4.1 INTRODUCTION: In this chapter, we compute the operation of removing a vertex and analyze the effect of this operation on $\gamma_{tstd}$. As we are dealing with Total Domination theory, we remove only those vertices whose removal does not yield isolated vertices in the resultant sub graph.

We have divided this chapter into three sections. In the first section, we compute the operation of removing a vertex ‘$v$’ having the property that $\{v\}$ is a color class every $\chi$ - Partition of a graph. In the second section, we compute the operation of removing a vertex ‘$v$’ having the property that $\{v\}$ is not a color class of any $\chi$ - Partition of a graph. In the third section, we pose conjectures.

Before going further we introduce the following notations:

**Notations:** Let $G = (V, E)$ be a graph.

1. $V^i = \{v \in V / G \setminus \{v\} \text{ has an isolated vertex}\}$

2. $V^0_{tstd} = \{v \in V/ \gamma_{tstd} (G \setminus \{v\}) = \gamma_{tstd} (G)\}$

3. $V^-_{tstd} = \{v \in V/ \gamma_{tstd} (G \setminus \{v\}) < \gamma_{tstd} (G)\}$

4. $V^+_{tstd} = \{v \in V/ \gamma_{tstd} (G \setminus \{v\}) > \gamma_{tstd} (G)\}$

Clearly, all these sets are mutually disjoint and their union is the vertex set $V$ of $G$. 

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4.2 REMOVING A VERTEX ‘v’ HAVING THE PROPERTY THAT \{v\} IS A COLOR CLASS OF EVERY \chi - PARTITION OF A GRAPH:

We first state some preliminary results.

**RESULT 4.1.1:** Let \( G = (V, E) \) be a graph and \( v \in V \). Then \( \chi (G \setminus \{v\}) \leq \chi (G) \).

**RESULT 4.1.2:** Let \( G = (V, E) \) be a graph and \( v \in V \). If \( \chi (G \setminus \{v\}) < \chi (G) \) then \( \chi (G \setminus \{v\}) + 1 = \chi (G) \).

Now we mention one important theorem about a vertex ‘v’ having the property that \{v\} is color class of every \chi - Partition of G.

**THEOREM 4.1.3:** Let \( G = (V, E) \) be a graph of order \( n \) and \( v \in V \). Then \{v\} is a color class of every \chi - Partition of G if and only if \( \deg (v) = n - 1 \).

**PROOF:** Assume \{v\} is a color class of every \chi - Partition of G. Suppose \( \deg (v) \neq n - 1 \). Then \( v \) is not adjacent to some vertex \( u \). So \{u, v\} is independent set in G. Therefore \{v\} is not maximal independent set of G. Hence by theorem 3.2.8, \{v\} is not a color class of some \chi - Partition of G, which is a contradiction. Hence \( \deg (v) = n - 1 \).

Conversely, assume \( \deg (v) = n - 1 \). If \{v\} is not a color class of some \chi - Partition of G then there exists a vertex with which \( v \) is not adjacent, which is again a contradiction. Hence \{v\} is a color class of every \chi - Partition of G.

Now let us see some examples which shows that this number, \( \gamma_{\text{tstd}} \), may remain same, increase or decrease by removing a vertex of degree \( n - 1 \).( \( n \) is the order of the graph G)

**EXAMPLE 4.1.4:** For the given graph G, in fig. 4.1, \( \gamma_{\text{tstd}} (G \setminus \{u_3\}) = \gamma_{\text{tstd}} (G) \)

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EXAMPLE 4.1.5: For the below given graph $G$, $\gamma_{tstd}(G \setminus \{u_7\}) > \gamma_{tstd}(G)$.

EXAMPLE 4.1.6: For the given graph $G$, in fig. 4.5, $\gamma_{tstd}(G \setminus \{u_7\}) < \gamma_{tstd}(G)$. 
THEOREM 4.1.7: Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^i$ and $\{v\}$ is a color class of every $\chi$-Partition of $G$. If $v \in V_{\text{tstd}}$ then $\gamma_{\text{tstd}}(G \setminus \{v\}) + 1 = \gamma_{\text{tstd}}(G)$.

PROOF: Assume $v \in V_{\text{tstd}}$.

As $\{v\}$ is a color class of every $\chi$-Partition of $G$, $\chi(G \setminus \{v\}) + 1 = \chi(G)$. Let $D$ be a $\gamma_{\text{tstd}}$-Set of $G \setminus \{v\}$ for the $\chi$ - 1 - Partition $\{V_1, V_2, \ldots, V_{\chi-1}\}$ of $G \setminus \{v\}$. Note that $\deg(v) = n - 1$. So $D$ is a total dominating set of $G$. Also $D \cup \{v\}$ is a total dominating color transversal set of $G$ for the $\chi$ - Partition $\{\{v\}, V_1, V_2, \ldots, V_{\chi-1}\}$ of $G$. So $\gamma_{\text{tstd}}(G) \leq \gamma_{\text{tstd}}(G \setminus \{v\}) + 1$. Therefore as $\gamma_{\text{tstd}}(G \setminus \{v\}) < \gamma_{\text{tstd}}(G)$, we have $\gamma_{\text{tstd}}(G \setminus \{v\}) + 1 = \gamma_{\text{tstd}}(G)$.

Now we are at a stage where we can introduce our first theorem about necessary and sufficient condition under which $\gamma_{\text{tstd}}$ number for a graph decrease when a vertex, that is a color class of every $\chi$ - Partition of $G$, is removed.

THEOREM 4.1.8: Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^i$ and $\{v\}$ is a color class of every $\chi$ - Partition of $G$. Then $v \in V_{\text{tstd}}$ if and only if there exists a total
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dominating set, say \( D \), of \( G \setminus \{v\} \) and a \( \chi - 1 \) - Partition \( \{W_1, W_2, \ldots, W_{\chi-1}\} \) of \( G \setminus \{v\} \) such that \( |D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1 \).

PROOF: Assume \( v \in V_{\text{tsd}}^- \).

As \( \{v\} \) is a color class of every \( \chi \) - Partition of \( G \), \( \chi_t (G) = 2 \). By theorem 1.2.7, \( \gamma_{\text{tsd}} (G) = \chi(G) \). Then by theorem 2.1.5, there exists a total dominating set \( S \) of \( G \) and a \( \chi \) - Partition \( \{V_1, V_2, \ldots, V_\chi\} \) of \( G \) such that \( |S \cap V_i| = 1, \forall i = 1, 2, \ldots, \chi \).

Let \( \{v\} = V_1 \) (without loss of generality). Now \( v \in V_{\text{tsd}}^- \), \( \gamma_{\text{tsd}} (G \setminus \{v\}) + 1 = \gamma_{\text{tsd}}(G) \) (by theorem 4.1.7). So \( \gamma_{\text{tsd}} (G \setminus \{v\}) + 1 = \chi(G) \). Therefore \( \chi(G \setminus \{v\}) = \chi(G) - 1 = \gamma_{\text{tsd}}(G \setminus \{v\}) \). Then by theorem 2.1.5, there exists a total dominating set \( D \) of \( G \setminus \{v\} \) and a \( \chi - 1 \) - Partition \( \{W_1, W_2, \ldots, W_{\chi-1}\} \) of \( G \setminus \{v\} \) such that \( |D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1 \).

Conversely, assume that there exists a total dominating set, say \( D \), of \( G \setminus \{v\} \) and a \( \chi - 1 \) - Partition \( \{W_1, W_2, \ldots, W_{\chi-1}\} \) of \( G \setminus \{v\} \) such that \( |D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1 \). Then \( D \) becomes total dominating color transversal set of \( G \setminus \{v\} \). So \( \gamma_{\text{tsd}}(G \setminus \{v\}) \leq \chi(G \setminus \{v\}) \). But \( \chi(G \setminus \{v\}) \leq \gamma_{\text{tsd}}(G \setminus \{v\}) \). Therefore \( \gamma_{\text{tsd}}(G \setminus \{v\}) = \chi(G \setminus \{v\}) < \chi(G) = \gamma_{\text{tsd}}(G) \). Hence \( v \in V_{\text{tsd}}^- \).

THEOREM 4.1.9: Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \not\in V^i \) and \( \{v\} \) is a color class of every \( \chi \) - Partition of \( G \). Then \( v \in V_{\text{tsd}}^0 \) if and only if following two conditions are satisfied:

(1) There exists a total dominating set, say \( D \), of \( G \setminus \{v\} \) and a \( \chi - 1 \) - Partition \( \{V_1, V_2, \ldots, V_{\chi-1}\} \) of \( G \setminus \{v\} \) such that \( |D \cap V_i| = 1, \forall i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1 \) and \( |D \cap V_j| = 2 \), for some \( j \neq i \).

(2) \( \gamma_{\text{tsd}}(G \setminus \{v\}) > \chi(G \setminus \{v\}) \).

PROOF: As \( \{v\} \) is a color class of every \( \chi \) - Partition of \( G \), \( \chi_t (G) = 2 \). By theorem 1.2.7, \( \gamma_{\text{tsd}}(G) = \chi(G) \). Also \( \chi(G \setminus \{v\}) = \chi(G) - 1 \).
Assume $v \in V_{tstd}^0$. So $\chi(G) = \gamma_{tstd} (G) = \gamma_{tstd} (G \setminus \{v\})$. So there exists a total dominating set, say $D$, of $G \setminus \{v\}$ and a $\chi$ - 1 - Partition $\{V_1, V_2, \ldots, V_{\chi-1}\}$ of $G \setminus \{v\}$ such that $|D \cap V_i| = 1$, $\forall$ $i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1$ and $|D \cap V_j| = 2$, for some $j \neq i$. So (1) holds. Also, as $\gamma_{tstd} (G \setminus \{v\}) = \gamma_{tstd} (G) = \chi(G) > \chi(G \setminus \{v\})$. So (2) holds.

Conversely, assume (1) and (2). Then $\chi(G) - 1 = \chi(G \setminus \{v\}) < \gamma_{tstd} (G \setminus \{v\}) \leq |D| = \chi(G)$. So $\gamma_{tstd} (G \setminus \{v\}) = \chi(G) = \gamma_{tstd} (G)$. Hence $v \in V_{tstd}^0$.

**COROLLARY 4.2.0:** Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^I$ and $\{v\}$ is a color class of every $\chi$ - Partition of $G$. Then $v \in V_{tstd}^0$ if and only if there exists a $\gamma_{tstd}$ - Set, say $D$, of $G \setminus \{v\}$ and a $\chi$ - 1 partition $\{V_1, V_2, \ldots, V_{\chi-1}\}$ of $G \setminus \{v\}$ such that $|D \cap V_i| = 1$, $\forall$ $i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1$ and $|D \cap V_j| = 2$, for some $j \neq i$.

**THEOREM 4.2.1:** Let $G = (V, E)$ be a graph and $v, u \in V$ with $v \notin V^I$. If $\{v\}$ is not a color class of any $\chi$ - Partition of $G$ and $\{u\}$ is a color class of every $\chi$ - Partition of $G$ then $v \in V_{tstd}^0$ with $\gamma_{tstd} (G \setminus \{v\}) = \gamma_{tstd} (G) = \chi(G)$.

**PROOF:** As $\{u\}$ is a color class of every $\chi$ - Partition of $G$, $\gamma_t (G) = 2$. By theorem 1.2.7, $\gamma_{tstd} (G) = \chi(G)$. Also $\chi(G \setminus \{v\}) = \chi(G)$, by theorem 3.5.3. Note that $\gamma_t (G \setminus \{v\}) = 2$. So $\gamma_{tstd} (G \setminus \{v\}) = \chi(G \setminus \{v\}) = \chi(G) = \gamma_{tstd} (G)$ and hence $v \in V_{tstd}^0$.

**EXAMPLE 4.2.2:** For the given graph $G$, $\gamma_{tstd}(G) = \gamma_{tstd}(G \setminus \{v\}) = 3 = \chi(G)$.

![Fig. 4.7](image1)

![Fig. 4.8](image2)
REMARK 4.2.3: Converse of above theorem 4.2.1 is not true, in general. Take an example of graph $C_4$ in which no vertex is a color class any $\chi$ - Partition of $G$ but still $\gamma_{tstd}(G) = \gamma_{tstd}(G \setminus \{v\}) = \chi(G) = 2$.

We end this section of removing a vertex of degree $n - 1$ by given following examples that reflects the fact that the total dominating color transversal number of a graph may decrease by removing any vertex of the graph. Also we provide an example of a graph for which this number may increase by more than 1 by removing such vertex.

EXAMPLE 4.2.4: Consider $G = K_n$, ($n \geq 3$) ($K_n$ is a complete graph with n vertices). In this graph every vertex has degree $n - 1$. Clearly, $\gamma_{tstd}(G \setminus \{v\}) < \gamma_{tstd}(G), \forall v \in V$.

EXAMPLE 4.2.5: Consider $G = W_{13}$ (The wheel graph with 13 vertices). Then $\gamma_{tstd}(G) = 3$ and $\gamma_{tstd}(G \setminus \{v\}) = 6$, where $v$ is the hub vertex. Hence $\gamma_{tstd}$ number may increase by more than 1 by removal of a vertex.

4.3 REMOVING A VERTEX ‘$v$’ HAVING THE PROPERTY THAT \{v\} IS NOT A COLOR CLASS OF ANY $\chi$ - PARTITION OF A GRAPH:

We first of all provide examples that reflects the fact that $\gamma_{tstd}$ may decrease, increase or remain same by removing such a vertex.

EXAMPLE 4.2.6: For the given graph $G$, in fig. 4.9, $\gamma_{tstd}$ decreases by removing vertex $u_9$. 
$G$

Fig. 4. 9

$\gamma_{tstd} (G) = 5$

$G \setminus \{u_9\}$

Fig. 4.10

$\gamma_{tstd} (G \setminus \{u_9\}) = 4$
EXAMPLE 4.2.7: For the given graph $G$, in Fig. 4.11, $\gamma_{\text{tstd}}$ increases by removing a vertex $u_7$.

$G$

Fig. 4.11

$\gamma_{\text{tstd}}$ - Set of $G$ is $\{u_4, u_5, u_7\}$ and hence $\gamma_{\text{tstd}} (G) = 3$

$G \setminus \{u_7\}$

Fig. 4.12

$\gamma_{\text{tstd}}$ - Set of $G \setminus \{u_7\}$ is $\{u_2, u_3, u_4, u_5\}$ and hence $\gamma_{\text{tstd}} (G \setminus \{u_7\}) = 4$

EXAMPLE 4.2.8: For the graph $G = P_4$ (a path graph with four vertices), we know that $\gamma_{\text{tstd}} (G) = 2$. Note that if $v$ is a pendant vertex of $G$ then $G \setminus \{v\}$ is a path graph with three vertices and $\gamma_{\text{tstd}} (G \setminus \{v\}) = 2$. Hence $\gamma_{\text{tstd}} (G) = \gamma_{\text{tstd}} (G \setminus \{v\})$. 

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It is to be noted that in examples 4.2.6, 4.2.7 and 4.2.8 we have removed only those vertices \( w \) such that \( \{w\} \) is not a color classes of any \( \chi \) - Partition of the respective graphs.

Let us proceed towards our first theorem about decrement of total dominating color transversal number by removal of such a vertex ‘\( v \)’ having the property that \( \{v\} \) is not a color class of any \( \chi \) - Partition of \( G \).

Throughout this section we remove only those vertices whose degree is less than the chromatic number of the graph.

First we mention the following definition.

**DEFINITION 4.2.9: (TOTAL CHROMATIC NEIGHBOURHOOD OF A VERTEX)** Let \( G = (V, E) \) be a graph. Let \( S \) be a total dominating color transversal set of \( G \) for some \( \chi \)- Partition \( \Pi = \{V_1, V_2, \ldots, V_\chi\} \) of \( G \) and \( v \in S \) with \( v \in V_i \), for some \( V_i \in \Pi \). Then the total chromatic neighbourhood of \( v \) with respect to \( S \) and \( \Pi \) is denoted and defined as \( \chi^T(v, S, \Pi) = \{u \in S/ u \in V_i \text{ and } S \setminus \{u\} \text{ does not have an isolate vertex}\} \).

We here note that \( v \) may not belong to \( \chi^T(v, S, \Pi) \).

**THEOREM 4.3.0:** Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \notin V^i \), \( \{v\} \) is not a color class of any \( \chi \)- Partition of \( G \) and \( \deg(v) < \chi(G) \). Then \( v \in V_{tstd}^- \) if and only if the following two conditions are satisfied:

1. There exists a \( \gamma_{tstd} \)- Set \( S \) of \( G \) not containing \( v \).
2. There exists \( u \in N(v) \) such that \( u \in \chi^T(w, S, \Pi) \), for some vertex \( w \neq u \) and \( \chi \)- Partition \( \Pi \) of \( G \) for which \( S \) is a transversal.

**PROOF:** As \( \{v\} \) is not a color class of any \( \chi \)- Partition of \( G \), \( \chi(G \setminus \{v\}) = \chi(G) \). (by theorem 3.5.3)

Assume \( v \in V_{tstd}^- \).

Let \( D \) be a \( \gamma_{tstd} \)- Set of \( G \setminus \{v\} \) for some \( \chi \)- Partition \( \Pi \) = \{ \( V_1 \), \( V_2 \), \ldots, \( V_\chi \) \} of \( G \setminus \{v\} \).
As \( \text{deg} (v) < \chi (G) \), \( \{v\} \) cannot meet all the color classes of \( G \setminus \{v\} \). So there exists a color class, say \( V_i \), of \( G \setminus \{v\} \) such \( \{v\} \cap V_i = \emptyset \). Then \( D \) is a transversal of \( G \) of the \( \chi \) – Partition \( \Pi = \{ V_1 \cup \{v\}, V_2, \ldots, V_{\chi} \} \) of \( G \). Note that \( v \notin D \). Also \( D \) do not contain any neighbour of \( v \), for otherwise \( \gamma_{\text{std}} (G) \leq \gamma_{\text{std}} (G \setminus \{v\}) \). Trivially, \( S = D \cup \{u\} \) is the total dominating color transversal set of \( G \) for some \( u \in N (v) \). So \( \gamma_{\text{std}} (G) \leq \gamma_{\text{std}} (G \setminus \{v\}) + 1 \). Therefore by \( \gamma_{\text{std}} (G \setminus \{v\}) < \gamma_{\text{std}} (G) \), we have \( \gamma_{\text{std}} (G \setminus \{v\}) + 1 = \gamma_{\text{std}} (G) \). Hence \( S \) is a \( \gamma_{\text{std}} \) - Set of \( G \) not containing \( v \). Also note that \( u \in N (v) \) such that \( u \in \chi^T (w, S, \Pi) \), for some vertex \( w \neq u \) and \( \chi \) – Partition \( \Pi \) of \( G \) for which \( S \) is a transversal. Hence (1) and (2).

Conversely, assume (1) and (2).

Let \( D = S \setminus \{u\} \).

Claim: \( D \) is a total dominating color transversal set of \( G \setminus \{v\} \).

Note that \( D \) is a transversal of \( G \setminus \{v\} \). Also \( D \) has no isolates.

Let \( x \in V \setminus \{v\} \).

If \( x = u \) then as \( S \) is a total dominating set of \( G \) there exists \( y \in S \setminus \{u\} = D \), such that \( y \) is adjacent to \( x \). So let us assume \( x \neq u \). If there does not exist any \( y \in S \setminus \{u\} = D \) such that \( x \) is adjacent to \( y \) then \( x \) is adjacent to \( u \) only in \( S \). Hence \( D = S \setminus \{u\} \) has isolates, which is a contradiction. Hence there exists some \( y \in S \setminus \{u\} = D \) such that it is adjacent to \( x \). Therefore \( D \) is total dominating color transversal set of \( G \setminus \{v\} \). Therefore we have \( \gamma_{\text{std}} (G \setminus \{v\}) \leq |D| < |S| = \gamma_{\text{std}} (G) \). Hence \( v \in V_{\text{std}}^{-} \).

**COROLLARY 4.3.1:** Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \notin V^I \), \( \{v\} \) is not a color class of any \( \chi \) - Partition of \( G \) and \( \text{deg} (v) < \chi (G) \). If \( v \in V_{\text{std}}^{-} \) then \( \gamma_{\text{std}} (G \setminus \{v\}) + 1 = \gamma_{\text{std}} (G) \).

**PROOF:** It is obvious by the construction of set \( S \), in the ‘if’ part, in above theorem 4.3.0.
EXAMPLE 4.3.2:

\[ G = P_5 \]

Fig. 4.13

\[ S = \gamma_{\text{tstd}} \text{ - Set of } G = \{u, v_1, w\} \text{ and } u \in \chi^T(w, S, \Pi). \]
\[ D = \gamma_{\text{tstd}} \text{ - Set of } G \setminus \{v\} = \{v_1, w\}. \]

Hence \( \gamma_{\text{tstd}} (G) = 3 \) and \( \gamma_{\text{tstd}} (G \setminus \{v\}) = 2. \) Note that \( \deg (v) = 1 < \chi (G) = 2. \)

THEOREM 4.3.3: Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \notin V^i \) and \( \{v\} \) is not a color class of any \( \chi \) - Partition of \( G. \) If \( v \in V^+_{\text{tstd}} \) then the following two conditions are satisfied:

(1) Every \( \gamma_{\text{tstd}} \) - Set of \( G \) contains \( v. \)

(2) If \( S \subset V \setminus N[v] \) such that \( |S| = \gamma_{\text{tstd}} (G), \) then \( S \) is not a total dominating color transversal set of \( G \setminus \{v\}. \)

PROOF: As \( \{v\} \) is not a color class of any \( \chi \) - Partition of \( G, \chi (G \setminus \{v\}) = \chi (G). \) In such case, a total dominating color transversal set, not containing \( v, \) of \( G \) is also transversal of some \( \chi \) - Partition of \( G \setminus \{v\}. \)

Assume \( v \in V^+_{\text{tstd}}. \)

Suppose (1) is not true. Then there exists a \( \gamma_{\text{tstd}} \) - Set, say \( D, \) of \( G \) not containing \( v. \) Then \( D \) is also a total dominating color transversal set of \( G \setminus \{v\}. \) Hence we have \( \gamma_{\text{tstd}} (G \setminus \{v\}) \leq \gamma_{\text{tstd}} (G), \) which is a contradiction. So (1) is true.

Suppose (2) is not true. If \( S \subset V \setminus N[v] \) such that \( |S| = \gamma_{\text{tstd}} (G), \) then \( S \) is a total dominating color transversal set of \( G \setminus \{v\}. \) So \( \gamma_{\text{tstd}} (G \setminus \{v\}) \leq \gamma_{\text{tstd}} (G) = |S|, \) which is a contradiction. So (2) is true.

THEOREM 4.3.4: Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \notin V^i \) and \( \{v\} \) is not a color class of any \( \chi \) - Partition of \( G \) and \( \deg (v) < \chi (G). \) Then \( v \in V^+_{\text{tstd}} \) if the following two conditions are satisfied:
(1) Every $\gamma_{tstd} - \text{Set of } G$ contains $v$.

(2) If $S \subseteq V \setminus N[v]$ such that $|S| = \gamma_{tstd} (G)$, then $S$ is not a total dominating color transversal set of $G \setminus \{v\}$.

**PROOF**: Assume (1) and (2).

It is enough to show that $v \notin V^0_{tstd}$ and $v \notin V^0_{tstd}$.

If $v \notin V^{-}_{tstd}$ then by theorem 4.3.1, there exists a $\gamma_{tstd}$-Set of $G$ not containing $v$, which is a contradiction. So let us assume that $v \notin V^0_{tstd}$. Then $\gamma_{tstd} (G \setminus \{v\}) = \gamma_{tstd} (G)$. Let $D$ be a $\gamma_{tstd}$-Set of $G \setminus \{v\}$. Then $v \notin D$. Note that $|D| = \gamma_{tstd} (G)$. So if $D \subseteq V \setminus N[v]$ then by (2), $D$ cannot be a total dominating color transversal set of $G \setminus \{v\}$. So there exists $u \in N(v)$ such that $u \notin D$. So $D$ is a total dominating set of $G$. Also $\deg(v) < \chi(G)$, then $\{v\}$ cannot meet all the color classes of $G$. $D$ is a transversal of some $\chi$-Partition of $G$. Therefore $D$ is a total dominating color transversal set of $G$. Hence $D$ is a $\gamma_{tstd}$-Set of $G$ as $|D| = \gamma_{tstd} (G)$, which is a contradiction to (1) as $v \notin D$. Therefore $v \notin V^0_{tstd}$. Hence the theorem.

**THEOREM 4.3.5**: Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^i$ and $\{v\}$ is not a color class of any $\chi$-Partition of $G$ and $\deg(v) < \chi(G)$. Then $v \in V^+_{tstd}$ if and only if the following two conditions are satisfied:

(1) Every $\gamma_{tstd} - \text{Set of } G$ contains $v$.

(2) If $S \subseteq V \setminus N[v]$ such that $|S| = \gamma_{tstd} (G)$ then $S$ is not a total dominating color transversal Set of $G \setminus \{v\}$.

**PROOF**: Obvious by theorem 4.3.3 and theorem 4.3.4.
EXAMPLE 4.3.6:

\begin{center}
\begin{tikzpicture}
\node[circle, draw, fill=black!50, inner sep=2pt, label=above:2] (v) at (0,0) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=left:1] (u1) at (-1,-1) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=right:3] (u3) at (1,-1) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=below:1] (u4) at (-1,-2) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=right:2] (u5) at (1,-2) {};
\draw (v) -- (u1);
\draw (v) -- (u3);
\draw (u1) -- (u4);
\draw (u1) -- (u5);
\draw (u3) -- (u5);
\end{tikzpicture}
\end{center}

\text{Fig. 4.14} \quad G

\begin{center}
\begin{tikzpicture}
\node[circle, draw, fill=black!50, inner sep=2pt, label=above:2] (v) at (0,0) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=left:1] (u1) at (-1,-1) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=right:3] (u3) at (1,-1) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=below:1] (u4) at (-1,-2) {};
\node[circle, draw, fill=black!50, inner sep=2pt, label=right:2] (u5) at (1,-2) {};
\draw (v) -- (u1);
\draw (v) -- (u3);
\draw (u1) -- (u4);
\draw (u2) -- (u5);
\draw (u3) -- (u5);
\end{tikzpicture}
\end{center}

\text{Fig. 4.15} \quad G \setminus \{v\}

\[ \gamma_{\text{std}}(G) = 3 \text{ and } \gamma_{\text{std}}(G \setminus \{v\}) = 4 \]

We end this section by providing an example of a graph in which removal of any vertex, keeps the total dominating color transversal number same.

EXAMPLE 4.3.7: For cycle graph \( G = C_4 \), \( \gamma_{\text{std}}(G \setminus \{v\}) = \gamma_{\text{std}}(G), \forall \ v \in V. \)

4.4 CONJECTURES: We found some results true on the basis of speculations and rigorous research work. But we do not have the proof. We pose these results as Conjectures. We provide examples that strengthens these conjectures.

CONJECTURE I: Let \( G = (V, E) \) be a graph and \( v \in V \) with \( v \notin V^i \). If \( v \in V_{\text{std}}^i \) then \( \gamma_{\text{std}}(G \setminus \{v\}) + 1 = \gamma_{\text{std}}(G) \).

EXAMPLE 4.3.8: Consider graph G in Fig. 3.7, Chapter 3.
In graph G,
\[ \Rightarrow u_4 \notin V^i. \]
\[ \Rightarrow \{u_4\} \text{ is a not a color class of any } \chi \text{- Partition of } G. \]
\[ \Rightarrow \deg(u_4) = 3 = \chi(G). \]
EXAMPLE 4.3.9:

In graph $G$,
$\Rightarrow u_6 \notin V^i$.
$\Rightarrow \{u_6\}$ is not a color class of any $\chi$ – Partition of $G$.
$\Rightarrow$ deg $(u_6) = 5 > 4 = \chi (G)$.
$\Rightarrow$ One can easily see that $\gamma_{tstd} (G \setminus \{u_6\}) = 4$. And $\gamma_{tstd} (G \setminus \{u_6\}) + 1 = 4 + 1 = 5 = \gamma_{tstd} (G)$. Here we note that in $G \setminus \{u_6\}$, assigning color ‘4’ to $u_3$ vertex and not changing colors of other vertices of $G \setminus \{u_6\}$, we obtain a $\gamma_{tstd} = \text{Set} \{u_1, u_2, u_3, u_5\}$ of $G \setminus \{u_6\}$.

CONJECTURE II: Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^i$, $\{v\}$ is not a color class of any $\chi$ - Partition of $G$. If $v \in V_{tstd}^-$ if and only if the following two conditions are satisfied:
(1) There exists a $\gamma_{tstd}$ - Set $S$ of $G$ not containing $v$. 

\[ \Rightarrow \gamma_{tstd} (G \setminus \{u_4\}) + 1 = 3 + 1 = 4 = \gamma_{tstd} (G). \]
There exists $u \in N(v)$ such that $u \in \chi^T(w, S, \Pi)$, for some vertex $w \neq u$ and $\chi$ – Partition $\Pi$ of $G$ for which $S$ is a transversal.

**EXAMPLE 4.4.0:**

In graph $G$,

- $v \notin V^i$.
- $\{v\}$ is not a color class of any $\chi$ – Partition of $G$.
- $\deg(v) = 3 = \chi(G)$.

$S = \gamma_{tstd}$ - Set of $G = \{v_1, v_2, u, w\}$ and $u \in \chi^T(w, S, \Pi)$. $D = \gamma_{tstd}$ - Set of $G \setminus \{v\} = \{v_1, u, w\}$. Hence $\gamma_{tstd}(G) = 4$ and $\gamma_{tstd}(G \setminus \{v\}) = 3$.

**CONJECTURE III:** Let $G = (V, E)$ be a graph and $v \in V$ with $v \notin V^i$ and $\{v\}$ is not a color class of any $\chi$ - Partition of $G$. Then $v \in V_{tstd}^+$ if and only if the following two conditions are satisfied:

1. Every $\gamma_{tstd} –$ Set of $G$ contains $v$.
2. If $S \subset V \setminus N[v]$ such that $|S| = \gamma_{tstd}(G)$, then $S$ is not a total dominating color transversal set of $G \setminus \{v\}$.
EXAMPLE 4.4.1:

In graph $G$,

$\Rightarrow v \not\in V^i$.

$\Rightarrow \{v\}$ is not a color class of any $\chi$–Partition of $G$.

$\Rightarrow \deg (v) = 3 = \chi (G)$.

$\Rightarrow \gamma_{tstd}(G) = 3$ and $\gamma_{tstd}(G \setminus \{v\}) = 4$.

Example 4.4.2:

In graph $G$,

$\Rightarrow v \not\in V^i$.

$\Rightarrow \{v\}$ is not a color class of any $\chi$–Partition of $G$. 
\[ \Rightarrow \deg (v) = 4 > 3 = \chi (G). \]
\[ \Rightarrow \gamma_{tstd}(G) = 3 \text{ and } \gamma_{tstd}(G \setminus \{v\}) = 4. \]

**CONJECTURE IV:** Let \( G = (V, E) \) be a graph and \( v, u \in V \) with \( v, u \notin V_i \). If \( v \in V_{tstd}^- \) and \( u \in V_{tstd}^+ \) then \( u \) and \( v \) cannot be adjacent.

**CONJECTURE V:** There is no graph \( G = (V, E) \) with \( |V| = |V_{tstd}^+| \).

**4.5 CONCLUDING REMARKS:** In section 4.3, we removed only those vertices of a graph whose degree is less than the chromatic number of the graph. For the discussion of the stability of total dominating color transversal number of a graph, in general, some more preliminaries about \( \chi \) – Partition of a graph are required to be developed and hence becomes an open problem. Some conjectures are posed whose proofs become open problem for any researcher.

In the penultimate chapter, we introduce and discuss about different properties of an absolutely new coloring of vertices of a graph called complementary coloring.