet, like a paramagnetic substance has unpaired electrons. However in addition to the electrons, intrinsic magnetic moments has the tendency to be parallel for an applied field. Thus even in the absence of an applied field the magnetic moments of the electrons in the material spontaneously line up parallel to one another. Every ferromagnetic substance has its own individual temperature called the Curie temperature or Curie point, above which it loses its ferromagnetic properties. This is because the thermal tendency to disorder overwhelms the energy-lowering due to ferromagnetic order. Ferromagnetism only occurs in a few substances the common ones are iron, nickel, cobalt their alloys and some alloys of rare earth metals.

Magnetic systems are fascinating nonlinear dynamical systems [80,81]. They exhibit both coherent and chaotic structure depending on the nature of the inter-
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actions involved. They are characterized by different kinds of interactions [82-85] such as

(i) bilinear, biquadratic or higher order exchange interactions between magnetic ions;

(ii) anisotropic interactions due to crystal field effects in the form of easy axis (uniaxial and biaxial) and easy plane;

(iii) interactions between magnetic ions and constant, rotating or oscillating external magnetic fields and

(iv) relativistic interactions (Gilbert damping) due to spin orbit coupling and interaction between magnetic and conduction electrons. The above magnetic interactions may be accommodated in appropriate spin Hamiltonian models, like the Ising, XY, XYZ, Heisenberg and XXZ.

2.2 The Heisenberg Model

2.2.1 Definition of the Model

The problem in which we are interested is that of the Heisenberg model of localized spins on a lattice [86]

\[ H = - \sum_{(i,j)} J_{ij} \vec{S}_i \cdot \vec{S}_j. \]  

(2.1)

Here \(i\) and \(j\) refer to sites on a lattice, \(\vec{S}_i = (S_i^x, S_i^y, S_i^z)\) are quantum mechanical operators and \(J_{ij}\) is a number characterizing the strength of the exchange inter-
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action between any pair of spins. We imagine that only nearest neighbour spins interact and choose the notation \( \sum_{(i,j)} \) to represent a summation over all nearest neighbour sites-each interaction being counted once.

If rather than quantum mechanical operators the spins are thought of as a classical vectors, then \( J_{ij} > 0 \) and \( J_{ij} < 0 \) favour parallel and antiparallel orientations of neighbouring vectors \( \vec{S}_i \) and \( \vec{S}_j \) respectively, i.e. the two cases correspond to ferromagnetism and antiferromagnetism respectively. Energy minimization is achieved for \( J_{ij} > 0 \) by allowing all spin vectors to point along a certain direction, which we can take as the \( z \)-direction. For \( J_{ij} < 0 \), the energy is minimized when all nearest neighbour spin vectors are antiparallel to each other. The Heisenberg model representing parallel \( J_{ij} > 0 \) and antiparallel \( J_{ij} < 0 \) spin orientation is clearly illustrated in Figure 2.1.

We will not here go into how other examples of Heisenberg models can arise as low-energy descriptions of different systems. Thus in the remainder of this we will simply consider the Heisenberg ferro and antiferromagnetic model of interacting spins and explore its properties without dwelling more on the origin of the model itself.

2.2.2 Heisenberg Ferromagnetic Spin Chain

In magnetism, one of the most important systems is the ferromagnetic (FM) in which spins of all the atoms in the ground state are oriented in one direction (see
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\[ H = -\sum_{i,j} J_{ij} S_i \cdot S_j \]

**Figure 2.1:** Schematic representation of Heisenberg model.

- **Heisenberg exchange**
- **$J_1 > 0$** ferromagnetic
- **$J_1 < 0$** antiferromagnetic
Figure 2.2: Spin ordering in ferromagnets ($J_{ij} > 0$).

Figure 2.2). The interaction responsible for this parallel alignment of spins was proposed by Heisenberg [86] and is called the 'exchange interaction'. It depends on the fact that electron obey Fermi-Dirac statistics and therefore the overall wave function of the system of electrons must be antisymmetric. Assuming that there is an energy $-J_{ij}$ between the nearest neighbours $i$ and $j$ associated with the exchange interaction, one obtains an effective Hamiltonian for the spins $\vec{S}_i$ in the form (2.1).

Having defined the ground state of the FM spin system, one wants naturally to understand the behaviour of the excited states. The energy required to produce such excitations may come from thermal motions, from a deliberate disturbance of the lattice structure and so on. These excited states of the ferromagnets, like any other many body system, may be represented in terms of elementary excitations.
2.3 Magnetic Models and Equations of Motion

As is well known, intrinsically nonlinear problems in magnetic systems as in other branches have been treated traditionally in the framework of linearization schemes to obtain spatially extended waves called, spin waves. Quantization of these spin waves leads to the corresponding elementary excitations—magnons. However, the evolution equations governing the dynamics of the FM systems are in general highly nonlinear difference/differential equations. So, the traditional perturbation treatment in terms of the spin waves often break down as it is impossible to reach the nonlinear sector of the solution space perturbatively in terms of linear modes. It has been realized in recent times [81,87-90] that the elementary nonlinear excitations of these magnetic systems are often characterized by the spatially compact solitons in their own right for appropriate magnetic interactions, and sometimes also by chaotic structure. Solitons are in one dimensional magnets, magnons and solitons are both required for the dynamics and in two dimensional magnets, other excitations like instantons, vortices, and magnetic bubble have been found to be important topological excitations.

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2.3.1 Models

The Heisenberg spin Hamiltonian (2.1) does not show any preferred direction and has complete rotational symmetry. If the directional effects $\alpha, \beta,$ and $\gamma$ are
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included, the Hamiltonian takes the form

\[ H = - \sum_{i,j} \left[ \alpha S_i^x S_j^x + \beta S_i^y S_j^y + \gamma S_i^z S_j^z \right]. \tag{2.2} \]

Hamiltonian (2.2) represents the XYZ model. Different values of \( \alpha, \beta, \gamma \) in Eq.(2.2) leads to different models such as Ising (\( \alpha = \beta = 0 \)), XY (\( \gamma = 0 \)), Heisenberg (\( \alpha = \beta = \gamma \)), easy axis (\( \gamma > \alpha \simeq \beta \)) and easy plane(\( \gamma < \alpha \simeq \beta \)).

The effect of an external magnetic field \( B \) can be included by a term \( \mu \mathbf{B} \cdot \sum_i \mathbf{S}_i \) to the right hand side of (2.2), \( \mu \) being the gyromagnetic ratio. For example, the Hamiltonian for the system with uniaxial anisotropy in an external magnetic field can be given in the form

\[ H = - \sum_{i,j} J_{ij} (\mathbf{S}_i \cdot \mathbf{S}_j) + A \sum_i (S_i^z)^2 - \mu \mathbf{B} \cdot \sum_i \mathbf{S}_i, \tag{2.3} \]

where \( J \) is the exchange integral, \( A \) corresponds to the single site anisotropy coefficient with easy axis along the Z-direction.

Recently, there has been considerable interest in the study of magnetic systems whose localized moments interact in a way that goes beyond the usual Heisenberg model. In such a case, the Heisenberg model is recovered for spin \( S=1 \). However, for higher spins, the most general isotropic Hamiltonian involving two site exchange interactions is the addition of biquadratic exchange interaction. The Hamiltonian for this case including anisotropy and external magnetic field takes
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the form

\[ H = - \sum_{i,j} J(\vec{S}_i , \vec{S}_j) + K(\vec{S}_i , \vec{S}_j)^2 + A \sum_i (S_i^z)^2 - \mu \vec{B} \cdot \sum_i \vec{S}_i, \]  

(2.4)

where K is the biquadratic exchange parameter. Because of the isotropic nature of the exchange interaction \( J_{ij} \) has been replaced by J.

Spin being a quantum entity, the effort in representing the dynamics of the magnetic chain is then directed towards obtaining the low lying eigenvalues and eigen states of the corresponding quantum mechanical spin Hamiltonian [80,91-93]. However, many real magnetic systems have large spin quantum numbers 'S' so that the classical approximation in which the quantum operator of the spin angular momentum is replaced by a classical spin vector may be valid down to quite low temperatures. Hence for many realistic cases, the quantum fluctuation ceases and a classical approximation is reasonable.

### 2.3.2 Equations of Motion

The nature of nonlinear spin excitations in Heisenberg ferromagnetism can be understood through the dynamical equation derived from the associated model Hamiltonian H [94]. The Heisenberg equation of motion [95] for the quantum spin operator \( S_i \) can be written as

\[ i\hbar \frac{d\vec{S}_i}{dt} = [\vec{S}_i, H], \quad (i) = 1, 2, ..., N, \]  

(2.5)
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The square bracket in the right hand side represents the quantum mechanical commutator. In the classical limit $\hbar \to 0$ and $\vec{S} \to \infty$, $(\vec{S}_i)_{\text{quantum}} \to (\vec{S}_i)_{\text{classical}}$ and the quantum mechanical commutator is replaced by the classical poisson bracket [96]. That is

$$\left(\frac{1}{\hbar}\right)[\vec{S}_i, H] \quad \hbar \to 0 \quad \{\vec{S}_i, H\}_{PB} \quad (2.6)$$

The dynamics of spins can also be considered from a classical point of view, by developing a Hamiltonian formalism from the outset [97]. Here the spin is considered as a basic dynamical or canonical variable and suitable canonical equations can be obtained in equivalence with a spinless non-relativistic particle [98]. The classical poisson bracket is thus globalized to include spin [99-103]. Hence, in the classical limit, the Heisenberg equation of motion (Eq.(2.5)) becomes

$$\frac{d\vec{S}_i}{dt} = \{\vec{S}_i, H\}_{PB} \quad (2.7)$$

The above equation involves the assumption that the length of each spin vector does not change with time and that the equation of motion corresponds to the assumption of rigidity of the spins. Hence we assume that all the spins will have unit length and the spin angular momentum is represented by three component unit vector \(\vec{S}_i = (\vec{S}_i^x, \vec{S}_i^y, \vec{S}_i^z)\) and \(S_i^2 = 1\).

The dynamics of spins can also be considered from an altogether classical point of view by developing a Hamiltonian formalism from the outset. Generalizing the classical poisson bracket by including spin as a canonical variable, suitable
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canonical equations can be obtained in analogy with the spinless non relativistic particles. Thus, for given functions \( f(\vec{S}_i) \) and \( f'(\vec{S}_i) \) of the spin variable, the poisson bracket may be defined in terms of the spin components as

\[
\{f, g\}_{PB} = \sum_{i=1}^{N} \sum_{\alpha, \beta, \gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} \frac{\partial f}{\partial S_{i\alpha}} \frac{\partial g}{\partial S_{i\beta}} S_{i\gamma}.
\] (2.8)

Here \( \varepsilon_{\alpha\beta\gamma} \) is the complete antisymmetric Levi-civita tensor which can be defined as the permutation tensor of rank three dimensional space [104]. It can be easily checked that the spin canonical Poisson bracket (2.8) satisfies the same algebraic relations as the usual canonical Poisson bracket [96,105].

\[
\{f, g\}_{PB} = -\{g, f\}_{PB}.
\] (2.9)

If \( h \) is another arbitrary function of spins, then

\[
\{f, g + h\}_{PB} = \{f, g\}_{PB} + \{f, h\}_{PB},
\] (2.10)

\[
\{f, gh\}_{PB} = g\{f, h\}_{PB} + \{f, g\}_{PB}h.
\] (2.11)

Also the Jacobi identity

\[
\{f, \{g, h\}_{PB}\}_{PB} + \{g, \{h, f\}_{PB}\}_{PB} + \{h, \{f, g\}_{PB}\}_{PB} = 0
\] (2.12)
holds. As a result, the canonical equation of motion can again be written in the form (2.7).

Now, the equation for the FM system (2.4) can be derived from either of the above approaches. Using Eq.(2.7), the equation of motion takes the form

\[
\frac{d\vec{S}_i}{dt} = \vec{S}_i \wedge \left\{ J(\vec{S}_{i+1} + \vec{S}_{i-1}) + K J [\vec{S}_{i+1}(\vec{S}_i, \vec{S}_{i+1}) + \vec{S}_{i-1}(\vec{S}_i, \vec{S}_{i+1})] \right\} - 2AS_z \hat{n}
\]
where $\hat{n} = (0, 0, 1)$.

### 2.4 Continuum Limit

In the above considerations, we described the Heisenberg ferromagnet in a discrete lattice. Since the nonlinear character of Eq. (2.13) renders a general discussion difficult, one may consider the continuum limit of the system which is considered to be a satisfactory description in the long wave length, low temperature limit. We allow the lattice parameter 'a' between the nearest neighbour sites to approach zero, assume a slow variation of $S_i$ over a lattice distance. Considering a simple n-dimensional lattice with lattice constant vector 'a' and replacing $\vec{S}_i(t)$ by $\vec{S}(r, t)$, where $r = (r_1,..., r_n)$, $\hat{r} = \hat{n}a$, we may introduce the following series expansion

$$\vec{S}_{i\pm 1}(t) = \vec{S}(r, t) \pm a \nabla \vec{S} + \frac{a^2}{2!} (\nabla^2 \vec{S}) \pm \frac{a^3}{3!} (\nabla^3 \vec{S}) + \frac{a^4}{4!} (\nabla^4 \vec{S}) + ..., \hspace{1cm} (2.14)$$

where the n-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + .... + \frac{\partial^2}{\partial r_n^2}. \hspace{1cm} (2.15)$$

Using Eqs. (2.14) and (2.15) in Eq. (2.13), the resultant equation of motion up to order $O(a^4)$ (after rescaling of $t$ to $t/a^2 J$) takes the form

$$\frac{\partial \vec{S}}{\partial t} = \vec{S}\{\nabla^2 \vec{S} + \gamma_1 \nabla^4 \vec{S} - \gamma_2 [((\nabla \vec{S})^2 \nabla \vec{S}^2 + 2(\nabla \vec{S} \cdot \nabla \vec{S}) \nabla \vec{S})] \}
\quad - 2AS_n^z + \mu B, \hspace{1cm} (2.16)$$
2.5 Other Generalizations

where \( \gamma_1 = \frac{a^2}{17}, \quad \gamma_2 = \frac{K a^2}{(1+2K)} \).

Generalizing, one obtains the celebrated Landau-Lifshitz equation (LLE) without dissipation in the form

\[
\frac{\partial \vec{S}}{\partial t} = \vec{S} \wedge F_{\text{eff}}
\]  

(2.17)

where \( F_{\text{eff}} \) includes all the four types of magnetic interactions discussed earlier.

Eq.(2.17) was originally derived by Landau and Lifshitz from phenomenological arguments in 1935 [106] to describe the magnetic excitations in ferromagnets.

2.5 Other Generalizations

The above models have all been derived from various physical considerations. In addition, one can consider several mathematical generalizations of the Heisenberg FM spin chain. For example, with the Hamiltonian

\[
H = -2 \sum_{i,j} \log(1 + \vec{S}_i \cdot \vec{S}_j),
\]

(2.18)

Ishimori [114] have considered the generalized spin system

\[
\frac{d\vec{S}_i}{dt} = \vec{S}_i \wedge \left\{ \frac{\vec{S}_{i+1}}{1 + \vec{S}_i \cdot \vec{S}_{i+1}} + \frac{\vec{S}_{i-1}}{1 + \vec{S}_i \cdot \vec{S}_{i-1}} \right\},
\]

(2.19)

which is completely integrable. Interestingly, the continuum limit in the lowest order \( O(a^2) \) of (2.19) coincides with that of the standard Heisenberg model (\( \gamma_1 = \gamma_2 = 0 \)) in Eq.(2.17).
2.6 Heisenberg Models: A Semiclassical Approach

Further, one can consider certain site dependent generalized Heisenberg spin Hamiltonian in the form

\[ H = -J \sum_{i,j} \vec{s}_i \cdot f_i \vec{s}_j. \]  (2.20)

On deriving the equation of motion for (2.20) using (2.7), in the continuum limit and in (1+1) dimensions, the spin evolution equation up to order \( O(a^2) \) can be written as

\[ \vec{s}_t = f(x) \vec{s} \wedge \vec{s}_{xx} + f_x \vec{s} \wedge \vec{s}_x \]  (2.21)

where \( f \) is some scalar function of \( x \).

2.6 Heisenberg Models: A Semiclassical Approach

2.6.1 Ferromagnetic Spin System

The Heisenberg Hamiltonian for one dimensional (1D) FM involving nearest neighbour spin-spin exchange interaction is as follows [107]:

\[ \tilde{H} = -\sum_i [J(\tilde{s}_i \tilde{s}_{i+1})], \]  (2.22)

where \( \tilde{s}_i = (S_i^x, S_i^y, S_i^z) \) represents the spin operator at the lattice site \( i \). Now, by introducing the spin operator \( \hat{s}_i = \frac{\tilde{s}_i}{\hbar} \) and defining the spin ladder (raising and lowering) operators \( \hat{s}_i^\pm = \hat{s}_i^x \pm i\hat{s}_i^y \), the spin Hamiltonian (2.22) in the dimensionless form can be written as

\[ H = -\sum_i \left[ \frac{J}{2S^2} \left( \hat{s}_i^+ \hat{s}_{i+1}^- + \hat{s}_i^- \hat{s}_{i+1}^+ + 2\hat{s}_i^z \hat{s}_{i+1}^z \right) \right]. \]  (2.23)
While writing Eq.(2.23) we have defined \( H = \frac{\hat{\mu}}{\hbar^2 S^2} \) and \( J = \tilde{J} \). Different models of 1D spin-\( \frac{1}{2} \) ordered FM chains with exchange interaction were proven to be exactly solvable with a complete description of their energy spectrum and eigen functions [91,108]. However, many real magnetic materials are characterized by higher values of spins, and solving them quantum mechanically is really a challenging task. An exact solution for spin chains with spin values greater than \( \frac{1}{2} \) does not exist. The excitation spectrum of spin systems with integer spins exhibit a gap [109]. On the other hand, advantageously, the higher values of spins reduce the quantum fluctuation and, hence, a semiclassical description of the models becomes meaningful in these cases. Since we have to bosonize the Hamiltonian in the semiclassical treatment, we need to express the Hamiltonian by using the Holstein-Primakoff (H-P) representation [110] as

\[
\hat{S}^+_n = (2S)^{\frac{1}{2}} [1 - \frac{a_n^\dagger a_n}{2S}]^{\frac{1}{2}} a_n, \quad (2.24)
\]

\[
\hat{S}^-_n = (2S)^{\frac{1}{2}} a_n^\dagger [1 - \frac{a_n a_n^\dagger}{2S}]^{\frac{1}{2}} \quad (2.25)
\]

and

\[
\hat{S}^z_n = [S - a_n^\dagger a_n]. \quad (2.26)
\]

The bosonic operators \( a_n \) and \( a_n^\dagger \) satisfy the usual commutation relations,

\[
[a_m, a_n^\dagger] = \delta_{mn} \quad (2.27)
\]
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and

\[ [a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0. \quad (2.28) \]

The ground state expectation value of \( a_n^\dagger a_n \) is small compared to \( 2S \) in low temperature limit and therefore, we use the semiclassical expansions for \( \hat{S}_n^+ \) and \( \hat{S}_n^- \) in the following form:

\[ \frac{\hat{S}_n^+}{S} = \sqrt{2} \left[ 1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - O(\epsilon^4) \right] \epsilon a_n, \quad (2.29) \]
\[ \frac{\hat{S}_n^-}{S} = \sqrt{2} \epsilon a_n^\dagger \left[ 1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - O(\epsilon^4) \right], \quad (2.30) \]

where \( \epsilon = \frac{1}{\sqrt{S}} \) is a small dimensionless parameter. By using Eqs.(2.29) and (2.30), Eq.(2.23) can be written as a power series in \( \epsilon \).

\[ H = -\sum_i \left[ J + \epsilon^2 J [(a_i a_{i+1}^\dagger + a_{i+1}^\dagger a_i - a_{i+1} a_i - a_i a_{i+1}^\dagger)] \right. \\
\left. \quad -\frac{\epsilon^4}{4} J [a_i a_{i+1}^\dagger a_{i+1}^\dagger a_i + a_i^\dagger a_i a_{i+1}^\dagger + a_{i+1}^\dagger a_{i+1} a_i + a_{i+1} a_i a_i^\dagger a_{i+1}^\dagger + a_i^\dagger a_i a_i^\dagger a_{i+1} + 4a_i a_i^\dagger a_{i+1}^\dagger a_{i+1}^\dagger] \right]. \quad (2.31) \]

The spin dynamics can be expressed in terms of the Heisenberg equation of motion for the boson operators by substituting Hamiltonian (2.31) in the following equation of motion.

\[ i\hbar \frac{\partial a_n}{\partial t} = [a_n, H] = F(a_n^\dagger, a_n, a_{n+1}^\dagger, a_{n+1}). \quad (2.32) \]

Using the Glauber’s coherent state representation [111] for the bosonic operators,

\[ \langle u| a_n^\dagger = \langle u| a_n^* |u\rangle = u_n |u\rangle, \quad |u\rangle = \Pi_n |u_n\rangle \] with \( \langle u|u\rangle = 1 \), where \( u_n \) is the
coherent amplitude of the operator $a_n$ for the system in the state $|u\rangle$, and write down the equation of motion for the average $\langle u|a_n|u\rangle$ as

$$\frac{du_j}{dt} = \epsilon^2 J[2u_j - u_{j-1} - u_{j+1}] + \frac{\epsilon^4}{4} J[2|u_j|^2(u_{j+1} + u_{j-1})]
+ u_j^2(u_{j+1}^* + u_{j-1}^*) + |u_{j+1}|^2u_{j+1} + |u_{j-1}|^2u_{j-1}
- 4(|u_{j+1}|^2 + |u_{j-1}|^2)u_j.$$

The above equation describes the nonlinear spin dynamics of 1D FM spin system in the semiclassical limit.

### 2.6.2 Anisotropic Heisenberg Ferromagnetic Spin System

If the anisotropic energy density $H_{ani}$ varies as the square of the components of the magnetization vector $\vec{S}$, then the Hamiltonian for an anisotropic Heisenberg ferromagnet [112] is written as

$$H = -J \sum_{<i,j>} (\vec{S}_i \cdot \vec{S}_j) + k_1 \sum_i (S^z_i)^2,$$

(2.34)

Here the z-axis coincides with the anisotropic axis and $K_1$ represents anisotropic parameter. The one-dimensional version of associated LL equation in the classical continuum limit can be derived through poisson bracket $\frac{d\vec{S}_i}{dt} = [\vec{S}_i, H]$ which is given by

$$\vec{S}_t = \vec{S} \times [JS_{xx}^* - 2k_1S^z\hat{k}].$$

(2.35)

Here $\hat{k} = (0,0,1)$. The integrability of the above model was established in refs. [113, 114] and multi-soliton solutions were first obtained in refs. [115, 116] by the
Hirota method, later, in refs. [117-120] multi-soliton and even more complicated solutions were obtained through various methods.

### 2.6.3 Anisotropic Heisenberg Ferromagnetic Spin System with D-M interaction

The Hamiltonian associated with one-dimensional anisotropic Heisenberg ferromagnet involving nearest neighbour spin-spin exchange interaction and D-M [121-125] interaction is given by

\[
H = -\sum_{<i,j>} [J(\vec{S}_i \cdot \vec{S}_j) + \vec{D}.(\vec{S}_i \times \vec{S}_j)] + k_1 \sum_i (S_i^z)^2. \tag{2.36}
\]

Here \(K_1\) represents an anisotropic parameter. The corresponding spin evolution equation is

\[
\vec{S}_t = \vec{S} \times [J\vec{S}_t^{xx} - (\vec{D} \times \vec{S}_t^z) - 2k_1 S_t^z \hat{k}]. \tag{2.37}
\]

The dynamics of the above spin system Eq.(2.37) is studied separately in the isotropic and anisotropic cases by choosing \(\vec{D} = D\vec{m}\) here \(\vec{m} = (1,1,1)\). In the isotropic case, i.e., when \(k_1 = 0\), Eq.(2.36) is found to be integrable and the elementary spin excitations are governed by solitons when the effective field due to weak interaction is considered within a small angle cone.

In an anisotropic case, the system is not integrable; however the system is found to be integrable under long wavelength approximation and for small angle
2.6 Heisenberg Models: A Semiclassical Approach

variation of spins since the weak anisotropic axis lies parallel to the easy axis of magnetization and the elementary spin excitations are governed by solitons [122].

2.6.4 Anisotropic Heisenberg Ferromagnetic Spin System with bilinear, biquadratic exchange interaction

The Hamiltonian for the anisotropic Heisenberg ferromagnetic spin system with bilinear, biquadratic exchange interaction is written as

\[ H = - \sum_{\langle i,j \rangle} [J_i \langle \vec{S}_i \cdot \vec{S}_j \rangle + k_1 \sum_{\langle i,j \rangle} (\vec{S}_i \cdot \vec{S}_j)^2 ] \quad (2.38) \]

and the corresponding classical LL spin evolution equation is given by

\[ \vec{S}_t = \vec{S} \times [\vec{S}_{xx} + \gamma_1 \vec{S}_{xxxx} + \gamma_2 [(\vec{S} \cdot \vec{S}_{xx}) \vec{S}_{xx} + \frac{2}{3} (\vec{S} \cdot \vec{S}_{xxx}) \vec{S}_x]], \quad (2.39) \]

where \( \gamma_1 = \frac{a^2}{12}, \quad \gamma_2 = \frac{K a^2}{(1+2K)} \) and \( a \) is the lattice parameter. Solitary waves in one-dimensional ferromagnets have been reported due to magnon-magnon interactions [126, 127] and are connected with the lattice deformation. The existence of solitary waves in an anisotropic Heisenberg spin chain with biquadratic exchange interaction under the Dyson-Maleev representation has been reported in [128]. It is found that such nonlinear excitations appearing as a solution of NLS equation with high degree of nonlinearity are absent in the case of isotropic spin chain. In ref. [129], it was shown that isotropic ferromagnetic spin chain with biquadratic exchange exhibits nonlinear spin excitations in the form of soliton under the HP transformation in the semi-classical limit.
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2.6.5 Heisenberg Ferromagnetic Spin System with inhomogeneous exchange interaction

The Hamiltonian for the Heisenberg ferromagnetic spin system with inhomogeneous Heisenberg exchange interaction is written as

\[
H = -J \sum_{i=1}^{n-1} f_i (\vec{S}_i \cdot \vec{S}_{i+1}),
\]

(2.40)

where \( f_i \) refers to the site dependent inhomogeneity and the corresponding spin evolution equation of motion in this case is given by

\[
\frac{d\vec{S}_i}{dt} = J f_i (\vec{S}_i \times \vec{S}_{i+1}) + J f_{i-1} (\vec{S}_i \times \vec{S}_{i-1}).
\]

(2.41)

A continuum description in which \( \vec{S}_i \rightarrow \vec{S}_i(x, t), f_i \rightarrow f(x, t) \) is suitable when \( \vec{S}_i, f_i \) vary slowly over the lattice separation \( a \). Inserting Taylor expansions of \( \vec{S}_i(x + a, t) \) and \( f(x - a, t) \) in Eq.(2.41), we get the following inhomogenous LL equation as

\[
\vec{S}_t = f(\vec{S} \times \vec{S}_{xx}) + f_x (\vec{S} \times \vec{S}_x),
\]

(2.42)

which is proved by Lakshmanan to be equivalent to the conventional NLS equation [39]. Through painleve singularity structure analysis [130], it is proved that Eq.(2.42) is integrable only when the function \( f(x) \) is linear function of \( x \) and the underlying spin excitations are governed by solitons.
2.7 Recent Progress: A Review

During the past four decades important progress has been made in identifying nonlinear excitations of several magnetic systems like $\text{CsNiF}_3$, $\text{TMMC}$ in terms of solitons. Considering the continuum limit of the purely isotropic Heisenberg spin chain ($\gamma_1 = \gamma_2 = 0$), in one space, one time dimension, Eq.(2.17) becomes

$$\vec{S}_t = \vec{S} \wedge \vec{S}_{xx}. \tag{2.43}$$

Nakamura and Sasada [131] and Lakshmanan et al. [89] have pointed out the existence of solitary wave-like excitations in ferromagnets. However it was in 1977, that Lakshmanan [90] made a breakthrough by using simple concepts of classical differential geometry showing that Eq.(2.43) is a completely integrable solitonic system and is equivalent to the Nonlinear Schrödinger Equation (NLSE).

Soon afterwards, Takhtajan [132] pointed out its complete integrability by finding N-soliton solution using the IST method. Later, Zakharov and Takhtajan [133] established the above said equivalence to be a gauge equivalence. From a different point of view, Mikeska [134] approximated the evolution of the 1D magnetic systems with the easy plane anisotropy

$$\vec{S}_t = \vec{S} \wedge [\vec{S}_{xx} - 2AS^z\hat{n} + \mu B\hat{m}], \tag{2.44}$$

where $\hat{n} = (0, 0, 1), A > 0$ and $\hat{m}$ is a unit vector, to the integral sine-Gordon model in the limit $\frac{\mu B}{J} \gg 1$, which prompted several experimental investigations.
on \( \text{CsNiF}_3 \). Since then the field has grown rapidly.

From the above, it is clear that some but impressive progress has been made in the understanding of the nonlinear dynamics of \((1+1)\)-dimensional continuum and discrete spin systems.

The above studies on nonlinear spin excitations in terms of solitary waves and solitons in the Heisenberg model of ferromagnets with different magnetic interactions are only in the classical limit \([90,93,131,132,134-165]\). For quantum spin systems, semiclassical treatment turns out to be a very suitable method for studying soliton excitations because of its consistency and validity \([166-176]\). Soliton spin excitations and solitary wave profiles have been identified in ferromagnets under semiclassical approximation in the long wavelength and low temperature limits, in which the spins are treated as bosonic operators under the H-P approximation \([110]\) in combination with Glauber’s coherent state representation \([111]\).

The soliton solution for these one dimensional magnets have been studied by several different approaches. In the classical approach \([134,135,137,177,179]\), general single soliton solutions are obtained for a continuum version of the classical linear Heisenberg chain. In a quantum spin system, a bosonic representation of the spin operators turns out to be a very suitable method for studying the solitary waves, because they allow one to include quantum corrections in a systematic way. In a spin-coherent representation \([180]\), one can work directly with
the operators, make no approximation to the Hamiltonian, and can develop an exact nonlinear equation for the quantum system [93]. The other coherent state treatments [136,181-183] use a severely truncated H-P expansion [110] for $S_j^+$ and further approximate the Hamiltonian to be biquadratic in boson operators. Working in the coherent state representation of Glauber [111] and making small amplitude and long wave approximations, one then finds solitary wave profiles identical to classical solitons, which is the so called semiclassical treatment.

From the above, it is clear that, impressive progress has been made in the understanding of the nonlinear dynamics of continuum Heisenberg FM spin system in classical and semiclassical limit. But the studies in higher dimensional FM spin system is in infant stage. Likewise, it appears we have unravelled only a small part of the nonlinear dynamics of 2D Heisenberg FM spin system and several problems remain to be investigated here.

2.8 Present Work

Inspired by the considerations put forth earlier, in our thesis we investigate the Soliton excitations and stability aspects in a (2+1) dimensional ferromagnetic spin system. In chapter 3, we study the soliton excitations in a lower order (2+1) dimensional FM spin system by including only bilinear exchange and anisotropic interactions. We study the dynamics under various physical circumstances in the
continuum limit. We use the HP transformation, coherent state ansatz and the perturbation technique (sine-cosine function method) to investigate solitary wave solutions. Next we present the dynamics of spin wave propagation by including the bilinear and anisotropic interactions in a 2-dimensional lattice and the results are presented in chapter 4. In chapter 5, we analyze the effect of biquadratic interactions in (2+1)-dimensional FM spin system. Also we study the dynamics of the corresponding model by generating equation of motion. For a specific choice of parameters, the subsequent dynamics is found to be governed by a perturbed nonlinear fourth order partial differential equation in (2+1) dimensions. In chapter 6, we investigate the nature of spin wave propagation in a square lattice model of FM spin system with biquadratic interactions. Intrinsic localized modes are constructed for different parametric choices. We also study the conditions for the soliton stability using linear stability analysis. In Chapter 7, we investigate the nature of nonlinear spin excitations in two dimensional discrete Heisenberg ferromagnetic spin system with bilinear interactions. The dynamics is found to be governed by a (2+1)-dimensional discrete nonlinear equation which possesses soliton solution. We analyze the stability of soliton using linear stability analysis. Next we investigate the localized excitations in higher dimensional discrete FM spin system including biquadratic interactions and the results are presented in chapter 8. We analyze the stability of soliton analytically and graphically using linear stability analysis. We again present the condition of MI
growth rate in large wavelength limit, short wavelength limit and the Brillouin boundary Zone. Typical dependence of the MI growth rate on the interaction parameter and perturbation amplitude of the system are shown graphically. In Chapter 9, we study in the continuum limit, the effect of inhomogeneity in a 2-dimensional FM spin system with biquadratic interaction. The dynamics associated with different magnetic interactions are governed by perturbed (2+1)-dimensional nonlinear higher order PDE which admit localized spin excitations. We employ the SC function method to construct soliton solutions to the nonlinear PDE. We investigate the behavior of the propagation of soliton for various types of inhomogeneities such as cubic, biquadratic, periodic and localized types and our results are analysed using SC function method. Finally in chapter 10, we investigate the dynamics of an inhomogeneous discrete FM spin system. The effect of periodic, localized and Gaussian inhomogeneities have been investigated using modulational instability analysis and the results have been discussed by dividing the region in to three: long wavelength limit, short wavelength limit and brillouin boundary zone.