Bivariate Tailed Distributions

Chapter 6

Bivariate Tailed Distributions

6.1 Introduction

In real life we encounter a number of situations where the multivariate forms of
tailed distributions are applied for their modeling. For example, consider the study
of reliability of a two component system. Let the random variables $X$ and $Y$ repre-
sent lifetimes of the components. Suppose that the system will fail because of the
instantaneous failure of the components. In this situation, we can apply the bivariate
tailed distributions to study probability behavior of the random vector $(X,Y)$. Tail
of the non negative random vector $(X,Y)$ includes the positive part of the sample
space excluding the point $(X = 0, Y = 0)$. Therefore in bivariate tailed distributions,
we express the probability distribution of $(X,Y)$ as mixture of a probability $'\sigma'$ at
\((X = 0, Y = 0)\) and the remaining part with probability \(1 - \sigma\). In Section 2, we introduce the tailed form of \(\text{BML}(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution and develop autoregressive process with bivariate tailed Mittag-Leffler marginals. A discrete analogue of bivariate tailed Mittag-Leffler distribution is studied in Section 3 and the corresponding autoregressive model is developed.

6.2 Bivariate Tailed Mittag-Leffler Distribution

We define now the tailed form of \(\text{BML}(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution given in (2.1).

**Definition 6.1.** A non negative random vector \((X, Y)\) is said to follow bivariate tailed Mittag-Leffler distribution if its Laplace transform is

\[
\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + (1 - \theta)\mu_1 \lambda_1^{\alpha_1} \mu_2 \lambda_2^{\alpha_2})}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + (1 - \theta)\mu_1 \lambda_1^{\alpha_1} \mu_2 \lambda_2^{\alpha_2}}; \quad 0 \leq \sigma < 1; \quad \mu_1, \mu_2 > 0; \quad 0 < \alpha_1, \alpha_2 \leq 1; \quad 0 \leq \theta \leq 1.
\]

It is denoted by \(\text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\).

A non negative random vector \((X, Y)\) following \(\text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution has a probability \(\sigma\) at \((X = 0, Y = 0)\) and for \((X > 0, Y > 0)\) it admits \((1 - \sigma)\) times the bivariate Mittag-Leffler distribution with Laplace transform in (2.1). That is, the following expression leads to (6.1)

\[
\phi(\lambda_1, \lambda_2) = \sigma + (1 - \sigma)\frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2}) - \theta \mu_1 \lambda_1^{\alpha_1} \mu_2 \lambda_2^{\alpha_2}}.
\]
When \( \sigma = 0 \), we get (2.1).

The Laplace transform of \( \text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \) distribution is

\[
\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}.
\]

(6.2)

In the following theorem we develop autoregressive models with marginals have \( \text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \) distribution.

**Theorem 6.1.** Consider a bivariate first order autoregressive process

\[
(X_0, Y_0) \overset{d}{=} \text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \text{ and for } n = 1, 2, 3, \ldots
\]

\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n) & \text{with probability } \rho \\
(a^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, a^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho 
\end{cases}
\]

(6.3)

where \( \{(\epsilon_n, \psi_n), n \geq 1\} \) is a sequence of independently and identically distributed random vectors. Suppose that \( \{(U_n, V_n), n \geq 1\} \) and \( \{(R_n, S_n), n \geq 1\} \) are two independent sequences distributed as \( \text{BTML}(a, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \) and \( \text{BTML}(\sigma', \mu_1', \mu_2', \alpha_1, \alpha_2, 1) \) respectively such that \( \sigma' = \frac{\sigma}{b}, \mu_1' = \mu_1 b \) and \( \mu_2' = \mu_2 b \) where \( b = a(\rho + (1 - \rho)\sigma) \). Then the process is stationary with marginals follow \( \text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \) distribution if and only if

\[
(\epsilon_n, \psi_n) \overset{d}{=} (U_n, V_n) + (R_n, S_n).
\]

(6.4)

**Proof.** Suppose that \( (X_n, Y_n), n \geq 1 \) have \( \text{BTML}(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \) distribution.

The Laplace transform of the model in (6.3) is

\[
\phi_{X_n Y_n}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon_n \psi_n}(\lambda_1, \lambda_2) + (1 - \rho)\phi_{X_{n-1} Y_{n-1}}(a^{\frac{1}{\alpha_1}} \lambda_1, a^{\frac{1}{\alpha_2}} \lambda_2)\phi_{\epsilon_n \psi_n}(\lambda_1, \lambda_2).
\]

(6.5)
If the process is stationary,\[
\phi_{X,Y}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) + (1 - \rho)\phi_{X,Y}(a_{1\lambda_1}^{1/2}, a_{1\lambda_2}^{1/2})\phi_{\epsilon,\psi}(\lambda_1, \lambda_2).
\]
Substituting \(\phi_{X,Y}(\lambda_1, \lambda_2)\), we get
\[
\frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})} = \rho \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) + (1 - \rho)\frac{1 + \sigma (a\mu_1 \lambda_1^{\alpha_1} + a\mu_2 \lambda_2^{\alpha_2})}{1 + (a\mu_1 \lambda_1^{\alpha_1} + a\mu_2 \lambda_2^{\alpha_2})}\phi_{\epsilon,\psi}(\lambda_1, \lambda_2)
\]
\[
= \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) \left( \frac{1 + (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})(a(\rho + (1 - \rho)\sigma))}{1 + a(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})} \right).
\]
Simplifying, we get
\[
\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \left( \frac{1 + a(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} \right) \left( \frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + b(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})} \right).
\]
where \(b = a(\rho + (1 - \rho)\sigma)\).

Now, finding the Laplace transform of (6.4)
\[
\phi_{U_n,V_n}(\lambda_1, \lambda_2) = \frac{1 + a(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})},
\]
\[
\phi_{R_n,S_n}(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}.
\]
Substituting \(\sigma' = \frac{\sigma}{b}\), \(\mu_1' = \mu_1 b\) and \(\mu_2' = \mu_2 b\) in \(\phi_{R_n,S_n}(\lambda_1, \lambda_2)\), we get
\[
\phi_{R_n,S_n}(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + b(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}.
\]
Thus we can see that Laplace transform of \((U_n, V_n) + (R_n, S_n)\) coincides with (6.6).

Conversely we can show that the process is stationary with marginals BTML \((\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) when \((\epsilon_n, \psi_n)\) satisfies (6.4). Suppose that \((X_0, Y_0)\) is distributed as BTML \((\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).

Put \(n = 1\) in (6.5)
\[
\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2) + (1 - \rho)\phi_{X_0,Y_0}(a_{1\lambda_1}^{1/2}, a_{1\lambda_2}^{1/2})\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2).
\]
Substituting the Laplace transform of \((X_0, Y_0)\) and \((\epsilon_1, \psi_1)\) and simplifying, we get

\[
\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

Hence by mathematical induction, we get the process is stationary with marginals follow BTML \((\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

As a remark of Theorem 6.1, we can obtain bivariate first order autoregressive process with marginals have tailed form of Moran’s bivariate exponential distribution given in (1.17).

We obtain the tailed form of Moran’s bivariate exponential distribution by putting \(\alpha_1 = \alpha_2 = 1\) in (6.1). Therefore, Laplace transform of Moran’s bivariate tailed exponential distribution is

\[
\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1 + \mu_2 \lambda_2 + (1 - \theta)\mu_1 \lambda_1 \mu_2 \lambda_2)}{1 + \mu_1 \lambda_1 + \mu_2 \lambda_2 + (1 - \theta)\mu_1 \lambda_1 \mu_2 \lambda_2},
\]

\[
0 \leq \sigma < 1; \mu_1, \mu_2 > 0; 0 \leq \theta \leq 1.
\]

We denote the distribution with the above Laplace transform as MBTE \((\sigma, \mu_1, \mu_2, \theta)\).

When \(\theta = 1\), we have

\[
\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma (\mu_1 \lambda_1 + \mu_2 \lambda_2)}{1 + (\mu_1 \lambda_1 + \mu_2 \lambda_2)}.
\]

In the following remark, we develop a bivariate first order autoregressive process with MBTE \((\sigma, \mu_1, \mu_2, 1)\) marginals.

**Remark 6.1.** Let \(\{(\epsilon_n, \psi_n), n \geq 1\}\) be a sequence of independently and identically distributed random vectors such that

\[
(\epsilon_n, \psi_n) = (U_n, V_n) + (R_n, S_n)
\]

(6.7)
where \(\{U_n, V_n\}, n \geq 1\) and \(\{R_n, S_n\}, n \geq 1\) are two independent sequences of independently and identically distributed random vectors. Consider a first autoregressive process with structure:

\[
\begin{align*}
(X_0, Y_0) & \overset{d}{=} MBTE(\sigma, \mu_1, \mu_2, 1) \quad \text{and for } n; 1, 2, 3, \ldots \\
(X_n, Y_n) = & \begin{cases} 
(\epsilon_n, \psi_n) & \text{with probability } \rho \\
(ax_{n-1} + \epsilon_n, aY_{n-1} + \psi_n) & \text{with probability } 1 - \rho
\end{cases}
\end{align*}
\tag{6.8}
\]

Then the process in (6.8) is stationary with marginals follow MBTE \((\sigma, \mu_1, \mu_2, 1)\) distribution if and only if \((\epsilon_n, \psi_n), n \geq 1\) satisfies (6.7) such that \((U_n, V_n)\) and \((R_n, S_n)\) are distributed as MBTE \((\sigma', \mu_1', \mu_2', 1)\) and MBTE \((\sigma', \mu_1', \mu_2', 1)\) respectively where 

\[
\sigma' = \frac{\sigma}{b}, \quad \mu_1' = \mu_1 b \quad \text{and} \quad \mu_2' = \mu_2 b \quad \text{and} \quad b = a(\rho + (1 - \rho)\sigma).
\]

Proof of the Remark 6.1 is omitted as it is obvious from the proof of Theorem 6.1.

A generalization of BTML \((\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution could be obtained by considering the tailed form of bivariate semi Mittag-Leffler distribution defined in (2.38). For this, we introduce a scale parameter \(\delta' (> 0)\) in BSML \((\alpha_1, \alpha_2, p)\) and therefore (2.38) becomes

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)}
\tag{6.9}
\]

where \(\xi(\lambda_1, \lambda_2)\) is as stated in Definition 2.3. Denoting this bivariate semi Mittag-Leffler distribution as BSML \((\delta, \alpha_1, \alpha_2, p)\).

Now, we define the tailed form of the distribution with Laplace transform in (6.9).
Definition 6.2. A non-negative random vector \((X, Y)\) is said to follow bivariate tailed semi Mittag-Leffler distribution if it has Laplace transform

\[
\phi(\lambda_1, \lambda_2) = \frac{1 + \delta \sigma \xi(\lambda_1, \lambda_2)}{1 + \delta \xi(\lambda_1, \lambda_2)}, \quad 0 \leq \sigma \leq 1.
\]  

(6.10)

and we denote the respective distribution as BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\). Note that (6.10) is obtained by assigning a probability \(\sigma\) at \((X = 0, Y = 0)\) and for \((X > 0, Y > 0)\), the probability distribution is \((1 - \sigma)\) times the bivariate semi Mittag-Leffler distribution with Laplace transform in (6.9). That is, (6.10) is obtained from

\[
\phi(\lambda_1, \lambda_2) = \sigma + (1 - \sigma)\frac{1}{1 + \delta \xi(\lambda_1, \lambda_2)}.
\]

In the following theorem we obtain a first order stationary autoregressive process with BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\) marginals.

**Theorem 6.2.** The bivariate first order tailed semi Mittag-Leffler autoregressive process can be defined as

\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n) & \text{with probability } \rho \\
(\alpha^{\lambda_1} X_{n-1} + \epsilon_n, \alpha^{\lambda_2} Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho,
\end{cases}
\]

with \(0 \leq a < 1, 0 \leq \rho \leq 1\), where \(\{\epsilon_n, \psi_n\}, n \geq 1\) is a sequence of independently and identically distributed random vectors. Then the process is stationary with BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\) marginals if and only if \((\epsilon_n, \psi_n), n \geq 1\) satisfies

\[
(\epsilon_n, \psi_n) = (U_n, V_n) + (R_n, S_n)
\]
where \( \{(U_n, V_n), n \geq 1\} \) and \( \{(R_n, S_n), n \geq 1\} \) are two independent sequences of independent and identical random vectors following BTSML \((\delta, a, \alpha_1, \alpha_2, \rho)\) and BTSML \((\delta', \sigma', \alpha_1, \alpha_2, \rho)\) distributions respectively, \(\delta' = \delta b\), \(\sigma' = \frac{\sigma}{b}\) and \(b = a(\rho + (1 - \rho)\sigma)\), provided \((X_0, Y_0)\) is distributed as BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\).

Proof of the Theorem 6.3 can obtained by following the arguments used in Theorem 6.1.

The following theorem gives a characterization of BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\).

**Theorem 6.3.** Consider a first order autoregressive process

\[
(X_n, Y_n) = (\sigma^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, \sigma^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n), \quad (6.11)
\]

for \(n = 1, 2, 3, \ldots, 0 < a < 1\).

When the process is stationary, the innovations \(\{(\epsilon_n, \psi_n, n \geq 1)\}\) is a sequence of independently and identically distributed random vectors according to BTSML \((\delta, \sigma, \alpha_1, \alpha_2, \rho)\) if and only if \(\{(X_n, Y_n), n \geq 1\}\) are distributed as BTSML \((\delta, \alpha_1, \alpha_2, \sigma)\).

**Proof.** The Laplace transform of (6.11) is

\[
\phi_{X_n,Y_n}(\lambda_1, \lambda_2) = \phi_{X_{n-1},Y_{n-1}}(\sigma^{\frac{1}{\alpha_1}} \lambda_1, \sigma^{\frac{1}{\alpha_2}} \lambda_2) \phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2).
\]

When the process is stationary

\[
\phi_{X,Y}(\lambda_1, \lambda_2) = \phi_{X,Y}(\sigma^{\frac{1}{\alpha_1}} \lambda_1, \sigma^{\frac{1}{\alpha_2}} \lambda_2) \phi_{\epsilon,\psi}(\lambda_1, \lambda_2). \quad (6.12)
\]

Suppose that \(\{(\epsilon_n, \psi_n), n \geq 1\}\) follow BTSML \((\sigma, \delta, \alpha_1, \alpha_2, \rho)\) distribution.
Therefore,

\[ \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1 + \delta \sigma \xi(\lambda_1, \lambda_2)}{1 + \delta(\lambda_1, \lambda_2)}. \]

Substituting \( \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) \) in (6.12) and using the fact that any bivariate Laplace transform, \( \phi(\lambda_1, \lambda_2) \) can be expressed as \( \frac{1}{1 + \delta \xi(\lambda_1, \lambda_2)} \), we get

\[ \frac{1}{1 + \delta \xi(\lambda_1, \lambda_2)} = \phi_{X,Y}(\sigma_{\lambda_1}, \sigma_{\lambda_2}) \frac{1 + \delta \sigma \xi(\lambda_1, \lambda_2)}{1 + \delta \xi(\lambda_1, \lambda_2)}. \]

Therefore,

\[ \phi_{X,Y}(\lambda_1, \lambda_2) = \frac{1}{1 + \delta \xi(\lambda_1, \lambda_2)}. \]

Hence \( \{(X_n, Y_n), n \geq 1\} \) are distributed as \( \text{BSML}(\delta, \alpha_1, \alpha_2, \sigma) \).

To prove the converse, substituting the Laplace transform of \( (X_n, Y_n), n \geq 1 \) in (6.12) and solving, we get

\[ \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1 + \delta \sigma \xi(\lambda_1, \lambda_2)}{1 + \delta \xi(\lambda_1, \lambda_2)}. \]

Thus \( (\epsilon_n, \psi_n), n \geq 1 \) are distributed according to \( \text{BTSML}(\delta, \sigma, \alpha_1, \alpha_2, p) \).

\[ \square \]

### 6.3 Bivariate Tailed Discrete Mittag-Leffler Distribution

We consider a non-negative integer random vector \( (X, Y) \) such that it assumes a probability \( \sigma \) when \( (X = 0, Y = 0) \) and for other pairs of values it admits \( 1 - \sigma \) times the bivariate discrete Mittag-Leffler distribution stated in (4.2). In the following
Definition 6.3. A non-negative integer valued random vector \((X, Y)\) is said to follow bivariate tailed discrete Mittag-Leffler distribution if its p.g.f. is

\[
P(s_1, s_2) = \frac{1 + \sigma(c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2} + (1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2})}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2} + (1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}},
\]

(6.13)

where \(c_1, c_2 > 0, \ 0 < \alpha_1, \alpha_1 \leq 1, \ 0 \leq \theta \leq 1, \ |s_1| \leq 1, \ |s_2| \leq 1.\)

It is denoted by BTDML \((\sigma, c_1, c_2, \alpha_1, \alpha_2, \theta)\). As stated earlier, we can obtain (6.13) from

\[
P(s_1, s_2) = \sigma + (1-\sigma) \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2} + (1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}.
\]

When \(\theta = 1\), (6.13) becomes

\[
P(s_1, s_2) = \frac{1 + \sigma(c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2})}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}.
\]

(6.14)

Now, in the following theorem we generate (6.14) as the distribution of random sum of independently and identically distributed random vectors according to bivariate Sibuya distribution with p.g.f. in (4.33). We assume that the number of summands follow the 'zero modified' geometric distribution in (1.21).

**Theorem 6.4.** Consider a sequence \(\{(X_i, Y_i), i \geq 1\}\) of independently and identically distributed random vectors. Let \(N\) be independent of \((X_i, Y_i), i \geq 1,\) and have the 'zero modified' geometric distribution in (1.21). Let \(U_N = \sum_{i=1}^{N} X_i\) and \(V_N = \sum_{i=1}^{N} Y_i\). Then
\((U_N, V_N)\) is distributed as BTDML \((\sigma, c, c, \alpha_1, \alpha_2, 1)\) where \(c = \frac{1 - p}{p}\) if and only if \((X_i, Y_i), i \geq 1\) follow bivariate Sibuya distribution with p.g.f. in (4.33).

**Proof.** Suppose that \((X_i, Y_i), i \geq 1\) are independently and identically distributed random vectors and have p.g.f. \(Q(s_1, s_2)\). Then the p.g.f. of \((U_N, V_N)\) is

\[
P(s_1, s_2) = E(s_1^{U_N} s_2^{V_N})
\]

\[
= \sum_{n=0}^{\infty} E(s_1^{X_1 + X_2 + \ldots + X_n} s_2^{Y_1 + Y_2 + \ldots + Y_n} / N = n) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} E(s_1^{X_1 + X_2 + \ldots + X_n} s_2^{Y_1 + Y_2 + \ldots + Y_n}) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} \left( E(s_1^{X_1} s_2^{Y_1}) \right)^n P(N = n)
\]

\[
= \sum_{n=0}^{\infty} (Q(s_1, s_2))^n P(N = n)
\]

\[
= \sigma + (1 - \sigma) \frac{p}{1 - (1 - p)Q(s_1, s_2)}.
\]  \(\text{(6.15)}\)

Putting \(Q(s_1, s_2) = 1 - (1 - s_1)^{\alpha_1} - (1 - s_2)^{\alpha_2}\) in (6.15), we get

\[
P(s_1, s_2) = \frac{1 + \sigma(c(1 - s_1)^{\alpha_1} + c(1 - s_2)^{\alpha_2})}{1 + c(1 - s_1)^{\alpha_1} + c(1 - s_2)^{\alpha_2}}
\]

where \(c = \frac{1 - p}{p}\). Hence we get \((U_N, V_N)\) is distributed as BTDML \((\sigma, c, c, \alpha_1, \alpha_2, 1)\).

In order to prove the converse, substituting \(P(s_1, s_2)\) in (6.15), we have

\[
\frac{1 + \sigma(c(1 - s_1^{\alpha_1}) + c(1 - s_2^{\alpha_2}))}{1 + c(1 - s_1^{\alpha_1}) + c(1 - s_2^{\alpha_2})} = \sigma + (1 - \sigma) \frac{p}{1 - (1 - p)Q(s_1, s_2)}.
\]

Solving, we get

\[
Q(s_1, s_2) = 1 - (1 - s_1)^{\alpha_1} - (1 - s_2)^{\alpha_2}.
\]
In the following theorem we develop a first order stationary autoregressive process in which the marginals follow BTDML \((\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)\) distribution.

**Theorem 6.5.** Let a bivariate autoregressive process have the structure

\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n) & \text{with probability } \rho \\
(a^{\frac{1}{a_1}} \diamond X_{n-1} + c_1, a^{\frac{1}{a_2}} \diamond Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho
\end{cases}
\quad (6.16)
\]

where \(\{ (\epsilon_n, \psi_n), n \geq 1 \} \) is a sequence of independently and identically distributed random vectors and satisfies

\[
(\epsilon_n, \psi_n) \overset{d}{=} (U_n, V_n) + (R_n, S_n).
\quad (6.17)
\]

where \(\{ (U_n, V_n), n \geq 1 \} \) and \(\{ (R_n, S_n), n \geq 1 \} \) are two independent sequences. Then the process is stationary with marginals follow BTDML \((\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)\) distribution if and only if \(\{ (U_n, V_n), n \geq 1 \} \) and \(\{ (R_n, S_n), n \geq 1 \} \) are distributed according to BTDML \((\sigma', c_1', c_2', \alpha_1, \alpha_2, 1)\) and BTDML \((\sigma', c_1', c_2', \alpha_1, \alpha_2, 1)\) respectively where

\[
\sigma' = \frac{\sigma}{b}, c_1' = c_1 b, c_2' = c_2 b, b = a(\rho + (1 - \rho)\sigma)
\]

and provided

\[
(X_0, Y_0) \overset{d}{=} \text{BTDML}(\sigma, c_1, c_2, \alpha_1, \alpha_2, 1).
\]

**Proof.** The p.g.f. of (6.16) is

\[
P_{X_n, Y_n}(s_1, s_2) = \rho P_{\epsilon_n, \psi_n}(s_1, s_2) + (1 - \rho) P_{X_{n-1}, Y_{n-1}}(1 - a^{\frac{1}{a_1}} + a^{\frac{1}{a_2}} s_1, 1 - a^{\frac{1}{a_2}} + a^{\frac{1}{a_2}} s_2) P_{\epsilon_n, \psi_n}(s_1, s_2).
\quad (6.18)
\]

Assume that the process \((X_n, Y_n, \geq 1)\) is stationary with BTDML \((\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)\) marginals. Then (6.18) becomes
Solving we get,

\[ P_{\epsilon, \psi}(s_1, s_2) = \left( \frac{1 + a(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}{1 + c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2}} \right) \left( \frac{1 + \sigma(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}{1 + b(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})} \right) \]

where \( b = a(\rho + (1 - \rho)\sigma) \). Now finding the p.g.f. of (6.17)

\[ P_{U_n, V_n}(s_1, s_2) = \frac{1 + a(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}{1 + c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2}}. \]

\[ P_{R_n, S_n}(s_1, s_2) = \frac{1 + \sigma'(c_1'(1 - s_1)^{a_1} + c_2'(1 - s_2)^{a_2})}{1 + c_1'(1 - s_1)^{a_1} + c_2'(1 - s_2)^{a_2}}. \]

Replacing \( \sigma', c_1', c_2' \) as given, we get

\[ P_{R_n, S_n}(s_1, s_2) = \frac{1 + \sigma(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}{1 + b(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}. \]

Since \( \{(U_n, V_n), n \geq 1\} \) and \( \{(R_n, S_n), n \geq 1\} \) are two independent sequences of random vectors, we get that p.g.f. of (6.17) coincides with (6.19).

In order to prove the converse, assume that \( (\epsilon_n, \psi_n), n \geq 1 \) satisfies (6.17). Put \( n = 1 \) in (6.18), we get

\[ P_{X_1, Y_1}(s_1, s_2) = \rho \phi_{\epsilon_1, \psi_1}(s_1, s_2) + (1 - \rho)P_{X_0, Y_0}(1 - a^{\frac{1}{a_1}} + a^{\frac{1}{a_1}} s_1, 1 - a^{\frac{1}{a_2}} + a^{\frac{1}{a_2}} s_2)P_{\epsilon_1, \psi_1}(s_1, s_2). \]

Substituting the p.g.f. of \( (X_0, Y_0) \) and \( (\epsilon_1, \psi_1) \) and simplifying, we get

\[ P_{X_1, Y_1}(s_1s_2) = \frac{1 + \sigma(c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2})}{1 + c_1(1 - s_1)^{a_1} + c_2(1 - s_2)^{a_2}}. \]

By induction, we get the process is stationary with marginals have BTDML \((\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)\) distribution.