6.1 INTRODUCTION: In the first six chapters we talked about dominating or total dominating set that is a transversal of some $\chi$ - Partition of a graph. After having introduced, in the last chapter, a new coloring called complementary coloring that yields the unique $\chi^c$ - Partition, discussion about dominating and total dominating set that is a transversal of the $\chi^c$ - Partition is expected. Hence we introduce two new sets called dominating complementary color transversal set and total dominating complementary color transversal set of a graph.

We begin with definitions.

6.2 DEFINITIONS:

DEFINITION 6.1.1: (DOMINATING COMPLEMENTARY COLOR TRANSVERSAL SET) Let $G = (V, E)$ be a graph. Then a dominating set $S \subseteq V$ is called a dominating complementary color transversal set of $G$ if it is transversal of the $\chi^c$ – Partition of $G$.

DEFINITION 6.1.2: (MINIMUM DOMINATING COMPLEMENTARY COLOR TRANSVERSAL SET/ DOMINATING COMPLEMENTARY COLOR TRANSVERSAL NUMBER) Let $G = (V, E)$ be a graph. Then a dominating complementary color transversal set $S$ of $G$ is called a minimum dominating
complementary color transversal set of $G$ if $|S| = \text{minimum} \{ |D| : D \text{ is a dominating complementary color transversal set of } G \}$. Here $S$ is called $\gamma_{\text{std}}^C$ — Set and its cardinality, denoted by $\gamma_{\text{std}}^C(G)$ or just by $\gamma_{\text{std}}^C$, is called the dominating complementary color transversal number of $G$.

**DEFINITION 6.1.3: (TOTAL DOMINATING COMPLEMENTARY COLOR TRANSVERSAL SET)** Let $G = (V, E)$ be a graph. Then a total dominating set $S \subseteq V$ is called a total dominating complementary color transversal set of $G$ if it is transversal of the $\chi^C$ — Partition of $G$.

**DEFINITION 6.1.4: (MINIMUM TOTAL DOMINATING COMPLEMENTARY COLOR TRANSVERSAL SET/ TOTAL DOMINATING COMPLEMENTARY COLOR TRANSVERSAL NUMBER)** Let $G = (V, E)$ be a graph. Then a total dominating complementary color transversal set $S$ of $G$ is called a minimum total dominating complementary color transversal set of $G$ if $|S| = \text{minimum} \{ |D| : D \text{ is a total dominating complementary color transversal set of } G \}$. Here $S$ is called $\gamma_{\text{tstd}}^C$ — Set and its cardinality, denoted by $\gamma_{\text{tstd}}^C(G)$ or just by $\gamma_{\text{tstd}}^C$, is called the total dominating complementary color transversal number of $G$.

**6.3 RELATION BETWEEN $\chi^C$, $\gamma_{\text{std}}^C$, AND $\gamma_{\text{tstd}}^C$:**

Before mentioning the results reflecting relations between the above variants we need few other results. These results have their own importance as well. We mention them first.

**THEOREM 6.1.5:** Let $G = (V, E)$ be a graph of order $n$. Then for any $v \in V$, $\{v\}$ is a color class of $G$ if and only if $\gamma(G) = 1$.

**PROOF:** Obvious
**THEOREM 6.1.6:** Let $G$ be a graph.

(1) If $\chi^c(G) \geq 2$ then $\gamma_t(G) = 2$.

(2) If $\chi^c(G) = 1$ then $\gamma(G) > 1$.

**PROOF:** (1) Let $\chi^c(G) \geq 2$ and $\Pi = \{V_1, V_2, \ldots, V_{\chi^c} \}$ be the $\chi^c$-Partition of a graph $G$.

Let $u \in V_i$ and $v \in V_j$, for some $i \neq j$. Then $u$ and $v$ are adjacent and $u$ is adjacent to all the vertices that are not in $V_i$ and $v$ is adjacent to all the vertices that are not in $V_j$. So $\{u, v\}$ is total dominating set of $G$. Hence $\gamma_t(G) = 2$.

(2) Let $\chi^c(G) = 1$. If $\gamma(G) = 1$ then there exists a vertex $v$ of $G$ such that $\{v\}$ is a color class of $G$. Clearly, $\chi^c(G) \geq 2$, which is a contradiction. Hence $\gamma(G) > 1$.

**REMARK 6.1.7:** Converse of (1) of above theorem is not true, in general. For example consider the Path graph $P_4$. $\gamma_t(P_4) = 2$ but $\chi^c(P_4) = 1$. Also converse of (2) is not true, in general. For example consider $C_4$. $\gamma(C_4) = 2 > 1$ but $\chi^c(C_4) = 2$.

**RESULT 6.1.8:** For any graph $G = (V, E)$, $\gamma_{std}^c(G) \leq \gamma_{tstd}^c(G)$.

**PROOF:** Obvious as a total dominating set of $G$ is always a dominating set of $G$.

**THEOREM 6.1.9:** For any graph $G = (V, E)$.

(1) If $\chi^c(G) = 1$ then $\gamma_{std}^c(G) = \gamma(G)$ and $\gamma_{tstd}^c(G) = \gamma_t(G)$.

(2) If $\chi^c(G) \geq 2$ then $\gamma_{std}^c(G) = \gamma_{tstd}^c(G) = \chi^c(G)$.

**PROOF:** Let $\chi^c(G) = 1$ then definitely by the definition $\gamma_{std}^c(G) = \gamma(G)$ and $\gamma_{tstd}^c(G) = \gamma_t(G)$. Hence (i).

Let $\chi^c(G) \geq 2$. Then note that any transversal of the $\chi^c$–Partition of $G$ will be a total dominating set of $G$ and hence also dominating set of $G$. So $\chi^c(G) \leq \gamma_{std}^c(G) \leq \gamma_{tstd}^c(G) = \chi^c(G)$. Hence $\gamma_{std}^c(G) = \gamma_{tstd}^c(G) = \chi^c(G)$. Hence (ii).
COROLLARY 6.2.0: For any graph $G = (V, E)$, $\gamma_{std}^c(G) = \max \{\gamma(G), \chi^c(G)\}$ and $\gamma_{tstd}^c(G) = \max \{\gamma_t(G), \chi^c(G)\}$.

THEOREM 6.2.1: Let $G = (V, E)$ be a graph of order $n$. If $G$ has two or more support vertices then $\chi^c(G) = 1$.

PROOF: Let $u$ and $v$ be two distinct support vertices of $G$. Let $x$ and $y$ be two distinct pendant vertices of $G$ that are adjacent to $u$ and $v$ respectively. Clearly $n \geq 4$. As $x$ is not adjacent to any vertex in $V \setminus \{u\}$, all the vertices in $V \setminus \{u\}$ must be in same color class with $x$. Also as $y$ is not adjacent to $u$, color class of $y$ and $u$ must be same. Therefore $u$ is also in same color class with $x$. Hence all the vertices are in one color class. Hence $\chi^c(G) = 1$.

COROLLARY 6.2.2: If a graph $G = (V, E)$ has two or more support vertices then $\gamma_{std}^c(G) = \gamma(G)$ and $\gamma_{tstd}^c(G) = \gamma_t(G)$.

REMARK 6.2.3: Converse of above theorem is not true, in general. For example consider cycle graph $G = C_5$. $C_5$ does not have any support vertex, yet $\chi^c(G) = 1$.

THEOREM 6.2.4 [2]: Let $G$ be a graph of order $n$ with no isolated vertices. Then $\gamma(G) \leq \frac{n}{2}$.

THEOREM 6.2.5: Let $G = (V, E)$ be a graph of order $n$. Then $\gamma_{std}^c(G) = n$ if and only if $\chi^c(G) = n$.

PROOF: Assume that $\gamma_{std}^c(G) = n$. If $\chi^c(G) \geq 2$ then $\gamma_{std}^c(G) = n = \chi^c(G)$ and we are done.

Claim: $\chi^c(G) \neq 1$. 

120
Suppose $\chi^C(G) = 1$. Then $\gamma_{\text{std}}^C(G) = \max \{\gamma^C(G), \chi^C(G)\} = \gamma(G) = n$, which is a contradiction to theorem 6.2.4, $\gamma(G) \leq \frac{n}{2} < n$. So $\chi^C(G) \neq 1$.

Hence we have $\chi^C(G) = n$.

Converse is obvious.

**THEOREM 6.2.6:** Let $G = (V, E)$ be a graph of order $n$. Then $\gamma_{\text{std}}^C(G) = n - 1$ if and only if $\chi^C(G) = n - 1$.

**PROOF:** The proof is analogous to the proof of above theorem.

**THEOREM 6.2.7:** Let $G = (V, E)$ be a graph of order $n$. If $\chi^C(G) = n - 2$ then $\gamma_{\text{std}}^C(G) = n - 2$.

**REMARK 6.2.8:** Converse of above theorem is not true, in general. For example consider $G = P_4$, the path graph with four vertices. $\gamma_{\text{std}}^C(G) = 2 = n - 2$ but $\chi^C(G) = 1 \neq 2 = n - 2$.

**THEOREM 6.2.9:** Let $G = (V, E)$ be a graph of order $n \geq 3$. Then $\gamma_{\text{std}}^C(G) = n$ if and only if $\chi^C(G) = n$.

**PROOF:** Assume that $\gamma_{\text{std}}^C(G) = n$. If $\chi^C(G) \geq 2$ then $\gamma_{\text{std}}^C(G) = n = \chi^C(G)$ and we are done.

Claim: $\chi^C(G) \neq 1$.

Suppose $\chi^C(G) = 1$. Then $\gamma_{\text{std}}^C(G) = \max \{\gamma^C(G), \chi^C(G)\} = \gamma^C(G) = n$, which is contradiction to theorem 3.4.9, $\gamma^C(G) \leq \frac{2n}{3} < n$. So $\chi^C(G) \neq 1$.

Hence we have $\chi^C(G) = n$.

Converse is obvious.
REMARK 6.3.0: If G = (V, E) is a graph of order 2 then G = P_2, the path graph with two vertices and hence γ_{tstd}^C(G) = 2 = χ^C(G).

THEOREM 6.3.1: Let G = (V, E) be a graph of order n ≥ 3. Then γ_{tstd}^C(G) = n - 1 if and only if χ^C(G) = n - 1.

PROOF: The proof is analogous to the proof of above theorem 6.2.9.

THEOREM 6.3.2: Let G = (V, E) be a graph of order n. If χ^C(G) = n - 2 then γ_{tstd}^C(G) = n - 2.

REMARK 6.3.3: Converse of above theorem is not true, in general. For example consider the graph G = P_4, the path graph with four vertices. γ_{tstd}^C(G) = 2 = n - 2 but χ^C(G) = 1 ≠ 2 = n - 2.

6.4 CONCLUDING REMARKS: By the results it seems that both dominating complementary color transversal number and total dominating complementary color transversal number loses their identity as they are actually domination number, total domination number or complementary chromatic number of a graph. This is due to the fact that for the graphs with complementary chromatic number greater than or equal to two, a transversal itself becomes dominating / total dominating set. This property is actually eye-catching. It was our effort to relate these two numbers with each other as well as with complementary chromatic number. This has actually produced amazing results.
LIST OF SYMBOLS

- $G = (V, E)$: A graph with vertex $V$ and edge set $E$
- $V (G)$: Vertex set of a graph $G$
- $E (G)$: Edge set of a graph $G$
- $| V |$: Cardinality of a set $V$
- $\deg (v)$: Degree of a vertex $v$
- $N (v)$: Open neighbourhood of a vertex $v$
- $N [v]$: Closed neighbourhood of a vertex $v$
- $\delta (G)$ or $\delta$: $\min \{ \deg (v): v \text{ is a vertex of } G \}$
- $\Delta (G)$ or $\Delta$: $\max \{ \deg (v): v \text{ is a vertex of } G \}$
- $P_n$: Path graph with $n$ vertices
- $C_n$: Cycle graph with $n$ vertices
- $K_n$: Complete graph with $n$ vertices
- $W_n$: Wheel graph with $n$ vertices
- $d_G (u, v)$ or $d (u, v)$: Distance between vertices $u$ and $v$ of a graph $G$
- $e (v)$: Eccentricity of a vertex $v$
- $\text{rad} (G)$: Radius of a graph $G$
- $\text{Diam} (G)$: Diameter of a graph $G$
- $<S>$: The induced sub graph of $G$ whose vertex set is $S$ and whose edge set is the subset of $E(G)$ consisting of those edges with both end vertices in $S$.
- $G \times H$: Kronecker product of graphs $G$ and $H$
- $G \Box H$: Cartesian product of graphs $G$ and $H$
- $G \boxtimes H$: Strong product of graphs $G$ and $H$
- $G[H]$: Lexicographic product of graphs $G$ and $H$
- $G \vee H$: Join of graphs $G$ and $H$
- $G \odot H$: Corona of graphs $G$ and $H$
- $\overline{G}$: Complement graph of a graph $G$
- $\omega (G)$ or $\omega$: Clique number of a graph $G$
<table>
<thead>
<tr>
<th>Symbol/Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma - \text{Set}$</td>
<td>Minimum dominating set of a graph</td>
</tr>
<tr>
<td>$\gamma (G)$ or $\gamma$</td>
<td>Domination number of a graph G</td>
</tr>
<tr>
<td>$\gamma_t - \text{Set}$</td>
<td>Minimum total dominating set of a graph</td>
</tr>
<tr>
<td>$\gamma_t (G)$ or $\gamma_t$</td>
<td>Total domination number of a graph G</td>
</tr>
<tr>
<td>$\chi (G)$ or $\chi$</td>
<td>Chromatic number of a graph G</td>
</tr>
<tr>
<td>$\gamma_{\text{std}} - \text{Set}$</td>
<td>Minimum dominating color transversal set of a graph</td>
</tr>
<tr>
<td>$\gamma_{\text{std}} (G)$ or $\gamma_{\text{std}}$</td>
<td>Dominating color transversal number of a graph G</td>
</tr>
<tr>
<td>$\gamma_{t\text{std}} - \text{Set}$</td>
<td>Minimum total dominating color transversal set of a graph</td>
</tr>
<tr>
<td>$\gamma_{t\text{std}} (G)$ or $\gamma_{t\text{std}}$</td>
<td>Total dominating color transversal number of a graph G</td>
</tr>
<tr>
<td>$\gamma_{tg} (G)$</td>
<td>Total global domination number of a graph G</td>
</tr>
<tr>
<td>$G^k$</td>
<td>Spanning super graph of a graph G obtained by adding the edges between vertices u and v of G when $d(u, v) \leq k$.</td>
</tr>
<tr>
<td>$W_{p+q}$</td>
<td>Generalised wheel graph</td>
</tr>
<tr>
<td>$H_n$</td>
<td>Helm graph</td>
</tr>
<tr>
<td>$P(n, 1)$</td>
<td>Generalised Petersen graph</td>
</tr>
<tr>
<td>$\equiv$</td>
<td>Congruence relation</td>
</tr>
<tr>
<td>$G \setminus {v}$</td>
<td>Sub graph of G obtained by removing a vertex ‘v’ from G</td>
</tr>
<tr>
<td>$G \setminus e$</td>
<td>Sub graph of G obtained by removing an edge ‘e’ from G.</td>
</tr>
<tr>
<td>$G + e$</td>
<td>Super graph of G obtained by adding edge e between two non adjacent vertices of G</td>
</tr>
<tr>
<td>$W^i$</td>
<td>${v \in V / G \setminus {v} \text{ is disconnected}}$</td>
</tr>
<tr>
<td>$E^i$</td>
<td>${e \in E / G \setminus e \text{ is disconnected}}$</td>
</tr>
<tr>
<td>$V^i$</td>
<td>${v \in V / G \setminus {v} \text{ has an isolated vertex}}$</td>
</tr>
<tr>
<td>$V^0_{t\text{std}}$</td>
<td>${v \in V / \gamma_{t\text{std}} (G \setminus {v}) = \gamma_{t\text{std}} (G)}$</td>
</tr>
<tr>
<td>$V^-_{t\text{std}}$</td>
<td>${v \in V / \gamma_{t\text{std}} (G \setminus {v}) &lt; \gamma_{t\text{std}} (G)}$</td>
</tr>
<tr>
<td>$V^+_{t\text{std}}$</td>
<td>${v \in V / \gamma_{t\text{std}} (G \setminus {v}) &gt; \gamma_{t\text{std}} (G)}$</td>
</tr>
<tr>
<td>$\chi^T (v, S, \Pi)$</td>
<td>Total chromatic neighbourhood of a vertex v with respect to total dominating color transversal set S that is a transversal some $\chi$ - Partition $\Pi$ of the graph.</td>
</tr>
</tbody>
</table>
- $\chi_c^c(G)$, $\chi^c$ or $cchr(G)$: Complementary chromatic number of a graph $G$
- $\gamma_{std}^c$: Minimum dominating complementary color transversal set of a graph
- $\gamma_{std}^c(G)$ or $\gamma_{std}^c$: Dominating complementary color transversal number of a graph $G$
- $\gamma_{tstd}^c$: Minimum total dominating complementary color transversal set of a graph
- $\gamma_{tstd}^c(G)$ or $\gamma_{tstd}^c$: Total dominating complementary color transversal number of a graph $G$