10

A Square Lattice Model of Inhomogeneous Ferromagnetic Spin System

10.1 Introduction

In recent times, discrete nonlinear equations (differential difference or mapping) have also got wide attention in the context of both analytical and numerical studies from the integrability and nonintegrability point of view. As real systems are discrete in nature, it is then appropriate and quite natural to study the dynamics of discrete models as well. Hence in the chapter, we investigate the dynamics of an inhomogeneous discrete FM spin system. The effect of periodic, localized and Gaussian inhomogeneities have been investigated using Modulational Instability analysis and the results have been discussed by dividing the region into...
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10.2.1 Analytical results

In this section, we consider Eq.(9.10) and assume that it admits monochromatic wave solution of the form given in Eq.(7.8). The dispersion relation is then becomes

\[ \omega = -\epsilon^2 \left( (2A + 6A') - f_{n,m}(J + J_1 + J_2) - g_{n,m}(\dot{J} + \dot{J}_1 + \dot{J}_2) \right. \]

\[ \left. (\cos q_1 + i\sin q_1)(Jf_{n,m} + 2\dot{J}_1 g_{n,m}) + (\cos q_2 + i\sin q_2) \right. \]

\[ \left. (J_1 f_{n,m} + 2\dot{J}_1 g_{n,m}) + (\cos(q_1 + q_2) + i\sin(q_1 + q_2))(J_2 f_{n,m} + 2\dot{J}_2 g_{n,m}) + J_1 f_{n-1,m} (\cos q_1 - i\sin q_1) + J_2 f_{n-1,m} (\cos q_2 - i\sin q_2) \right. \]

\[ \left. + \dot{J}_1 g_{n-1,m} (\cos q_1 - i\sin q_1) + \dot{J}_2 g_{n-1,m} (\cos q_2 - i\sin q_2) \right. \]

\[ \left. + J_1 f_{n,m-1} + J_2 f_{n,m-1} - 2(\dot{J}_1 f_{n-1,m} + \dot{J}_2 f_{n-1,m} + \dot{J}_1 f_{n,m-1} + \dot{J}_2 f_{n,m-1}) \right) \].

(10.1)

For simplicity, we set \( q_1 = q_2 = q \) and analyze the effect of a small perturbation on the plane wave using the solution is given in Eq.(7.10).
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in Eq.(9.10) and obtain a system of linearized equation for perturbations \( \eta_{n,m} \):

\[
i \dot{\eta}_{n,m} + \epsilon^2 u_0 \epsilon \eta_{n,m} \left[ (\cos q + i \sin q) (-f_{n,m} J - f_{n,m} J_1) - A - 2A' - 
2g_{n,m} J' - 2g_{n,m} J'_1 - Jf_{n-1,m} - f_{n,m-1} J_1 - 2g_{n-1,m} J' - 2g_{n,m-1} J'_1 
+ Jf_{n-1,m} \eta_{n-1,m} + 2J' g_{n-1,m} \eta_{n-1,m} + Jf_{n,m-1} \eta_{n,m-1} + 2gJ_1 \eta_{n,m+1} + 
2g_{n,m} J' \eta_{n,m+1} + fJ \eta_{n+1,m} + 2gJ' \eta_{n+1,m} + \right] + \left[ (\cos 2q + i \sin 2q) (-f_{n,m} J_2 
- 2g_{n,m} J_5 - Jf_{n-1,m-1} - Jf_{n-1,m-1} \eta_{n-1,m-1} + 2J' g_{n-1,m-1} \eta_{n-1,m-1} + 
2g_{n,m} \eta_{n+1,m+1} \right] = 0. 
\] (10.2)

Furthermore, we assume a general solution of the above-mentioned system as given by Eq.(7.6). Substituting Eq.(7.6) in Eq. (10.2) and dropping higher order terms, we obtain the linearized evolution equation as

\[
\Omega^2 B_1 B_2^* + \Omega (B_1 N + B_2^* M) + MN = 0 
\] (10.3)

with

\[
M = 2\epsilon^2 u_0 B_1 \left[ (J_2 f_{n,m} - J_1 f_{n,m} - 2g_{n,m} J' - 2g_{n,m} J'_1 - 2A - 4A') (\cos q - i \sin q) 
- (\cos 2q + i \sin 2q) (J_2 f_{n,m} + 2J' g_{n,m})(\cos q + i \sin q) \cos q [Jf_{n,m} + J_1 f_{n,m} 
+ 2g_{n,m} J' + 2g_{n,m} J'_1] + (J_2 f_{n,m} + 2J' g_{n,m}) \cos 2q (\cos 2q + i \sin 2q) + \sin q 
(\sin q + i \cos q) [Jf_{n,m} + J_1 f_{n,m} + 2g_{n,m} J' + 2g_{n,m} J'_1] [J_2 f_{n-1,m-1} + 2J' g_{n-1,m-1}] 
- \cos 2q + \cos 2q \cos 2Q + i \sin 2q - i \cos 2Q \sin 2q 
- \cos 2q \sin 2Q - \sin 2q \sin 2Q] + [Jf_{n-1,m} + J_1 f_{n,m-1} + J' g_{n-1,m}] 
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\[ + J'_1 g_{n,m-1} \left[ - \cos q + \cos q \cos Q + i \sin q - i \cos Q \sin q - \cos q \sin Q \right. \]
\[ \left. - \sin q \sin Q \right] \], \quad (10.4) \]

\[ N = -2 \varepsilon^2 u_0 B_2^2 \left[ (J f_{n,m} - J_{1f,n,m} - 2 g_{n,m} J' - 2 g_{n,m} J'_1 - 2 A - 4 A') (\cos q - i \sin q) \right. \]
\[ - (\cos 2q + i \sin 2q) (J_2 f_{n,m} + 2 J'_2 g_{n,m}) (\cos q + i \sin q) \cos q [J f_{n,m} + J_1 f_{n,m} \]
\[ + 2 g_{n,m} J' + 2 g_{n,m} J'_1 + (J_2 f_{n,m} + 2 J'_2 g_{n,m}) \cos 2q (\cos 2q + i \sin 2q) + \sin q \]
\[ (\sin q + i \cos q) [J f_{n,m} + J_1 f_{n,m} + 2 g_{n,m} J' + 2 g_{n,m} J'_1] [J f_{n-1,m-1} + \]
\[ 2 J'_2 g_{n-1,m-1}] - \cos 2q + \cos 2q \cos 2Q + i \sin 2q - i \cos 2Q \sin 2q \]
\[ - \cos 2q \sin 2Q - \sin 2q \sin 2Q \right] + [J f_{n-1,m} + J_{1f,n-1,m} + J' g_{n-1,m} \]
\[ + J'_1 g_{n,m-1}] - \cos q + \cos q \cos Q + i \sin q - i \cos Q \sin q - \cos q \sin Q \]
\[ - \sin q \sin Q \right] \]. \quad (10.5) \]

Solving Eq. (10.3) yields
\[ \Omega = \frac{- (B_1 N + B_2^2 M) \pm \sqrt{(B_1 N + B_2^2 M)^2 - 4 B_1 B_2^2 M N}}{2 B_1 B_2^2}. \quad (10.6) \]

Since the dispersion relation Eq.(10.6) is complex, the instability of the system is defined by the imaginary part of \( \Omega(Q) \), i.e. \( G \equiv |Im(\Omega(Q))| \). It should be noted that the presence of a trigonometric function in the dispersion relation (10.6) makes a main difference between the continuous and discrete systems.

From Eq.(10.6), we analyze the condition of the instability gain for the unstaggered (long wavelength limit), staggered (short wavelength limit) and the Brillouin boundary zone.
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10.2.1.1 Long wavelength limit

First we consider the unstaggered case (long wavelength limit) for wave number $q = 0$. The dispersion relation becomes

$$
\Omega = \epsilon^2 u_0 i ((- (J + J_1) f_{n,m} \sin Q - 2 g_{n,m} \sin Q (J' + J'_1) - f_{n,m} \sin 2Q
$$

$$(J_2 + 2J'_2) + (Jf_{n-1,m} + J_1f_{n,m-1}) \sin Q + (J_2f_{n-1,m-1} + J'_2g_{n-1,m-1}) \sin 2Q + 2 \sin Q (J'g_{n-1,m} + J'_1g_{n,m-1})).
$$

(10.7)

The growth rate at $q = 0$ for different values of periodic, localized and Gaussian Inhomogeneity is plotted in Figures (10.1 -10.3). In Figure 10.1 We study the effect of periodic inhomogeneity. $f_{m,n} = 1 + R_1(\cos[n] + \cos[m])$ and $g_{m,n} = 1 + R_2(\cos[m] + \cos[n])$. In the absence of inhomogeneity, $f = g = 1$, the system supports stable propagation of soliton for the choice of parametric values $J = J_1 = 12, J_2 = 8.5, J' = J'_1 = 6, J'_2 = 4.25, A = 0.1, A' = 1.03, \gamma = 1.2, U_0 = 0.1, R_1 = R_2 = 0$ [see Figure(10.1 (i))]. As the strength of inhomogeneities is increased as in Figure 10.1 [(i) $R_1 = R_2 = 0$ (ii) $R_1 = R_2 = 2.3$ (iii) $R_1 = R_2 = 4.7$ (iv) $R_1 = R_2 = 5.2$], amplitude of growth rate decreases. Depending upon the strength of inhomogeneity the growth rate diminishes and the stable propagation of soliton gets reduced in the range $-1.5 < Q < -.5$ and $2 < Q < 2.5$ for values of $Q$ beyond this region the soliton is not stable. Next to study the effect of localized inhomogeneity, we choose the functions $f_{m,n} = 1 + R_3(\tanh[n] + \tanh[m])$ and $g_{m,n} = 1 + R_4(\tanh[n] + \tanh[m])$ (See Figure 10.2). Again in the long
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Figure 10.1: Regions of modulational instability at $q = 0$ for different values of periodic inhomogeneity.

wavelength limit, an increase in $R_3$ and $R_4$ values makes the growth rate distorted in the Localized region with unstable modes developed subsequently. In Figure 10.3 we study the effect of Gaussian inhomogeneity $f_{n,m} = e^{-\frac{(n+\mu_1)^2}{2\sigma_1^2}} + e^{-\frac{(m+\mu_2)^2}{2\sigma_2^2}}$ and $g_{n,m} = e^{-\frac{(n+\mu_3)^2}{2\sigma_3^2}} + e^{-\frac{(m+\mu_4)^2}{2\sigma_4^2}}$. For $q = 0$, Gaussian inhomogeneity increases the amplitude and at the same time decreases the width of the MI gain, thus conserving the stability region. In the other two limits, this type of inhomogeneity distorts the localized region with unstable modes develop subsequently.

10.2.1.2 Short wavelength limit

Next we consider the staggered case (short wavelength limit) when the carrier wave number $q = \pi$. The dispersion relation in this case is written as.

$$
\Omega = \epsilon^2 u_0 i (\sin Q (f_{n,m}J + f_{n,m}J_1 + 2g_{n,m}J' + 2g_{n,m}J'_1) -
$$
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Figure 10.2: Regions of modulational instability at $q = 0$ for different values of localized inhomogeneity.

Figure 10.3: Regions of modulational instability at $q = 0$ for different values of Gaussian inhomogeneity.
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**Figure 10.4:** Regions of modulational instability at $q = \pi$ for different values of periodic inhomogeneity.

**Figure 10.5:** Regions of modulational instability at $q = \pi$ for different values of localized inhomogeneity.
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Figure 10.6: Regions of modulational instability at \( q = \pi \) for different values of Gaussian inhomogeneity.

Figure 10.7: Regions of modulational instability at \( q = \frac{\pi}{2} \) for different values of periodic inhomogeneity.
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Figure 10.8: Regions of modulational instability at $q = \frac{\pi}{2}$ for different values of localized inhomogenity.

Figure 10.9: Regions of modulational instability at $q = \frac{\pi}{2}$ for different values of Gaussian inhomogenity.
Using the dispersion relation (10.8), the MI gain for \( q = \pi \) is plotted in Figures (10.4 -10.6). Due to the influence of Periodic, localized and Gaussian Inhomogeneity, the MI is enhanced and the characteristic bandwidths of these response functions are increased and pulse splitting occurs in the fundamental soliton propagation. Moreover, the MI gain band has a unique maximum growth rate.

### 10.2.1.3 At the Brillouin boundary zone

The dispersion relation is

\[
\Omega = \epsilon^2 u_0 (\sin Q(f_{n,m}J + f_{n,m}J_1 + 2g_{n,m}J' + 2g_{n,m}J'_1) + 2i \\
\sin 2Q(f_{n,m}J_2 + 2g_{n,m}J'_2 - f_{n-1,m-1}J_2 + 2g_{n-1,m-1}J'_2) + \\
\sin Q(J' g_{n-1,m} - J_1 g_{n,m-1}).
\]  

(10.9)

Using the dispersion relation (10.9), the MI gain for \( q = \frac{\pi}{2} \) is plotted in Figures (10.7 - 10.9). At the Brillouin zone boundary one finds that the growth rate increases by increasing the strength of periodic, Localized and Gaussian Inhomogeneity as in the previous case. This shows the suppression of soliton stability at the Brillouin boundary zone for particular values of \( Q \) which results in the creation of localized pulses.
10.3 Conclusion

In this chapter, we investigate the nonlinear spin dynamics of a 2-dimensional inhomogeneous discrete FM spin system. Using the resulting higher order discrete NLS equation, we study the mechanism of formation of localized modes through modulational instability analysis and discuss the physical properties of the localized modes. The effect of periodic, localized and Gaussian inhomogeneities have been investigated and the results have been discussed by dividing the region into three: long wavelength limit $q = 0$, short wavelength limit $q = \pi$ and Brillouin boundary zone $q = \frac{\pi}{2}$. The results of the stability analysis show the influence of the different types of inhomogeneities on the stability of soliton propagation.