CHAPTER - 3
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SKOLEM MEAN LABELING

In this chapter we prove that, the three star \( K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if \( |n-m-\ell| \leq 3 \). Also the four star \( K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} = 2K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) with \( |n-m-2\ell| \leq 3 \) and the five star \( K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} = 3K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) with \( |n-m-3\ell| \leq 3 \) are skolem mean graphs.

3.1 Definition: A graph \( G \) with \( p \) vertices and \( q \) edges is called a mean graph if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from 0, 1, 2, \ldots, \( q \) in such a way that when each edge \( e = uv \) is labeled with \( \frac{f(u) + f(v)}{2} \) if \( f(u) + f(v) \) is even and \( \frac{f(u) + f(v) + 1}{2} \) if \( f(u) + f(v) \) is odd, then the resulting edge labels are distinct. The labeling \( f \) is called a mean labeling of \( G \).

3.2 Definition: A graph \( G \) with \( p \) vertices and \( q \) edges is said to be a Skolem mean graph if there exists a function \( f \) from the vertex set of \( G \) to \( \{1, 2, \ldots, p\} \) such that the induced map \( f^* \) from the edge set of \( G \) to \( \{2, 3, \ldots, p\} \) defined by
\[
f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}
\]

, then the edges get distinct labels from the set \( \{2,3,\ldots,p\} \). The labeling \( f \) is called a skolem mean labeling of \( G \).

In the following theorem we prove that the three star \( K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if \( |n - m - \ell| \leq 3 \).

3.3 Theorem: The three star \( K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if \( |n - m - \ell| \leq 3 \).

Proof: Consider the graph \( G = K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \).

The condition \( |n - m - \ell| \leq 3 \) implies \( \ell + m - 3 \leq n \leq \ell + m + 3 \)

That is, there are seven cases viz. \( n = \ell + m + 3, n = \ell + m + 2, n = \ell + m + 1, n = \ell + m, n = \ell + m - 1, n = \ell + m - 2 \) and \( n = \ell + m - 3 \). Let us prove that in each of these cases the graph \( G \) is a skolem mean graph.

Case 1: Let \( n = \ell + m + 3 \).

Let \( V = V_1 \cup V_2 \cup V_3 \) where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\} \),

\( V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\} \) and

\( V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m + 3\} \) be the vertex set of \( G \).
Let
\[ E = \{uu_i : 1 \leq i \leq \ell\} \cup \{vv_i : 1 \leq i \leq m\} \cup \{ww_i : 1 \leq i \leq n = \ell + m + 3\} \]
be the edge set of G.

G has \( \ell + m + n + 3 = 2n \) vertices and \( \ell + m + n = 2n - 3 \) edges.
The vertex labeling
\[ f : V(G) \rightarrow \{1, 2, 3, \ldots, \ell + m + n + 3 = 2n\} \]
is defined as follows:
\[
\begin{align*}
  f(u) &= 1; \\
  f(v) &= 3; \\
  f(w) &= \ell + m + n + 3 = 2n; \\
  f(u_i) &= 2i + 3 \quad 1 \leq i \leq \ell; \\
  f(v_i) &= 2\ell + 2i + 3 \quad 1 \leq i \leq m; \\
  f(w_i) &= 2i \quad 1 \leq i \leq n - 1 = \ell + m + 2; \\
  f(w_n) &= f(w_{\ell+m+3}) = \ell + m + n + 2 = 2n - 1
\end{align*}
\]
The corresponding edge labels are as follows:
The edge labels of \( uu_i \) is \( 2 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 3, 4, \ldots, \ell + 2 \)); \( vv_i \) is \( \ell + 3 + i \) for \( 1 \leq i \leq m \) (edge labels are \( \ell + 4, \ell + 5, \ldots, \ell + m + 3 = n \)) and \( ww_i \) is \( n + i \) for \( 1 \leq i \leq (n - 1) \) and of \( ww_i \) is \( 2n \). The edge labels of \( ww_i \) are \( n + 1, n + 2, \ldots, 2n \).
The edge labels hence are $3, 4, \ldots, \ell + 2, \ell + 4, \ell + 5, \ldots, \ell + m + 3 = n, n + 1, n + 2, \ldots, 2n$.

Hence the induced edge labels of $G$ are distinct.

Hence $G$ is a skolem mean graph.

Example:

![Diagram of graphs $K_{1,4} \cup K_{1,5} \cup K_{1,12}$]

Figure 3.3.1

Case 2: Let $n = \ell + m + 2$. 
Let \( V = V_1 \cup V_2 \cup V_3 \) where \( V_1 = \{ u \} \cup \{ u_i : 1 \leq i \leq \ell \} \),

\[ V_2 = \{ v \} \cup \{ v_i : 1 \leq i \leq m \} \] and

\[ V_3 = \{ w \} \cup \{ w_i : 1 \leq i \leq n = \ell + m + 2 \} \]

be the vertex set of \( G \).

Let

\[ E = \{ u_i u : 1 \leq i \leq \ell \} \cup \{ vv_i : 1 \leq i \leq m \} \cup \{ w w_i : 1 \leq i \leq n = \ell + m + 2 \} \]

be the edge set of \( G \).

\( G \) has \( \ell + m + n + 3 = 2n + 1 \) vertices and \( \ell + m + n = 2n - 2 \) edges.

The vertex labeling

\[ f : V(G) \to \{ 1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 1 \} \]

is defined as follows:

\[ f(u) = 1; \quad f(v) = 2; \]

\[ f(w) = \ell + m + n + 3 = 2n + 1; \]

\[ f(u_i) = 2i + 2 \text{ for } 1 \leq i \leq \ell; \]

\[ f(v_i) = 2\ell + 2i + 2 \text{ for } 1 \leq i \leq m; \]

\[ f(w_i) = 2i + 1 \text{ for } 1 \leq i \leq n - 1 \text{ and} \]

\[ f(w_n) = \ell + m + n + 2 = 2n. \]

The corresponding edge labels are as follows:
The edge labels of $uu_i$ is $2 + i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, \ldots, \ell+2$); $vv_j$ is $\ell + 2 + i$ for $1 \leq i \leq m$ (edge labels are $\ell+3, \ell+4, \ldots, \ell+m+2 = n$) and $ww_i$ is $n+1+i$ for $1 \leq i \leq (n-1)$ and of $ww_n$ is $2n+1$. The edge labels of $ww_i$ are $n+2, n+3, \ldots, 2n+1$.

The edge labels hence are $3, 4, \ldots, \ell+2, \ell+3, \ell+4, \ldots, \ell+m+2=n, n+2, n+3, \ldots, 2n$ and $2n+1$.

Hence the induced edge labels of $G$ are distinct.

Hence the graph $G$ is a skolem mean graph.
Example:

\[ K_{1,4} \cup K_{1,5} \cup K_{1,11} \]

Figure 3.3.2

Case 3: Let \( n = \ell + m + 1 \).

Let \( V = V_1 \cup V_2 \cup V_3 \) where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\} \), \( V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\} \) and \( V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m + 1\} \)

be the vertex set of \( G \).
Let
\[ E = \{uu_i : 1 \leq i \leq \ell\} \cup \{vv_i : 1 \leq i \leq m\} \cup \{ww_i : 1 \leq i \leq n = \ell + m + 1\} \]

be the edge set of G. G has \( \ell + m + n + 3 = 2n + 2 \) vertices and \( \ell + m + n = 2n - 1 \) edges.

The vertex labeling \( f: V(G) \rightarrow \{1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 2\} \) is defined as follows:

\[
\begin{align*}
    f(u) &= 1; \\
    f(v) &= 2; \\
    f(w) &= \ell + m + n + 3 = 2n + 2; \\
    f(u_i) &= 2i + 1 \quad 1 \leq i \leq \ell; \\
    f(v_i) &= 2\ell + 2i + 1 \quad 1 \leq i \leq m; \\
    f(w_i) &= 2i + 2 \quad 1 \leq i \leq n - 1 \text{ and} \\
    f(w_n) &= \ell + m + n + 2 = 2n + 1.
\end{align*}
\]

The corresponding edge labels are as follows:

The edge labels of \( uu_i \) is \( 1 + i \) for \( 1 \leq i \leq \ell \) (edge labels are 2, 3, \ldots, \ell+1); \( vv_i \) is \( \ell + 2 + i \) for \( 1 \leq i \leq m \) (edge labels are \( \ell + 3, \ell + 4, \ldots, \ell + m + 2 = n + 1 \)) and \( ww_i \) is \( n + 2 + i \) for \( 1 \leq i \leq (n - 1) \) and of \( ww_n \) is 2n + 2. The edge labels of \( ww_i \) are \( n + 3, n + 4, \ldots, 2n + 1 \) and 2n+2.

The edge labels hence are 2, 3, \ldots, \ell+1, \ell+3, \ldots, \ell + m + 2 = n + 1, n + 3, n + 4, \ldots, 2n+2.

These induced edge labels of G are distinct.
Hence the graph $G$ is a skolem mean graph.

Example:

\[
\begin{array}{c}
1,4 & 1,5 & 1,10 \\
\end{array}
\]

\[
K_{1,4} \cup K_{1,5} \cup K_{1,10}
\]

Figure 3.3.3

Case 4: Let $n = \ell + m$.

Let $V = V_1 \cup V_2 \cup V_3$ where

\[
V_1 = \{u\} \cup \{v_i : 1 \leq i \leq \ell\},
\]

\[
V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\} \text{ and}
\]

\[
V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m\}
\]

be the vertex set of $G$. 
Let

\[ E = \{ uu_i : 1 \leq i \leq \ell \} \cup \{ vv_i : 1 \leq i \leq m \} \cup \{ ww_i : 1 \leq i \leq n = \ell + m \} \]

be the edge set of \( G \). \( G \) has \( \ell + m + n + 3 = 2n + 3 \) vertices and \( \ell + m + n = 2n \) edges.

The vertex labeling

\[ f: V(G) \rightarrow \{ 1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 3 \} \]

is defined as follows:

- \( f(u) = 1 \);
- \( f(v) = 3 \);
- \( f(w) = \ell + m + n + 3 = 2n + 3 \);
- \( f(u_i) = 2i \quad 1 \leq i \leq \ell \);
- \( f(v_j) = 2\ell + 2i \quad 1 \leq i \leq m \);
- \( f(w_i) = 2i + 3 \quad 1 \leq i \leq n - 1 \) and
- \( f(w_n) = \ell + m + n + 2 = 2n + 2 \).

The corresponding edge labels are as follows:

The edge labels of \( uu_i \) is \( 1+i \) for \( 1 \leq i \leq \ell \) (edge labels are 2, 3, ..., \( \ell + 1 \)); \( vv_j \) is \( \ell + 2 + i \) for \( 1 \leq i \leq m \) (edge labels are \( \ell + 3 \), \( \ell + 4 \), ..., \( \ell + m + 2 = n + 2 \)) and \( ww_i \) is \( n + 3 + i \) for \( 1 \leq i \leq (n - 1) \) and of \( ww_n \) is \( 2n + 3 \). (edge labels of \( ww_i \) are \( n + 4 \), \( n + 5 \), ..., \( 2n + 2 \), \( 2n + 3 \)).

The edge labels hence are 2, 3, ..., \( \ell + 1 \), \( \ell + 3 \), ..., \( \ell + m + 2 = n + 2 \), \( n + 4 \), \( n + 5 \), ..., \( 2n + 2 \), \( 2n + 3 \).

These induced edge labels of \( G \) are distinct.
Hence the graph $G$ is a skolem mean graph.

**Example:**

$$K_{1,7} \cup K_{1,8} \cup K_{1,15}$$

**Figure 3.3.4**

**Case 5:** Let $n = \ell + m - 1$.

Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\} \text{ and }$$

$$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m - 1\}$$

be the vertex set of $G$. 
Let
\[ E = \{ uu_i : 1 \leq i \leq \ell \} \cup \{ vv_i : 1 \leq i \leq m \} \cup \{ ww_i : 1 \leq i \leq n = \ell + m + 1 \} \]

be the edge set of G. G has \( \ell + m + n + 3 = 2n + 4 \) vertices and \( \ell + m + n = 2n + 1 \) edges.

The vertex labeling \( f : V(G) \rightarrow \{1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 4\} \) is defined as follows:

\[
\begin{align*}
    f(u) &= 2; \quad f(v) = 4; \\
    f(w) &= \ell + m + n + 3 = 2n + 4; \\
    f(u_i) &= 2i - 1 \quad 1 \leq i \leq \ell; \\
    f(v_i) &= 2\ell + 2i - 1 \quad 1 \leq i \leq m; \\
    f(w_i) &= 2i + 4 \quad 1 \leq i \leq n - 1 = \ell + m - 2; \\
    f(w_n) &= f(w_{\ell + m - 1}) = \ell + m + n + 2 = 2n + 3
\end{align*}
\]

The corresponding edge labels are as follows:

The edge labels of \( uu_i \) is \( 1 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2, 3, \ldots, \ell + 1 \)); \( vv_i \) is \( \ell + 2 + i \) for \( 1 \leq i \leq m \) (edge labels are \( \ell + 3, \ell + 4, \ldots, \ell + m + 2 = n + 3 \)) and \( ww_i \) is \( n + 4 + i \) for \( 1 \leq i \leq (n - 1) \) and of \( ww_n \) is \( 2n + 4 \). The edge labels of \( ww_i \) are \( n + 5, n + 6, \ldots, 2n + 3, 2n+4 \).

The edge labels hence are \( 2, 3, \ldots, \ell + 1, \ell + 3, \ldots, \ell + m + 2 = n + 3, n + 5, n + 6 \ldots, 2n + 3, 2n+4 \).
These induced edge labels of $G$ are distinct.

Hence the graph $G$ is a skolem mean graph.

**Example:**

![Graph Image](image)

Figure 3.3.5 ($K_{1,5} \cup K_{1,6} \cup K_{1,10}$)

**Case 6:** Let $n = \ell + m - 2$.

Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\}$ and

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m - 2\}$
be the vertex set of $G$.

Let

$$E = \{uu_i : 1 \leq i \leq \ell\} \cup \{vv_i : 1 \leq i \leq m\} \cup \{ww_i : 1 \leq i \leq n = \ell + m - 2\}$$

be the edge set of $G$. $G$ has $\ell + m + n + 3 = 2n + 5$ vertices and $\ell + m + n = 2n + 2$ edges.

The vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 5\}$ is defined as follows:

$$f(u) = 2; \quad f(v) = 3;$$

$$f(w) = \ell + m + n + 3 = 2n + 5;$$

$$f(u_i) = 2i - 1 \quad 1 \leq i \leq \ell;$$

$$f(v_i) = 2\ell + 2i \quad 1 \leq i \leq m;$$

$$f(w_i) = 2i + 3 \quad 1 \leq i \leq n = \ell + m - 2$$

The corresponding edge labels are as follows:

The edge labels of $uu_i$ is $1 + i$ for $1 \leq i \leq \ell$ (edge labels are 2, 3, $\ldots$, $\ell + 1$); $vv_i$ is $\ell + 2 + i$ for $1 \leq i \leq m$ (edge labels are $\ell + 3$, $\ell + 4$, $\ldots$, $\ell + m + 2 = n + 4$) and $ww_i$ is $n + 4 + i$ for $1 \leq i \leq n$. The edge labels of $ww_i$ are $n + 5$, $n + 6$, $\ldots$, $2n + 4$.

The edge labels hence are 2, 3, $\ldots$, $\ell + 1$, $\ell + 3$, $\ldots$, $\ell + m + 2 = n + 4$, $n + 5$, $n + 6$, $\ldots$, $2n + 4$. 47
The induced edge labels of G are distinct.
Hence the graph G is a skolem mean graph.

Example:

\[ K_{1,5} \cup K_{1,7} \cup K_{1,10} \]

Figure 3.3.6
Case 7: Let $n = \ell + m - 3$.

Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq m\}$ and

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq n = \ell + m - 2\}$

be the vertex set of $G$.

Let

$E = \{uu_i : 1 \leq i \leq \ell\} \cup \{vv_i : 1 \leq i \leq m\} \cup \{ww_i : 1 \leq i \leq n = \ell + m - 3\}$

be the edge set of $G$. $G$ has $\ell + m + n + 3 = 2n + 6$ vertices and

$\ell + m + n = 2n + 3$ edges.

The vertex labeling

$f: V(G) \rightarrow \{1, 2, 3, \ldots, \ell + m + n + 3 = 2n + 6\}$ is defined

as follows:

$f(u) = 2; \quad f(v) = 4;$

$f(w) = \ell + m + n + 3 = 2n + 6;$

$f(u_i) = 2i - 1 \quad 1 \leq i \leq \ell;$

$f(v_i) = 2\ell + 2i - 1 \quad 1 \leq i \leq m;$

$f(w_i) = 2i + 4 \quad 1 \leq i \leq n = \ell + m - 3$

The edge labels of $uu_i$ is $1 + i$ for $1 \leq i \leq \ell$ (edge labels are $2, 3, \ldots, \ell + 1$); $vv_i$ is $\ell + 2 + i$ for $1 \leq i \leq m$ (edge labels are $\ell + 3, \ell + 4, \ldots, \ell + m + 2 = n + 5$) and $ww_i$ is
\( n + 5 + i \) for \( 1 \leq i \leq n \). The edge labels of \( \omega \ell \) are \( n + 6, n + 7, \ldots, 2n + 5 \).

The edge labels hence are \( 2, 3, \ldots, \ell + 1, \ell + 3, \ldots, \ell + m + 2 = n + 5, n + 6, n + 7 \ldots, 2n + 5 \).

These induced edge labels of graph \( G \) are distinct.

Hence \( G \) is a skolem mean graph.

**Example:**

![Diagram](image)

\( K_{1,5} \cup K_{1,8} \cup K_{1,10} \)

*Figure 3.3.7*
Now, we prove that the four star graph

\[ K_{1,1} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \]

is a skolem mean graph if

\[ |n - m - 2\ell| \leq 3. \]

**3.4 Theorem:** The four star graph \( G = K_{1,1} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} = 2K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if

\[ |n - m - 2\ell| \leq 3. \]

**Proof:** Consider the graph \( G = K_{1,1} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \).

The condition \( |n - m - 2\ell| \leq 3 \) implies that

\[ 2\ell + m - 3 \leq n \leq 2\ell + m + 3 \]

That is, there are seven cases viz.

\( n = 2\ell + m + 3 \), \( n = 2\ell + m + 2 \), \( n = 2\ell + m + 1 \), \( n = 2\ell + m \), \( n = 2\ell + m - 1 \), \( n = 2\ell + m - 2 \) and \( n = 2\ell + m - 3 \).

Let us prove in each of these cases the graph \( G \) is a skolem mean graph.

**Case 1:** Let \( n = 2\ell + m + 3 \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \)

where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\} \),

\[ V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\} , \]

\[ V_3 = \{w\} \cup \{w_i : 1 \leq i \leq m\} \] and

\[ V_4 = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m + 3\} . \]
Let $E = \{u_i, v_i : 1 \leq i \leq \ell\} \cup \{w_i : 1 \leq i \leq m\}$

$\cup \{x_i : 1 \leq i \leq n = 2\ell + m + 3\}$ be the edge set of $G$.

Then $G$ has $4\ell + 2m + 7$ vertices and $4\ell + 2m + 3$ edges.

The vertex labeling

$$f : V \rightarrow \{1, 2, 3, \ldots, 2\ell + m + n + 4 = 4\ell + 2m + 7\}$$

is defined as follows:

$$f(u) = 1; \quad f(v) = 2; \quad f(w) = 3;$$

$$f(x) = 4\ell + 2m + 6;$$

$$f(u_i) = 2i + 3 \quad 1 \leq i \leq \ell$$

$$f(v_i) = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell$$

$$f(w_i) = 4\ell + 2i + 3 \quad 1 \leq i \leq m$$

$$f(x_i) = 2i + 2 \quad 1 \leq i \leq n - 2 = 2\ell + m + 1$$

$$f(x_{n-1} = 2\ell + m + 2) = 4\ell + 2m + 5$$

$$f(x_{n=2\ell + m + 3}) = 4\ell + 2m + 7$$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $2 + i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, \ldots, \ell + 2$), $vv_i$ is $\ell + 3 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 4, \ell + 5, \ldots, 2\ell + 3$), $ww_i$ is $2\ell + 3 + i$ for $1 \leq i \leq m$ (edge labels are $2\ell + 4, 2\ell + 5, \ldots, 2\ell + m + 3$), $xx_i$ is $2\ell + m + 4 + i$ for $1 \leq i \leq n - 2 = 2\ell + m + 1$ (edge labels are
The induced edge labels of graph G are distinct. Hence G is a skolem mean graph.

Example:
**Case 2:** Let \( n = 2\ell + m + 2 \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \)

Where \( V_1 = \{ u \} \cup \{ u_i : 1 \leq i \leq \ell \} \),

\[
V_2 = \{ v \} \cup \{ v_i : 1 \leq i \leq \ell \},
\]

\( V_3 = \{ w \} \cup \{ w_i : 1 \leq i \leq m \} \) and

\[
V_4 = \{ x \} \cup \{ x_i : 1 \leq i \leq n = 2\ell + m + 2 \}
\]

Let \( E = \{ uu_i, vv_i : 1 \leq i \leq \ell \} \cup \{ ww_i : 1 \leq i \leq m \} \)

\( \cup \{ xx_i : 1 \leq i \leq n = 2\ell + m + 2 \} \) be the edge set of \( G \).

\( G \) has \( 4\ell + 2m + 6 \) vertices and \( 4\ell + 2m + 2 \) edges.

The vertex labeling

\[ f : V \to \{ 1, 2, 3, ..., 2\ell + m + n + 4 = 4\ell + 2m + 6 \} \]

is defined as follows:

\[ f(u) = 1; f(v) = 2; \]

\[ f(w) = 3; \]

\[ f(x) = 4\ell + 2m + 5; \]

\[ f(u_i) = 2i + 3 \quad 1 \leq i \leq \ell \]

\[ f(v_i) = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell \]

\[ f(w_i) = 4\ell + 2i + 3 \quad 1 \leq i \leq m \]

\[ f(x_i) = 2i + 2 \quad 1 \leq i \leq n = 2\ell + m + 2 \]

The corresponding edge labels are as follows:
The edge label of $uu_i$ is $2 + i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, 5, \ldots, \ell + 2$), $vv_i$ is $\ell + 3 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 4, \ell + 5, \ell + 6, \ldots, 2\ell + 3$), $ww_i$ is $2\ell + 3 + i$ for $1 \leq i \leq m$ (edge labels are $2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3$) and $xx_i$ is $2\ell + m + 4 + i$ for $1 \leq i \leq n = 2\ell + m + 2$ (edge labels are $2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, \ldots, 4\ell + 2m + 6$).

Hence, the edge labels are $3, 4, 5, \ldots, \ell + 2,$ $\ell + 4, \ell + 5, \ell + 6, \ldots, 2\ell + 3,$ $2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3,$ $2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7,$ $\ldots, 4\ell + 2m + 6.$

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,8} \cup K_{1,20} \]

**Figure 3.4.2**

**Case 3:** Let \( n = 2\ell + m + 1 \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \)

where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}, \)

\[ V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}, \]

\[ V_3 = \{w\} \cup \{w_i : 1 \leq i \leq m\} \]

and \( V_4 = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m + 1\} \)
Let \( E = \{uu_i, vv_i : 1 \leq i \leq \ell\} \cup \{ww_i : 1 \leq i \leq m\} \cup \{xx_i : 1 \leq i \leq n = 2\ell + m + 1\} \) be the edge set of \( G \).

Then \( G \) has \( 4\ell + 2m + 5 \) vertices and \( 4\ell + 2m + 1 \) edges.

The vertex labeling

\[
f : V \rightarrow \{1, 2, 3, \ldots, 2\ell + m + n + 4 = 4\ell + 2m + 5\}
\]

is defined as follows:

\[
f(u) = 1; \quad f(v) = 2; \quad f(w) = 3;
\]

\[
f(x) = 4\ell + 2m + 5;
\]

\[
f(u_i) = 2i + 3 \quad 1 \leq i \leq \ell
\]

\[
f(v_i) = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell
\]

\[
f(w_i) = 4\ell + 2i + 3 \quad 1 \leq i \leq m
\]

\[
f(x_i) = 2i + 2 \quad 1 \leq i \leq n = 2\ell + m + 1
\]

The corresponding edge labels are as follows:

The edge label of \( uu_i \) is \( 2 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 3, 4, 5, \ldots, \ell + 2 \)), \( vv_i \) is \( \ell + 3 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell + 4, \ell + 5, \ell + 6, \ldots, 2\ell + 3 \)), \( ww_i \) is \( 2\ell + 3 + i \) for \( 1 \leq i \leq m \) (edge labels are \( 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3 \)) and \( xx_i \) is \( 2\ell + m + 4 + i \) for \( 1 \leq i \leq n = 2\ell + m + 1 \) (edge labels are \( 2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, \ldots, 4\ell + 2m + 5 \))
Thus the edge labels are $3, 4, 5, \ldots, \ell + 2, \ell + 4, \ell + 5, \ldots, 2\ell + 3,$

$2\ell + 4, 2\ell + 5, \ldots, 2\ell + m + 3,$

$2\ell + m + 5, 2\ell + m + 6, \ldots, 4\ell + 2m + 5.$

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.

**Example:**

![Graphs](image)

$K_{1,5} \cup K_{1,5} \cup K_{1,9} \cup K_{1,20}$

**Figure 3.4.3**
**Case 4:** Let \( n = 2\ell + m \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \) where

\[
\begin{align*}
V_1 & = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}, \\
V_2 & = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}, \\
V_3 & = \{w\} \cup \{w_i : 1 \leq i \leq m\} \quad \text{and} \\
V_4 & = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m\}
\end{align*}
\]

Let \( E = \{uu_i, vv_i : 1 \leq i \leq \ell\} \cup \{ww_i : 1 \leq i \leq m\} \cup \{xx_i : 1 \leq i \leq n = 2\ell + m\} \) be the edge set of \( G \).

Then \( G \) has \( 4\ell + 2m + 4 \) vertices and \( 4\ell + 2m \) edges.

The vertex labeling

\[
f : V \rightarrow \{1, 2, 3, \ldots, 2\ell + m + n + 4 = 4\ell + 2m + 4\}
\]

defined as follows:

\[
\begin{align*}
f(u) & = 1; \quad f(v) = 2; \quad f(w) = 3; \\
f(x) & = 4\ell + 2m + 4; \\
f(u_i) & = 2i + 3 \quad 1 \leq i \leq \ell \\
f(v_i) & = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell \\
f(w_i) & = 4\ell + 2i + 3 \quad 1 \leq i \leq m \\
f(x_i) & = 2i + 2 \quad 1 \leq i \leq n = 2\ell + m
\end{align*}
\]

The corresponding edge labels are as follows:

The edge label of \( uu_i \) is \( 2 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 3, 4, 5, \ldots, \ell + 2 \)), \( vv_i \) is \( \ell + 3 + i \) for \( 1 \leq i \leq \ell \) (edge labels
are \( \ell + 4, \ell + 5, \ell + 6, \ldots, 2\ell + 3 \), \( \text{ww}_i \) is \( 2\ell + 3 + i \) for \( 1 \leq i \leq m \) (edge labels are \( 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3 \)) and \( \text{xx}_i \) is \( 2\ell + m + 3 + i \) for \( 1 \leq i \leq n = 2\ell + m \) (edge labels are \( 2\ell + m + 4, 2\ell + m + 5, 2\ell + m + 6, \ldots, 4\ell + 2m + 3 \)).

Thus the edge labels are \( 3, 4, 5, \ldots, \ell + 2, \ell + 4, \ell + 5, \ell + 6, \ldots, 2\ell + 3, 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3 \), \( 2\ell + m + 4, 2\ell + m + 5, 2\ell + m + 6, \ldots, 4\ell + 2m + 3 \).

These induced edge labels of graph \( G \) are distinct.

Hence \( G \) is a skolem mean graph.
Example:

Figure 3.4.4

Case 5: Let $n = 2\ell + m - 1$

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4$ be the vertex set of $G$

where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$, $V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$,

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq m\}$ and

$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m - 1\}$
Let $E = \{uu_i, vv_i : 1 \leq i \leq \ell \} \cup \{ww_i : 1 \leq i \leq m\} \\
\cup \{xx_i : 1 \leq i \leq n = 2\ell + m - 1\}$ be the edge set of $G$.

Then $G$ has $4\ell + 2m + 3$ vertices and $4\ell + 2m - 1$ edges.

The vertex labeling

$$f : V \to \{1, 2, 3, \ldots, 2\ell + m + n + 4 = 4\ell + 2m + 3\}$$

defined as follows:

$$f(u) = 1; \quad f(v) = 2; \quad f(w) = 3;$$
$$f(x) = 4\ell + 2m + 3;$$
$$f(u_i) = 2i + 2 \quad 1 \leq i \leq \ell$$
$$f(v_i) = 2\ell + 2i + 2 \quad 1 \leq i \leq \ell$$
$$f(w_i) = 4\ell + 2i + 2 \quad 1 \leq i \leq m$$
$$f(x_i) = 2i + 3 \quad 1 \leq i \leq n = 2\ell + m - 1$$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $2 + i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, 5, \ldots, \ell + 2$), $vv_i$ is $\ell + 2 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 3, \ell + 4, \ell + 5, \ldots, 2\ell + 2$), $ww_i$ is $2\ell + 3 + i$ for $1 \leq i \leq m$ (edge labels are $2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3$) and $xx_i$ is $2\ell + m + 3 + i$ for $1 \leq i \leq n = 2\ell + m - 1$ (edge labels are $2\ell + m + 4, 2\ell + m + 5, 2\ell + m + 6, \ldots, 4\ell + 2m + 2$).
Thus the edge labels are $3, 4, 5, \ldots, \ell + 2,$

$\ell + 3, \ell + 4, \ell + 5, \ldots, 2\ell + 2, 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3$

$2\ell + m + 4, 2\ell + m + 5, 2\ell + m + 6, \ldots, 4\ell + 2m + 2.$

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.

Example:
Case 6: Let \( n = 2\ell + m - 2 \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \)

Where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}, V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}, \)

\[ V_3 = \{w\} \cup \{w_i : 1 \leq i \leq m\} \text{ and} \]

\[ V_4 = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m - 2\} \]

Let \( E = \{uu_i, vv_i : 1 \leq i \leq \ell\} \cup \{ww_i : 1 \leq i \leq m\} \]

\[ \cup \{xx_i : 1 \leq i \leq n = 2\ell + m - 2\} \] be the edge set of \( G \).

Then \( G \) has \( 4\ell + 2m + 2 \) vertices and \( 4\ell + 2m - 2 \) edges.

The vertex labeling

\[ f : V \to \{1, 2, 3, ..., 2\ell + m + n + 4 = 4\ell + 2m + 2\} \]

is defined as follows:

\[ f(u) = 2; \quad f(v) = 4; \quad f(w) = 6; \]

\[ f(x) = 4\ell + 2m + 1; \]

\[ f(u_i) = 2i - 1 \quad 1 \leq i \leq \ell \]

\[ f(v_i) = 2\ell + 2i - 1 \quad 1 \leq i \leq \ell \]

\[ f(w_i) = 4\ell + 2i - 1 \quad 1 \leq i \leq m \]

\[ f(x_i) = 2i + 6 \quad 1 \leq i \leq n = 2\ell + m - 2 \]

The corresponding edge labels are as follows:

The edge label of \( uu_i \) is \( 1 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2, 3, 4, ..., \ell + 1 \)), \( vv_i \) is \( \ell + 2 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell + 3, \ell + 4, \ell + 5, ..., 2\ell + 2 \)), \( ww_i \) is \( 2\ell + 3 + i \) for
1 \leq i \leq m \text{ (edge labels are } 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3) \\
\text{and } xx_i \text{ is } 2\ell + m + 4 + i \text{ for } 1 \leq i \leq n = 2\ell + m - 2 \text{ (edge labels are } 2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, \ldots, 4\ell + 2m + 2). \\
Thus the edge labels are 2, 3, 4, \ldots, \ell + 1, \\
\ell + 3, \ell + 4, \ell + 5, \ldots, 2\ell + 2, 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3 \\
, 2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, \ldots, 4\ell + 2m + 2.

These induced edge labels of graph G are distinct.

Hence G is a skolem mean graph.
Example:

\[ K_{1,6} \cup K_{1,6} \cup K_{1,10} \cup K_{1,20} \]

**Figure 3.4.6**

Case 7: Let \( n = 2\ell + m - 3 \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \) be the vertex set of \( G \)

where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\} \),

\( V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\} \),

\( V_3 = \{w\} \cup \{w_i : 1 \leq i \leq m\} \) and

\( V_4 = \{x\} \cup \{x_i : 1 \leq i \leq n = 2\ell + m - 3\} \)
Let $E = \{uu_i, vv_i : 1 \leq i \leq \ell\} \cup \{ww_i : 1 \leq i \leq m\} \cup \{xx_i : 1 \leq i \leq n = 2\ell + m - 3\}$ be the edge set of $G$.

Then $G$ has $4\ell + 2m + 1$ vertices and $4\ell + 2m - 3$ edges.

The vertex labeling

$$f : V \rightarrow \{1, 2, 3, ..., 2\ell + m + n + 4 = 4\ell + 2m + 1\}$$

is defined as follows:

$$f(u) = 2; \quad f(v) = 4; \quad f(w) = 6;$$
$$f(x) = 4\ell + 2m + 1;$$
$$f(u_i) = 2i - 1 \quad 1 \leq i \leq \ell$$
$$f(v_i) = 2\ell + 2i - 1 \quad 1 \leq i \leq \ell$$
$$f(w_i) = 4\ell + 2i - 1 \quad 1 \leq i \leq m$$
$$f(x_i) = 2i + 6 \quad 1 \leq i \leq n = 2\ell + m - 3$$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $1 + i$ for $1 \leq i \leq \ell$ (edge labels are $2, 3, 4, ..., \ell + 1$), $vv_i$ is $\ell + 2 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 3, \ell + 4, \ell + 5, ..., 2\ell + 2$), $ww_i$ is $2\ell + 3 + i$ for $1 \leq i \leq m$ (edge labels are $2\ell + 4, 2\ell + 5, 2\ell + 6, ..., 2\ell + m + 3$) and $xx_i$ is $2\ell + m + 4 + i$ for $1 \leq i \leq n = 2\ell + m - 3$ (edge labels are $2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, ..., 4\ell + 2m + 1$).
Thus the edge labels are $2, 3, 4, \ldots, \ell + 1, \ell + 3, \ell + 4, \ell + 5, \ldots, 2\ell + 2, 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 2\ell + m + 3, 2\ell + m + 5, 2\ell + m + 6, 2\ell + m + 7, \ldots, 4\ell + 2m + 1$.

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.

Example:
In continuance of the previous two theorems, we prove that the five star graph $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is also a skolem mean graph if $|n - m - 3\ell| \leq 3$.

3.5 Theorem: The five star $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} = 3K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|n - m - 3\ell| \leq 3$.

Proof: $|n - m - 3\ell| \leq 3$ implies $3\ell + m - 3 \leq n \leq 3\ell + m + 3$.

That is, there are seven cases viz. $n = 3\ell + m + 3$, $n = 3\ell + m + 2$, $n = 3\ell + m + 1$, $n = 3\ell + m$, $n = 3\ell + m - 1$, $n = 3\ell + m - 2$, and $3\ell + m - 3$.

Let us prove in each of these cases the graph $G = K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph.

Case 1: Let $n = 3\ell + m + 3$.

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ be the vertex set of $G$

where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$, $V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$
$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq \ell\}$,
$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq m\}$ and
$V_5 = \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m + 3\}$. 


Let $E = \{uu_i, vv_i, ww_i : 1 \leq i \leq \ell\} \cup \{xx_i : 1 \leq i \leq m\}$

$\cup \{yy_i : 1 \leq i \leq n = 3\ell + m + 3\}$ be the edge set of $G$.

$G$ has $3\ell + m + n + 5 = 6\ell + 2m + 8$ vertices and $6\ell + 2m + 3$
edges.

The vertex labeling

$$f : V \to \{1, 2, \ldots, 3\ell + m + n + 5 = 6\ell + 2m + 8\}$$

is defined as follows:

- $f(u) = 1$;
- $f(v) = 2$;
- $f(w) = 3$;
- $f(x) = 5$;
- $f(y) = 6\ell + 2m + 8$;
- $f(u_i) = 2i + 5$ for $1 \leq i \leq \ell$
- $f(v_i) = 2\ell + 2i + 5$ for $1 \leq i \leq \ell$
- $f(w_i) = 4\ell + 2i + 5$ for $1 \leq i \leq \ell$
- $f(x_i) = 6\ell + 2i + 5$ for $1 \leq i \leq m$
- $f(y_i) = 2i + 2$ for $1 \leq i \leq n - 1 = 3\ell + m + 2$
- $f(y_{3\ell + m + 3}) = 6\ell + 2m + 7$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $3 + i$ for $1 \leq i \leq \ell$ (edge labels are $4, 5, \ldots, 3\ell + 3$), $vv_i$ is $\ell + 4 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 5, \ell + 6, \ldots, 2\ell + 4$), $ww_i$ is $2\ell + 4 + i$ for $1 \leq i \leq \ell$ (edge labels are $2\ell + 5, 2\ell + 6, \ldots, 3\ell + 4$), $xx_i$ is $3\ell + 5 + i$ for $1 \leq i \leq m$ (edge labels are $3\ell + 6, 3\ell + 7, \ldots, 3\ell + m + 5$) and
$y_{i}$ is $3\ell + m + 5 + i$ for $1 \leq i \leq n - 1 = 3\ell + m + 2$ (edge labels are $3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 7$) and $y_{3\ell + m + 3}$ is $6\ell + 2m + 8$.

The edge labels hence are $4, 5, \ldots, \ell + 3, \ell + 5, \ell + 6, \ldots, 2\ell + 4, 2\ell + 5, 2\ell + 6, \ldots, 3\ell + 4, 3\ell + 6, 3\ell + 7, \ldots, 3\ell + m + 5, 3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 7$ and $6\ell + 2m + 8$.

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,4} \cup K_{1,22} \]

Figure 3.5.1
Case 2: Let \( n = 3\ell + m + 2 \).

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \) be the vertex set of \( G \) where
\[ V_1 = \{ u \} \cup \{ u_i : 1 \leq i \leq \ell \}, \quad V_2 = \{ v \} \cup \{ v_i : 1 \leq i \leq \ell \}, \]
\[ V_3 = \{ w \} \cup \{ w_i : 1 \leq i \leq \ell \}, \quad V_4 = \{ x \} \cup \{ x_i : 1 \leq i \leq m \} \]
and \( V_5 = \{ y \} \cup \{ y_i : 1 \leq i \leq n = 3\ell + m + 2 \} \).

Let \( E = \{ uu_i, vv_i, ww_i : 1 \leq i \leq \ell \} \cup \{ xx_i : 1 \leq i \leq m \} \)
\[ \cup \{ yy_i : 1 \leq i \leq n = 3\ell + m + 2 \} \]
be the edge set of \( G \).

\( G \) has \( 3\ell + m + n + 5 = 6\ell + 2m + 7 \) vertices and \( 6\ell + 2m + 2 \) edges.

The vertex labeling
\[ f : V \to \{ 1, 2, \ldots, 3\ell + m + n + 5 = 6\ell + 2m + 7 \} \]
defined as follows:

\[ f(u) = 1; \quad f(v) = 2; \quad f(w) = 3; \]
\[ f(x) = 4; \quad f(y) = 6\ell + 2m + 6; \]
\[ f(u_i) = 2i + 3 \quad 1 \leq i \leq \ell \]
\[ f(v_i) = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell \]
\[ f(w_i) = 4\ell + 2i + 3 \quad 1 \leq i \leq \ell \]
\[ f(x_i) = 6\ell + 2i + 3 \quad 1 \leq i \leq m \]
\[ f(y_i) = 2i + 4 \quad 1 \leq i \leq n - 2 = 3\ell + m \]
\[ f(y_{3\ell+1}) = 6\ell + 2m + 5 \]
\[ f(y_{3\ell+2}) = 6\ell + 2m + 7 \]
The corresponding edge labels are as follows:

The edge label of $uu_i$ is $2+i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, \ldots, \ell+2$), $vv_i$ is $\ell+3+i$ for $1 \leq i \leq \ell$ (edge labels are $\ell+4, \ell+5, \ell+6, \ldots, 2\ell+3$), $ww_i$ is $2\ell+3+i$ for $1 \leq i \leq \ell$ (edge labels are $2\ell+4, 2\ell+5, \ldots, 3\ell+3$), $xx_i$ is $3\ell+4+i$ for $1 \leq i \leq m$ (edge labels are $3\ell+5, 3\ell+6, \ldots, 3\ell+m+4$) and $yy_i$ is $3\ell+m+5+i$ for $1 \leq i \leq 3\ell+m$ (edge labels are $3\ell+m+6, 3\ell+m+7, \ldots, 6\ell+2m+5$) $yy_{3\ell+m+1}$ is $6\ell+2m+6$ and $yy_{3\ell+m+3}$ is $6\ell+2m+7$.

The edge labels hence are $3, 4, \ldots, \ell+2, \ell+4, \ell+5, \ldots, 2\ell+3, 2\ell+4, 2\ell+5, \ldots, 3\ell+3, 3\ell+4, 3\ell+5, 3\ell+6, \ldots, 3\ell+m+4, 3\ell+m+6, 3\ell+m+7, \ldots, 6\ell+2m+5, 6\ell+2m+6$ and $6\ell+2m+7$.

These induced edge labels of graph G are distinct.

Hence G is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,22} \]

Figure 3.5.2
Case 3: Let \( n = 3\ell + m + 1 \).

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \) be the vertex set of \( G \) where

\[
\begin{align*}
V_1 &= \{u\} \cup \{u_i : 1 \leq i \leq \ell \}, \\
V_2 &= \{v\} \cup \{v_i : 1 \leq i \leq \ell \}, \\
V_3 &= \{w\} \cup \{w_i : 1 \leq i \leq \ell \}, \\
V_4 &= \{x\} \cup \{x_i : 1 \leq i \leq m\} \text{ and} \\
V_5 &= \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m + 1 \}.
\end{align*}
\]

Let \( E = \{uu_i, vv_i, ww_i : 1 \leq i \leq \ell\} \cup \{xx_i : 1 \leq i \leq m\} \cup \{yy_i : 1 \leq i \leq n = 3\ell + m + 1\} \) be the edge set of \( G \).

\( G \) has \( 3\ell + m + n + 5 = 6\ell + 2m + 6 \) vertices and \( 6\ell + 2m + 1 \) edges.

The vertex labeling \( f : V \to \{1, 2, ..., 3\ell + m + n + 5 = 6\ell + 2m + 6\} \) is defined as follows:

\[
\begin{align*}
f(u) &= 1; \\
f(v) &= 2; \\
f(w) &= 3; \\
f(x) &= 4; \\
f(y) &= 6\ell + 2m + 6; \\
f(u_i) &= 2i + 3 \quad 1 \leq i \leq \ell \\
f(v_i) &= 2\ell + 2i + 3 \quad 1 \leq i \leq \ell \\
f(w_i) &= 4\ell + 2i + 3 \quad 1 \leq i \leq \ell
\end{align*}
\]
\[ f(x_i) = 6\ell + 2i + 3 \quad 1 \leq i \leq m \]
\[ f(y_i) = 2i + 4 \quad 1 \leq i \leq n - 1 = 3\ell + m \]
\[ f(y_{3\ell+m+1}) = 6\ell + 2m + 5 \]

The corresponding edge labels are as follows:

The edge label of \( uu_i \) is \( 2+i \) for \( 1 \leq i \leq \ell \) (edge labels are 3, 4, ..., \( \ell + 2 \)), \( vv_i \) is \( \ell + 3 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell + 4, \ell + 5, \ell + 6, ..., 2\ell + 3 \)), \( ww_i \) is \( 2\ell + 3 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2\ell + 4, 2\ell + 5, ..., 3\ell + 3 \)), \( xx_i \) is \( 3\ell + 4 + i \) for \( 1 \leq i \leq m \) (edge labels are \( 3\ell + 5, 3\ell + 6, ..., 3\ell + m + 4 \)) and \( yy_i \) is \( 3\ell + m + 5 + i \) for \( 1 \leq i \leq 3\ell + m \) (edge labels are \( 3\ell + m + 6, 3\ell + m + 7, ..., 6\ell + 2m + 5 \)) \( yy_{3\ell+m+1} \) is \( 6\ell + 2m + 6 \).

The edge labels hence are 3, 4, ..., \( \ell + 2 \), \( \ell + 4, \ell + 5, ..., 2\ell + 3 \), \( 2\ell + 4, 2\ell + 5, ..., 3\ell + 3 \), \( 3\ell + 5, 3\ell + 6, ..., 3\ell + m + 4 \), 3\ell + m + 6, 3\ell + m + 7, ..., 6\ell + 2m + 5 and 6\ell + 2m + 6.

These induced edge labels of graph G are distinct.

Hence G is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,6} \cup K_{1,22} \]

Figure 3.5.3
Case 4: Let $n = 3\ell + m$.

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ be the vertex set of $G$ where $V_i = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$,

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq \ell\}$,

$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq m\}$

and $V_5 = \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m\}$.

Let $E = \{uu_i, vv_i, ww_i : 1 \leq i \leq \ell\} \cup \{xx_i : 1 \leq i \leq m\}$

$\cup \{yy_i : 1 \leq i \leq n = 3\ell + m\}$ be the edge set of $G$.

$G$ has $3\ell + m + n + 5 = 6\ell + 2m + 5$ vertices and $6\ell + 2m$ edges.

The vertex labeling

$f : V \rightarrow \{1, 2, ..., 3\ell + m + n + 5 = 6\ell + 2m + 5\}$ is defined as follows:

$f(u) = 1$; \quad $f(v) = 2$; \quad $f(w) = 3$;

$f(x) = 4$; \quad $f(y) = 6\ell + 2m + 5$;

$f(u_i) = 2i + 3 \quad 1 \leq i \leq \ell$

$f(v_i) = 2\ell + 2i + 3 \quad 1 \leq i \leq \ell$

$f(w_i) = 4\ell + 2i + 3 \quad 1 \leq i \leq \ell$

$f(x_i) = 6\ell + 2i + 3 \quad 1 \leq i \leq m$

$f(y_i) = 2i + 4 \quad 1 \leq i \leq n = 3\ell + m$
The edge label of $uu_i$ is $2 + i$ for $1 \leq i \leq \ell$ (edge labels are $3, 4, ..., \ell + 2$), $vv_i$ is $\ell + 3 + i$ for $1 \leq i \leq \ell$ (edge labels are $\ell + 4, \ell + 5, \ell + 6, ..., 2\ell + 3$), $ww_i$ is $2\ell + 3 + i$ for $1 \leq i \leq \ell$ (edge labels are $2\ell + 4, 2\ell + 5, ..., 3\ell + 3$), $xx_i$ is $3\ell + 4 + i$ for $1 \leq i \leq m$ (edge labels are $3\ell + 5, 3\ell + 6, ..., 3\ell + m + 4$) and $yy_i$ is $3\ell + m + 5 + i$ for $1 \leq i \leq 3\ell + m$ (edge labels are $3\ell + m + 6, 3\ell + m + 7, ..., 6\ell + 2m + 5$).

The edge labels hence are $3, 4, ..., \ell + 2, \ell + 4, \ell + 5, ..., 2\ell + 3$, $2\ell + 4, 2\ell + 5, ..., 3\ell + 3, 3\ell + 5, 3\ell + 6, ..., 3\ell + m + 4$, $3\ell + m + 6, 3\ell + m + 7, ..., 6\ell + 2m + 5$.

These induced edge labels of graph $G$ are distinct.

Hence $G$ is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,7} \cup K_{1,22} \]

Figure 3.5.4
**Case 5:** Let $n = 3\ell + m - 1$.

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ be the vertex set of $G$

where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$,

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq \ell\}$,

$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq m\}$

and $V_5 = \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m - 1\}$.

Let $E = \{uu_i, vv_i, ww_i : 1 \leq i \leq \ell\} \cup \{xx_i : 1 \leq i \leq m\}$

$\cup \{yy_i : 1 \leq i \leq n = 3\ell + m - 1\}$ be the edge set of $G$.

$G$ has $3\ell + m + n + 5 = 6\ell + 2m + 4$ vertices and $6\ell + 2m - 1$ edges.

The vertex labeling

$$f : V \rightarrow \{1, 2, ..., 3\ell + m + n + 5 = 6\ell + 2m + 4\}$$

is defined as follows:

$$f(u) = 1; \quad f(v) = 2; \quad f(w) = 4;$$

$$f(x) = 6; \quad f(y) = 6\ell + 2m + 4;$$

$$f(u_i) = 2i + 1 \quad 1 \leq i \leq \ell$$

$$f(v_i) = 2\ell + 2i + 1 \quad 1 \leq i \leq \ell$$

$$f(w_i) = 4\ell + 2i + 1 \quad 1 \leq i \leq \ell$$
\[ f(x_i) = 6\ell + 2i + 1 \quad 1 \leq i \leq m \]
\[ f(y_i) = 2i + 6 \quad 1 \leq i \leq n - 1 = 3\ell + m - 2 \]
\[ f(y_{3\ell+m-1}) = 6\ell + 2m + 3 \]

The edge label of \( uu_i \) is \( 1+i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2, 3, \ldots, \ell+1 \)), \( vv_i \) is \( \ell+2+i \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell+3, \ell+4, \ldots, 2\ell+2 \)), \( ww_i \) is \( 2\ell+3+i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2\ell+4, 2\ell+5, \ldots, 3\ell+3 \)), \( xx_i \) is \( 3\ell+4+i \) for \( 1 \leq i \leq m \) (edge labels are \( 3\ell+5, 3\ell+6, \ldots, 3\ell+m+4 \)) and \( yy_i \) is \( 3\ell+m+5+i \) for \( 1 \leq i \leq 3\ell+m-2 \) (edge labels are \( 3\ell+m+6, 3\ell+m+7, \ldots, 6\ell+2m+3 \)), \( yy_{3\ell+m-1} \) is \( 6\ell+2m+4 \).

Thus edge labels are \( 2, 3, \ldots, \ell+1, \ell+3, \ell+4, \ldots, 2\ell+2, 2\ell+4, 2\ell+5, \ldots, 3\ell+3, 3\ell+5, 3\ell+6, \ldots, 3\ell+m+4, 3\ell+m+6, 3\ell+m+7, \ldots, 6\ell+2m+3, 6\ell+2m+4 \)

These induced edge labels of graph \( G \) are distinct.

Hence \( G \) is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,6} \cup K_{1,20} \]

Figure 3.5.5
Case 6: Let $n = 3\ell + m - 2$.

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ be the vertex set of $G$

where $V_i = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$,

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq \ell\}$,

$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq m\}$ and

$V_5 = \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m - 2\}$.

Let $E = \{u_1, v_1, w_i : 1 \leq i \leq \ell\} \cup \{x_i : 1 \leq i \leq m\}$

$\cup \{y_i : 1 \leq i \leq n = 3\ell + m - 2\}$ be the edge set of $G$.

$G$ has $3\ell + m + n + 5 = 6\ell + 2m + 3$ vertices and $6\ell + 2m - 2$ edges.

The vertex labeling

$f : V \rightarrow \{1, 2, ..., 3\ell + m + n + 5 = 6\ell + 2m + 4\}$ is

defined as follows:

$f(u) = 1$; \hspace{1cm} $f(v) = 2$; \hspace{1cm} $f(w) = 4$;

$f(x) = 6$; \hspace{1cm} $f(y) = 6\ell + 2m + 3$;

$f(u_i) = 2i + 1$ \hspace{1cm} $1 \leq i \leq \ell$

$f(v_i) = 2\ell + 2i + 1$ \hspace{1cm} $1 \leq i \leq \ell$

$f(w_i) = 4\ell + 2i + 1$ \hspace{1cm} $1 \leq i \leq \ell$
\[ f(x_i) = 6\ell + 2i + 1 \quad 1 \leq i \leq m \]
\[ f(y_i) = 2i + 6 \quad 1 \leq i \leq n = 3\ell + m - 2 \]

The edge label of \( uu_i \) is \( 1+i \) for \( 1 \leq i \leq \ell \) (edge labels are 2,3,...,\( \ell+1 \)), \( vv_i \) is \( \ell + 2 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell + 3, \ell + 4, \ldots, 2\ell + 2 \)), \( ww_i \) is \( 2\ell + 3 + i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2\ell + 4, 2\ell + 5, \ldots, 3\ell + 3 \)), \( xx_i \) is \( 3\ell + 4 + i \) for \( 1 \leq i \leq m \) (edge labels are \( 3\ell + 5, 3\ell + 6, \ldots, 3\ell + m + 4 \)) and \( yy_i \) is \( 3\ell + m + 5 + i \) for \( 1 \leq i \leq 3\ell + m - 2 \) (edge labels are \( 3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 3 \)).

The edge labels hence are 2,3,...,\( \ell + 1 \), \( \ell + 3, \ell + 4, \ldots, 2\ell + 2 \), \( 2\ell + 4, 2\ell + 5, \ldots, 3\ell + 3 \), \( 3\ell + 5, 3\ell + 6, \ldots, 3\ell + m + 4 \), \( 3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 3 \).

These induced edge labels of graph \( G \) are distinct.

Hence \( G \) is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,7} \cup K_{1,20} \]

**Figure 3.5.6**
Case 7: Let $n = 3\ell + m - 3$.

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ be the vertex set of $G$ where $V_i = \{u\} \cup \{u_i : 1 \leq i \leq \ell\}$,

$V_2 = \{v\} \cup \{v_i : 1 \leq i \leq \ell\}$,

$V_3 = \{w\} \cup \{w_i : 1 \leq i \leq \ell\}$,

$V_4 = \{x\} \cup \{x_i : 1 \leq i \leq m\}$ and

$V_5 = \{y\} \cup \{y_i : 1 \leq i \leq n = 3\ell + m - 3\}$.

Let $E = \{uu_i, vv_i, ww_i : 1 \leq i \leq \ell\} \cup \{xx_i : 1 \leq i \leq m\}$

$\cup \{yy_i : 1 \leq i \leq n = 3\ell + m - 3\}$ be the edge set of $G$.

$G$ has $3\ell + m + n + 5 = 6\ell + 2m + 2$ vertices and $6\ell + 2m - 3$ edges.

The vertex labeling

$f : V \rightarrow \{1, 2, ..., 3\ell + m + n + 5 = 6\ell + 2m + 2\}$ is defined as follows:

$f(u) = 1; \quad f(v) = 3; \quad f(w) = 5; \quad f(x) = 7; \quad f(y) = 6\ell + 2m + 2; \quad f(u_i) = 2i \quad 1 \leq i \leq \ell$

$f(v_i) = 2\ell + 2i \quad 1 \leq i \leq \ell$

$f(w_i) = 4\ell + 2i \quad 1 \leq i \leq \ell$
\[ f(x_i) = 6\ell + 2i \quad 1 \leq i \leq m \]
\[ f(y_i) = 2i + 7 \quad 1 \leq i \leq n = 3\ell + m - 3 \]

The edge label of \( uu_i \) is \( 1+i \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2, 3, \ldots, \ell +1 \)), \( vv_i \) is \( \ell + 2 + 1 \) for \( 1 \leq i \leq \ell \) (edge labels are \( \ell + 3, \ell + 4, \ldots, 2\ell + 2 \)), \( ww_i \) is \( 2\ell + 3 + 1 \) for \( 1 \leq i \leq \ell \) (edge labels are \( 2\ell + 4, 2\ell + 5, \ldots, 3\ell + 3 \)), \( xx_i \) is \( 3\ell + 4 + i \) for \( 1 \leq i \leq m \) (edge labels are \( 3\ell + 5, 3\ell + 6, \ldots, 3\ell + m + 4 \)) and \( yy_i \) is \( 3\ell + m + 5 + i \) for \( 1 \leq i \leq 3\ell + m - 3 \) (edge labels are \( 3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 2 \)).

The edge labels hence are \( 2, 3, \ldots, \ell + 1, \ell + 3, \ell + 4, \ldots, 2\ell + 2, \]
\( 2\ell + 4, 2\ell + 5, \ldots, 3\ell + 3, 3\ell + 5, 3\ell + 6, \ldots, 3\ell + m + 4, \]
\( 3\ell + m + 6, 3\ell + m + 7, \ldots, 6\ell + 2m + 2. \)

These induced edge labels of graph \( G \) are distinct.

Hence \( G \) is a skolem mean graph.
Example:

\[ K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,6} \cup K_{1,18} \]

**Figure 3.5.7**

We conjecture that for \( r \geq 1 \), \( rK_{1,\ell} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if \( |n - m - r\ell| \leq 3 \).