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A Square Lattice Model of
Ferromagnetic Spin System with
Bilinear Interactions

7.1 Introduction

In real times, discrete nonlinear equations have also got wide attention in the
context of both analytical and numerical studies from the integrability and non-
integrability point of view. As real systems are discrete in nature, it is then
appropriate and quite natural to study the dynamics of discrete models as well.
Hence, in this chapter, we investigate the nature of nonlinear spin excitations in
two dimensional discrete Heisenberg ferromagnetic spin system with bilinear in-
teractions. The dynamics is found to be governed by a (2+1)-dimensional discrete
nonlinear equation which possesses soliton solution. We analyze the stability of
soliton using linear stability analysis. The analysis is carried out in 3 different regions: (i) long wavelength limit (ii) short wavelength limit and (iii) Brillouin boundary zone. In all the three cases, the effect of interaction parameters on soliton stability has also been discussed.

7.2 Dynamics

We consider for our study a square lattice model of Heisenberg Ferromagnetic spin system with bilinear and anisotropic interactions. The Hamiltonian for such a system is in Eq.(3.1) and the discrete dynamical equation is in Eq.(3.7). We study the stability of soliton associated with Eq.(3.7) using MI analysis and the details are given in the following section.

7.3 Linear Stability Analysis

7.3.1 Analytical results

Eq.(3.7) has an exact monochromatic wave solution of the form

\[
    u_{n,m} = u_0 e^{i(q_1 n + q_2 m - \omega t)},
\]

(7.1)

where \( u_0 \) is a constant real amplitude, \( \omega \) is the frequency, \( q_1 \) and \( q_2 \) are wave numbers. Substituting the monochromatic wave solution Eq.(7.1) in Eq. (3.7), we obtain the dispersion relation

\[
    \omega = 2\epsilon^2[(-A + J + J_1 + J_2) - \cos q_1 J + \cos q_2 J_1 + \cos(q_1 + q_2) J_2].
\]

(7.2)
For simplicity, we have set $q_1 = q_2 = q$. Next, we make a small perturbation in the amplitude of wave solution Eq.(7.1)

$$u_{n,m} = u_0(1 + \eta_{n,m}(t))e^{i(q_1n + q_2m - \omega t)} \quad (7.3)$$

and substitute in Eq.(3.7) to obtain

$$i\dot{\eta}_{n,m} + \varepsilon^2 u_0 \left[ \cos q(-J - J_1 - 2A)\eta_{n,m} + J_1(\eta_{n+1,m} + \eta_{n-1,m}) + J_1(\eta_{n,m+1} + \eta_{n,m-1}) \right] + \cos 2q \left[ (J_2(-2\eta_{n,m} + \eta_{n+1,m+1} + \eta_{n-1,m-1}) + \eta_{n-1,m} + \eta_{n,m+1} - \eta_{n,m-1}) + i\sin q \left[ J(\eta_{n+1,m} - \eta_{n-1,m}) + J_1(\eta_{n,m+1} - \eta_{n,m-1}) \right] + i\sin 2q \left[ \eta_{n+1,m+1} - \eta_{n-1,m-1} \right] \right] = 0. \quad (7.4)$$

Further more, we assume a general solution of the above-mentioned system as

$$\eta_{n,m}(t) = B_1e^{i(Q_1n + Q_2m - \Omega t)} + B_2^*e^{-i(Q_1n + Q_2m - \Omega t)}, \quad (7.5)$$

where $Q_1$ and $Q_2$ are wave numbers and $\Omega$ is the frequency of perturbation. Substituting Eq.(7.5) in Eq. (7.4) and dropping higher order terms, we obtain the linearized evolution equation as

$$\Omega^2 B_1 B_2^* + \Omega(B_1 S + B_2^* R) + RS = 0 \quad (7.6)$$

with

$$R = 2\varepsilon^2 u_0 B_1 \left[ (J + J_1 - 2A)(-\cos q + \cos q \cos Q - \sin q \sin Q) + J_2(-\cos 2q + \cos 2q \cos 2Q - \sin 2q \sin 2Q) \right] \quad (7.7)$$

$$S = -2\varepsilon^2 u_0 B_2^* \left[ (J + J_1 - 2A)(-\cos q + \cos q \cos Q - \sin q \sin Q) + J_2(-\cos 2q + \cos 2q \cos 2Q - \sin 2q \sin 2Q) \right] \quad (7.8)$$
solving Eq. (7.6) yields

$$\Omega = \frac{-(B_1S + B_2^*R) \pm \sqrt{(B_1S + B_2^*R)^2 - 4B_1B_2^*RS}}{2B_1B_2^*}. \quad (7.9)$$

The stability of the nonlinear FM spin wave is determined by the imaginary part of \( \Omega \) (growth rate). When the eigen value \( \Omega \) is imaginary, i.e. \([(B_1S + B_2^*R)^2 < 4B_1B_2^*RS\)], the perturbation grows exponentially, and the excited nonlinear FM spin wave exhibits the MI. As a result, from the value of \( \Omega \), one can determine the stable and unstable regions for different modulations. Next we analyze the condition of the MI gain for the unstaggered (long wavelength limit), staggered (short wavelength limit) and the brillouin boundary zone using Eq.(7.9).

### 7.3.1.1 Long wavelength limit

First we consider the unstaggered case (long wavelength limit) for wave number \( q = 0 \). The dispersion relation becomes

$$\Omega = 2\epsilon^2 u_0 \left( -(J + J_1 + J_2 - 2A) \right) + \cos(Q)(J + J_1) + J_2 \cos(2Q). \quad (7.10)$$

Figure. (7.1) represents the growth rate curve for different values of interaction parameter \( J \). The parameters chosen are \( J_1 = 12, J_2 = 8.5, A = 1, \epsilon = 0.1, u_0 = 0.1 \) and (i) \( J = 5 \) (ii) \( J = 8 \) (iii) \( J = 12 \) (iv) \( J = 15 \). It is observed that an increase in interaction parameter \( J \) increases the growth rate amplitude while the width remains constant. The other interaction parameter \( J_1 \) also exhibits similar effect and for the sake of compactness, we do not present the details here.
7.3 Linear Stability Analysis

Figure 7.1: Regions of modulational instability at $q = 0$ for different values of $J$

Figure. (7.2) portrays the variation of MI growth rate with bilinear interaction parameter $J_2$ for (i) $J_2 = 3$ (ii) $J_2 = 7$ (iii) $J_2 = 8.5$ and (iv) $J_2 = 15$. In fact MI growth rate appears to increase slightly with increasing $J_2$, keeping the band width constant. Thus increase in $J_2$ value augments the stability of soliton.

Figure. (7.3) represents the growth rate for various perturbation amplitudes (i) $u_0 = 0.5$ (ii) $u_0 = 0.6$ (iii) $u_0 = 0.7$ (iv) $u_0 = 0.8$. It is observed that higher values of perturbation amplitudes boost the growth rate with symmetrical lobes, i.e, the increasing value of $u_0$ expand the bandwidth of the growth rate curve and hence enhances soliton stability. In all the above three cases, instability occurs for specific values of the perturbation wave vector. In the long wavelength limit, instability occurs at $Q = 0, \pm 6$, irrespective of the different choices of parameters.
### 7.3 Linear Stability Analysis

**Table 7.1:** Table of instability points for $q=\pi$

<table>
<thead>
<tr>
<th>Sl.No.</th>
<th>Parameters</th>
<th>points of instability $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$J = 5$</td>
<td>$0, \pm 1.5, \pm 4.6$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$J = 8$</td>
<td>$0, \pm 1.4, \pm 4.7$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$J = 12$</td>
<td>$0, \pm 1.1, \pm 5.1$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$J = 18$</td>
<td>$0, \pm 0.9, \pm 5.5$</td>
</tr>
<tr>
<td>(i)</td>
<td>$J_2 = 3$</td>
<td>$0, \pm 6$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$J_2 = 7$</td>
<td>$0, \pm 0.8, \pm 5.5$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$J_2 = 8.5$</td>
<td>$0, \pm 1.1, \pm 5.2$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$J_2 = 15$</td>
<td>$0, \pm 1.8, \pm 6$</td>
</tr>
<tr>
<td>(i)</td>
<td>$u_0 = 0.5$</td>
<td>$0, \pm 1, \pm 5$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$u_0 = 0.6$</td>
<td>$0, \pm 1, \pm 5$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$u_0 = 0.7$</td>
<td>$0, \pm 1, \pm 5$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$u_0 = 0.8$</td>
<td>$0, \pm 1, \pm 5$</td>
</tr>
</tbody>
</table>

#### 7.3.1.2 Short wavelength limit

Next we consider the staggered case (short wavelength limit) when the carrier wave number $q = \pi$. The dispersion relation in this case is written as

$$\Omega = -2\varepsilon^2 u_0\left( - (J + J_1 + J_2 - 2A) \right) + \cos(Q)(J + J_1 - J_2 \cos(2Q)) \quad (7.11)$$

Using the dispersion relation (7.11), the MI gain for $q = \pi$ is plotted in Figures 7.4-7.6. Figure. (7.4) represents the growth rate for different values of $J$ (i) $J = 5$ (ii) $J = 8$ (iii) $J = 12$ (iv) $J = 15$ and Figure. (7.5) shows the growth rate for different values of $J_2$ (i) $J_2 = 3$ (ii) $J_2 = 7$ (iii) $J_2 = 8.5$ and (iv) $J_2 = 15$. 

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Table 7.2: Table of instability points for \( q = \frac{\pi}{2} \)

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Parameters</th>
<th>Points of instability ( Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( J = 5 )</td>
<td>( 0, \pm 3, \pm 3.6, \pm 5.9 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( J = 8 )</td>
<td>( 0, \pm 3.1, \pm 3.4, \pm 6 )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( J = 12 )</td>
<td>( 0, \pm 3.1, \pm 6 )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( J = 18 )</td>
<td>( 0, \pm 3.2, \pm 6 )</td>
</tr>
<tr>
<td>(i)</td>
<td>( J_2 = 3 )</td>
<td>( 0, \pm 3.1, \pm 6 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( J_2 = 7 )</td>
<td>( 0, \pm 3.1, \pm 5.5 )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( J_2 = 8.5 )</td>
<td>( 0, \pm 3.1, \pm 5.4 )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( J_2 = 15 )</td>
<td>( 0, \pm 3, \pm 6 )</td>
</tr>
<tr>
<td>(i)</td>
<td>( u_0 = 0.5 )</td>
<td>( 0, \pm 3, \pm 6 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( u_0 = 0.6 )</td>
<td>( 0, \pm 3, \pm 6 )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( u_0 = 0.7 )</td>
<td>( 0, \pm 3, \pm 6 )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( u_0 = 0.8 )</td>
<td>( 0, \pm 3, \pm 6 )</td>
</tr>
</tbody>
</table>

and Figure. (7.6) portrays the growth rate for various values of perturbation amplitudes (i) \( u_0 = 0.5 \) (ii) \( u_0 = 0.6 \) (iii) \( u_0 = 0.7 \) (iv) \( u_0 = 0.8 \). The instability points are given in Table 7.1.

7.3.1.3 At the Brillouin boundary zone

The dispersion relation is

\[
\Omega = 2e^2 u_0 ((J_2 + 2A) + \sin(Q)(J + J_1) + J_2 \cos(2Q)).
\]

(7.12)

Figures 7.7-7.9 represents the MI gain for \( q = \frac{\pi}{2} \). At the Brillouin zone
7.3 Linear Stability Analysis

Figure 7.2: Regions of modulational instability at \( q = 0 \) for different values of \( J_2 \)
(i) \( J_2 = 5 \), (ii) \( J_2 = 8 \), (iii) \( J_2 = 12 \), (iv) \( J_2 = 18 \).

Figure 7.3: Regions of modulational instability at \( q = 0 \) for different values of \( u_0 \)
(i) \( u_0 = 0.5 \), (ii) \( u_0 = 0.6 \), (iii) \( u_0 = 0.7 \), (iv) \( u_0 = 0.8 \).
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Figure 7.4: Regions of modulational instability at $q = \pi$ for different values of $J$

Figure 7.5: Regions of modulational instability at $q = \pi$ for different values of $J_2$
(i) $J_2 = 5$, (ii) $J_2 = 8$, (iii) $J_2 = 12$, (iv) $J_2 = 18$. 
7.3 Linear Stability Analysis

Figure 7.6: Regions of modulational instability at $q = \pi$ for different values of $u_0$
(i) $u_0 = 0.5$, (ii) $u_0 = 0.6$, (iii) $u_0 = 0.7$, (iv) $u_0 = 0.8$.

Figure 7.7: Regions of modulational instability at $q = \frac{\pi}{2}$ for different values of $J$
7.3 Linear Stability Analysis

Figure 7.8: Regions of modulational instability at $q = \frac{\pi}{2}$ for different values of $J_2$
(i) $J_2 = 5$, (ii) $J_2 = 8$, (iii) $J_2 = 12$, (iv) $J_2 = 18$.

Figure 7.9: Regions of modulational instability at $q = \frac{\pi}{2}$ for different values of $u_0$
(i) $u_0 = 0.5$, (ii) $u_0 = 0.6$, (iii) $u_0 = 0.7$, (iv) $u_0 = 0.8$. 
boundary one finds that the growth rate increases by increasing the value of the interaction parameters $J, J_2$ and amplitude $u_0$ as in the previous case. In addition we observe that in the given range of $Q(-6, 6)$, the width of MI region depends on the interaction parameter $J, J_2$ and perturbation amplitude $u_0$. Instability points are tabulated in Table 7.2. This shows the suppression of soliton stability in the Brillion boundary zone for particular values of $Q$ which results in the creation of localized pulses. Figure (7.7) depicts the growth rate for different values of $J$ (i) $J = 5$ (ii) $J = 8$ (iii) $J = 12$ (iv) $J = 15$ and Figure. (7.8) portrays the growth rate for (i) $J_2 = 3$ (ii) $J_2 = 7$ (iii) $J_2 = 8.5$ and (iv) $J_2 = 15$ and Figure. (7.9) represents the growth rate for perturbation amplitudes (i) $u_0 = 0.5$ (ii) $u_0 = 0.6$ (iii) $u_0 = 0.7$ (iv) $u_0 = 0.8$.

### 7.4 Conclusion

In this chapter, we investigate the stability aspects of soliton in a square lattice model of ferromagnetic spin system with bilinear and anisotropic interaction in the semiclassical limit. We analyse the linear stability of nonlinear plane waves in the presence of small perturbation. The analytical result shows that growth rate decreases with increase in amplitudes and the system parameters. We observe that higher values of perturbation amplitudes boost the growth rate with symmetrical lobes, i.e. the increasing values of amplitude expands the bandwidth
7.4 Conclusion

of the growth rate curve and hence enhances soliton stability.