2.1 INTRODUCTION: As total dominating color transversal set is a blend of total dominating set and χ – Partition, discussion about relation between \( \gamma_{tstd} \), \( \gamma_t \) and \( \chi \) is most likely. In this chapter, we determine the condition under which \( \gamma_{tstd} \) becomes equal or unequal to \( \gamma_t \) or \( \chi \). This chapter removes misconception that \( \gamma_t < \chi \) or \( \chi < \gamma_t \), respectively, implies that \( \gamma_{tstd} = \chi \) or \( \gamma_t \). We also give different examples to justify our theorems and statements. Also, we determine an upper bound of \( \gamma_{tstd} \) in terms of \( \chi \) and \( \gamma_t \). At last, we determine an upper bound of \( \gamma_{tstd} \) for perfect graphs.

2.2 RESULTS REGARDING EQUALITY OR INEQUALITY BETWEEN \( \gamma_{tstd} \), \( \gamma_t \) AND \( \chi \):

REMARK 2.1.1: We have already mentioned the theorems 1.1.7, 1.1.8 and 1.2.7 which indicates relation between \( \gamma_{tstd} \), \( \chi \) and \( \gamma_t \).

First let us note down one result which will prove very important elementary result later on.

RESULT 2.1.2: If a graph \( G \) has \( k \) distinct support vertices then \( \gamma_{tstd} (G) \geq k \).
PROOF: As any total dominating set of G must contain all the support vertices of the graph G, $\gamma_t(G) \geq k$ and hence $\gamma_{tstd}(G) \geq k$.

**EXAMPLE 2.1.3:** For the given graph G, $\gamma_t < \chi$ but $\gamma_{tstd} \neq \chi$.

![Graph G](image1)

$\gamma_t(G) = 3$ and $\chi(G) = 4$ but $\gamma_{tstd}(G) = 5 \neq 4 = \chi(G)$.

**EXAMPLE 2.1.4:** For the given graph G, $\gamma_t < \chi$ and $\gamma_{tstd} = \chi$.

![Graph G](image2)

$\gamma_t(G) = 3$, $\chi(G) = 4$ and $\gamma_{tstd}(G) = 4 = \chi(G)$.
THEOREM 2.1.5: Let $G$ be a graph. Then $\gamma_{tstd}(G) = \chi (G) > \gamma_t(G)$ if and only if the following two conditions are satisfied:

1. There exists a total dominating set $D$ of $G$ and a $\chi$ – Partition $\{V_1, V_2, \ldots, V_\chi\}$ of $G$ such that $|D \cap V_i| = 1, \forall i = 1, 2, \ldots, \chi$.
2. No $\gamma_t$ - Set of $G$ is a transversal of any $\chi$ – Partition of $G$.

PROOF: We know that $\gamma_{tstd}(G) \geq \chi (G)$.

Suppose (1) is not true. Then $\gamma_{tstd}(G) > \chi (G)$, which is a contradiction to $\gamma_{tstd}(G) = \chi (G)$. Hence (1) is true.

Suppose (2) is not true. Then there exists a $\gamma_t$ - Set of $G$ which is a transversal of some $\chi$ – Partition of $G$. Therefore $\gamma_{tstd}(G) = \gamma_t(G)$ which is a contradiction to $\gamma_{tstd}(G) > \gamma_t(G)$. Hence (2) is true.

Conversely, assume (1) and (2). (1) implies that $\gamma_{tstd}(G) \leq \chi (G)$ which implies that $\gamma_{tstd}(G) = \chi (G)$ and (2) implies that $\gamma_{tstd}(G) > \gamma_t(G)$. So $\chi (G) = \gamma_{tstd}(G) > \gamma_t(G)$.

EXAMPLE 2.1.6: For the given graph $G$, $\gamma_t = \chi = k$ but $\gamma_{tstd} \neq k$.

![Graph G](image)

$G$

Fig. 2.3

$\gamma_t(G) = 3, \chi (G) = 3$ but $\gamma_{tstd}(G) = 4 \neq 3 = \chi (G) = \gamma_t(G)$. 

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EXAMPLE 2.1.7: For the given graph $G$, $\gamma_{\text{tstd}}(G) = \chi(G) = \gamma_t(G)$.

![Graph G](image)

Fig. 2.4

$\gamma_{\text{tstd}}(G) = \chi(G) = \gamma_t(G) = 3$.

THEOREM 2.1.8: Let $G$ be a graph. Then $\gamma_{\text{tstd}}(G) = \chi(G) = \gamma_t(G)$ if and only if there exists a $\gamma_t$-Set $D$ of $G$ and a $\chi$-Partition $\{V_1, V_2, ..., V_\chi\}$ of $G$ such that $|D \cap V_i| = 1, \forall i = 1, 2, ..., \chi$.

PROOF: Obvious.

REMARK 2.1.9: By the above theorem, it is clear that if $\gamma_{\text{tstd}}(G) = \chi(G) = \gamma_t(G)$ then there exists a $\gamma_t$-Set of $G$ such that it is a transversal of some $\chi$-Partition of $G$. Converse is not true, in general. Following example justifies this.

EXAMPLE 2.2.0:
Here $\gamma_t$ – Set of $G$ is $\{u_1, u_2, u_3\}$. $\chi(G) = 2$, $\gamma_{tstd}(G) = 3 = \gamma_t(G) \neq \chi(G) = 2$.

**EXAMPLE 2.2.1:** For the given graph $G$, $\chi \prec \gamma_t$ but $\gamma_{tstd} \neq \gamma_t$.

![Graph G](image1.png)

$G$  \hspace{1cm}  Fig. 2.6

$\chi(G) = 3$, $\gamma_t(G) = 4$ but $\gamma_{tstd}(G) = 5 \neq \gamma_t(G)$.

**EXAMPLE 2.2.2:** For the given graph $G$, $\chi \prec \gamma_t$ and $\gamma_{tstd} = \gamma_t$.

![Graph G](image2.png)

$G$  \hspace{1cm}  Fig. 2.7

$\chi(G) = 3$, $\gamma_t(G) = 4$ and $\gamma_{tstd}(G) = 4 = \gamma_t(G)$.
THEOREM 2.2.3: Let G be a graph. Then $\gamma_{tstd}(G) = \gamma_t(G) > \chi(G)$ if and only if the following two conditions are satisfied:

1. There exists a $\gamma_t$-Set of G such that it is a transversal of some $\chi$-Partition $\{V_1, V_2, ..., V_\chi\}$ of G.
2. For every $\gamma_t$-Set $D$ of G satisfying (1), $|D \cap V_i| > 1$ for some $i \in \{1, 2, 3, ..., \chi\}$.

PROOF: Obvious.

2.3 BOUNDS OF $\gamma_{tstd}$ IN TERMS OF $\gamma_t$ AND $\chi$: In this section, we discuss about bounds of $\gamma_{tstd}$. Especially an upper bound of $\gamma_{tstd}$ in terms of $\gamma_t$ and $\chi$. We provide different examples to justify our results. We have already mentioned results 1.1.4, 1.1.5 and 1.1.6 about the bounds of $\gamma_{tstd}$ in the first chapter.

THEOREM 2.2.4: For any graph G, $\gamma_{tstd}(G) \leq \gamma_t(G) + \chi(G) - 2$.

PROOF: Let S be a $\gamma_t$-set of G. Then S has at least two vertices that are adjacent and so they are in different color classes of every $\chi$-Partition of G. Adding, to S, one vertex from each remaining $\chi$-2 color classes yields a total dominating color transversal set of G. Hence $\gamma_{tstd}(G) \leq \gamma_t(G) + \chi(G) - 2$.

REMARK 2.2.5: Above given upper bound is sharp. Following result 2.2.6 justifies this.

RESULT 2.2.6: Let G be a graph. If $\chi(G) = 2$ or $\gamma_t(G) = 2$ then $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$.

PROOF: If $\chi(G) = 2$ then $\gamma_{tstd}(G) = \gamma_t(G)$, by theorem 1.1.8. So $\gamma_{tstd}(G) = \gamma_t(G) + 2 - 2 = \gamma_t(G) + \chi(G) - 2$. If $\gamma_t(G) = 2$ then $\gamma_{tstd}(G) = \chi(G)$, by theorem 1.2.7. So $\gamma_{tstd}(G) = \chi(G) + 2 - 2 = \gamma_t(G) + \chi(G) - 2$. 

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**REMARK 2.2.7:** There are graphs $G$ for which $\chi(G) > 2$ and $\gamma_t(G) > 2$ but still $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$. We give the following an example to justify this.

**EXAMPLE 2.2.8:** Consider graph $G$ as in Fig. 2.3. $\gamma_{tstd}(G) = 4$, $\gamma_t(G) = 3$ and $\chi(G) = 3$. Trivially, $\gamma_{tstd}(G) = 4 = 3 + 3 - 2 = \gamma_t(G) + \chi(G) - 2$.

**THEOREM 2.2.9:** Let $G$ be a graph. Then $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$ if and only if the following two conditions are satisfied:

1. Every $\gamma_t$-Set of $G$ intersects exactly two color classes of every $\chi$-Partition of $G$.
2. Every $\gamma_t$-Set of $G$ is contained in a $\gamma_{tstd}$-Set of $G$.

**PROOF:** Assume $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$.

1. Suppose there exists a $\gamma_t$-Set $D$ of $G$ such that it intersects at least three color classes of some $\chi$-Partition $\{V_1, V_2, \ldots, V_\chi\}$ of $G$. Then adding, to $D$, one vertex from each remaining at most $\chi - 3$ color classes, yields a total dominating color transversal set of $G$. Hence $\gamma_{tstd}(G) \leq |D| + \chi(G) - 3 < \gamma_t(G) + \chi(G) - 2$. So we get a contradiction. Hence (1) is true.

2. Consider a $\gamma_t$-Set $D$ of $G$. We know, by (1), that every $\gamma_t$-Set meet exactly two color classes of every $\chi$-Partition of $G$. So by adding, to $D$, one vertex from each remaining $\chi - 2$ color classes yields a total dominating color transversal set of $G$. Note that this set is minimum with this property. So every $\gamma_t$-Set of $G$ is contained in a $\gamma_{tstd}$-Set of $G$. Conversely, assume (1) and (2). Clearly $\chi(G) - 2 \leq \gamma_{tstd}(G) - \gamma_t(G)$. So $\gamma_{tstd}(G) \geq \gamma_t(G) + \chi(G) - 2$. Therefore by theorem 2.2.4, $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$.

**COROLLARY 2.3.0:** Let $G$ be a graph. If there exists $\gamma_t$-Set of $G$ that intersects $k$ color classes of some $\chi$-Partition of $G$ then $\gamma_{tstd}(G) \leq \gamma_t(G) + \chi(G) - k$.

**REMARK 2.3.1:** Let $G$ be a graph. If every $\gamma_t$-Set of $G$ intersects exactly two color of every $\chi$-Partition of $G$ then it is not necessary that $\gamma_{tstd}(G) = \gamma_t(G) + \chi(G) - 2$. Follo-
EXAMPLE 2.3.2:

Here we note that \( \{u_2, u_6, u_{11}\} \) is a color class of every \( \chi \) - Partition of \( G \). Also \( G \) has unique \( \gamma_t \) - Set \( D = \{u_2, u_3, u_6, u_{11}\} \) and it intersects exactly two color classes of every \( \chi \) - Partition of \( G \). Clearly, \( \{u_2, u_5, u_6, u_{12}, u_{13}\} \) is a \( \gamma_{\text{tstd}} \) - Set of \( G \). Also \( \chi(G) = 4 \). Hence \( \gamma_{\text{tstd}}(G) = 5 \neq 6 = \gamma_t(G) + \chi(G) - 2 = 4 + 4 - 2 \).

REMARK 2.3.3: If every \( \gamma_t \) - Set of \( G \) is contained in a \( \gamma_{\text{tstd}} \) - Set of \( G \), then it is not necessary that \( \gamma_{\text{tstd}}(G) = \gamma_t(G) + \chi(G) - 2 \). See the following example 2.3.4.

EXAMPLE 2.3.4: Consider the graph \( G \) given in Fig. 2.9.
Graph G has exactly two \( \gamma_t \) – Sets. They are \( D_1 = \{ u_1, u_2, u_3, u_4 \} \) and \( D_2 = \{ u_1, u_2, u_3, u_7 \} \). Note that both are \( \gamma_{tstd} \) - Sets of G. So we can say that both \( D_1 \) and \( D_2 \) are contained in a \( \gamma_{tstd} \) – Set of G. But \( \gamma_{tstd}(G) = 4 \neq 5 = 4 + 3 - 2 = \gamma_t(G) + \chi(G) - 2 \).

2.4 BOUNDS OF \( \gamma_{tstd} \) FOR PERFECT GRAPHS: In this section, we assume that the graph G and its complement \( \bar{G} \) have no isolated vertex.

**THEOREM 2.3.5:** If G is a Perfect graph then \( \gamma_{tstd}(G) \leq \gamma_t(G) + \omega(G) - 2 \).

**PROOF:** Obvious, as for a Perfect G, \( \chi(G) = \omega(G) \) and by theorem 2.2.4.

**THEOREM 2.3.6 [14]:** A Simple graph G is Perfect if and only if \( \bar{G} \) is Perfect.

**THEOREM 2.3.7:** If G is a Perfect graph then \( \gamma_{tstd}(\bar{G}) \leq 2 \omega(G) + \omega(\bar{G}) - 2 \).

**PROOF:** By theorem 2.2.4, \( \gamma_{tstd}(\bar{G}) \leq \gamma_t(\bar{G}) + \chi(\bar{G}) - 2 \). By theorem 2.3.6, \( \chi(\bar{G}) = \omega(\bar{G}) \). And hence \( \gamma_{tstd}(\bar{G}) \leq \gamma_t(\bar{G}) + \omega(\bar{G}) - 2 \).

Claim: \( \gamma_t(\bar{G}) \leq 2 \omega(G) \)

Suppose \( S \subseteq V \) is a maximum clique of G. Hence \( |S| = \omega(G) \). Also S is a maximum independent set in \( \bar{G} \). Therefore S is a dominating set in \( \bar{G} \). If S is not a total dominating set in \( \bar{G} \) then S has at most \( |S| \) isolates. As \( \bar{G} \) is a graph with no isolated vertex, each vertex of S has adjacent vertex in \( \bar{G} \). So adding at most \( |S| \) vertices, the resultant set becomes a total dominating set of \( \bar{G} \). So \( \gamma_t(\bar{G}) \leq 2 |S| = 2 \omega(G) \). Hence \( \gamma_{tstd}(\bar{G}) \leq 2 \omega(G) + \omega(\bar{G}) - 2 \).

**REMARK 2.3.8:** Bound of \( \gamma_{tstd}(\bar{G}) \) in theorem 2.3.7 is sharp. Following example 2.3.9 justifies this.

**EXAMPLE 2.3.9:** We know that every bipartite graph is Perfect.
RELATION BETWEEN TOTAL DOMINATING COLOR TRANSVERSAL NUMBER, TOTAL DOMINATION NUMBER AND CHROMATIC NUMBER OF GRAPHS

\begin{center}
\begin{tabular}{cc}
\includegraphics[width=0.3\textwidth]{C4} & \includegraphics[width=0.3\textwidth]{C4bar} \\
$C_4$ & $\overline{C_4}$
\end{tabular}
\end{center}

Fig. 2.10  Fig. 2.11

$\gamma_{std}(\overline{C_4}) = 4$, $\omega(C_4) = 2$ and $\omega(\overline{C_4}) = 2$. So $\gamma_{std}(\overline{C_4}) = 4 = 2\omega(C_4) + \omega(\overline{C_4}) - 2$.

**THEOREM 2.4.0:** If $G$ is a Perfect graph then $\gamma_{std}(G) = k - 2$, where $k = \min \{\gamma_t(G) + \omega(G), 2\omega(\overline{G}) + \omega(G)\}$.

**PROOF:** By theorem 2.2.4, $\gamma_{std}(G) \leq \gamma_t(G) + \chi(G) - 2 = \gamma_t(G) + \omega(G) - 2$ ....... (1)

By theorem 2.3.7, $\gamma_{std}(\overline{G}) \leq 2\omega(G) + \omega(\overline{G}) - 2$................................. (2)

As $G$ is perfect, $\overline{G}$ is also Perfect (by theorem 2.3.6). Hence applying inequality (2) to $\overline{G}$, we obtain the following inequality: $\gamma_{std}(G) = \gamma_{std}(\overline{G}) \leq 2\omega(\overline{G}) + \omega(G) - 2$................................. (3)

From (1) and (3), $\gamma_{std} = k - 2$, where $k = \min \{\gamma_t(G) + \omega(G), 2\omega(\overline{G}) + \omega(G)\}$.

**2.5 CONCLUDING REMARKS:** This chapter, as chapter 1, gives broad idea of various techniques of obtaining total dominating color transversal set of a graph and will provide ready reference for any researcher. Many graphs are generated using the fact that any total dominating set of a graph must contain all the support vertices of the graph. This fact has actually helped us generate the graphs to provide counter examples as well.

Next chapter is about relation between dominating color transversal number and total dominating color transversal number of a graph.