CHAPTER - 2
CHAPTER-2

RELAXED MEAN LABELING

In this chapter relaxed mean labeling is discussed. The condition for a graph to be relaxed mean graph is that \( p = q + 1 \). We prove that any path \( P_m \) is a relaxed mean graph and \( K_{1,n} \) is not a relaxed mean graph for \( m \geq 5 \). Also, we prove that the two star with an extra edge in common \( (K_{1,m} \cup K_{1,n}) \) is a relaxed mean graph if and only if \( |m-n| \leq 5 \).

2.1 Definition: A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to be a relaxed mean graph if there exists a function \( f \) from the vertex set of \( G \) to \( \{0, 1, 2, \ldots, q-1, q+1\} \) such that the induced map \( f^* \) from the edge set of \( G \) to \( \{1, 2, 3, \ldots, q\} \) defined by

\[
f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd,} \end{cases}
\]

then the edges get distinct labels from \( 1, 2, 3, \ldots, q \).

2.2 Note: In a relaxed mean graph \( p = q +1 \). Trees are the only connected graphs satisfying the condition \( p = q + 1 \).

In the following theorem we show that any path is a relaxed mean graph.
2.3 Theorem: Any path \( P_m \) is a relaxed mean graph.

Proof: Let \( P_m \) be the path with edges \( u_0u_1, u_1u_2, u_2u_3 \ldots u_{m-1}u_m \).

Define a map \( f : V(P_m) \rightarrow \{0, 1, 2, \ldots, m-1, m+1\} \) by

\[
f(u_i) = i \quad \text{for} \quad 0 \leq i \leq m-1
\]

\[
f(u_m) = m+1.
\]

The corresponding edge labels are distinct which makes \( P_m \) a relaxed mean graph.

Example:

Relaxed mean labeling of \( P_7 \) is given below:

![Figure 2.3.1](image)

Next we show that \( K_{1,m} \) for \( m \geq 5 \) is not a relaxed mean graph. Since \( K_{1,1} \) and \( K_{1,2} \) are path graphs, they are relaxed mean graphs. and we give the relaxed mean labeling for \( K_{1,3} \) and \( K_{1,4} \) below:

Let \( G(V, E) = K_{1,3} \). Then \( |V| = p = 4 \) and \( |E| = q = 3 \) and \( q + 1 = 4 \).
Let $G(V, E) = K_{1, 4}$. Then $|V| = p = 5$ and $|E| = q = 4$ and $q + 1 = 5$.

Thus we have the following result:

**2.4 Result:** $K_{1, m}$ is a relaxed mean graph if $m \leq 4$

**2.5 Theorem:** If $m \geq 5$, then $K_{1, m}$ is not a relaxed mean graph.

**Proof.** Suppose that $G = K_{1, m}$, $m \geq 5$ is a relaxed mean graph. Then the distinct edge labels are $\{1, 2, 3, \ldots, m\}$. Let $(V_1, V_2)$ be the bipartition of $K_{1, m}$ with $V_1 = \{u\}$. To get the edge label $m$, we must have $m - 1$ and $m + 1$ or $m - 2$ and
m + 1 as the vertex labels of adjacent vertices. Clearly one of m − 2, m − 1 and m + 1 must be the label of u. In each case, since m ≥ 5, there will be no edge with label 1. This contradiction proves that $K_{1,m}$ is not a relaxed mean graph. Hence the theorem.

We define two stars and find the characterization for the relaxed mean labeling of two stars.

2.6 Definition: The two star is the disjoint union of $K_{1,m}$ and $K_{1,n}$. It is denoted by $K_{1,m} \cup K_{1,n}$.

2.7 Notation: The two star $G = (K_{1,m} \cup K_{1,n})$ with an extra edge in common is denoted as $K_{1,m} \cup' K_{1,n}$. The extra edge is between a vertex of $K_{1,m}$ and a vertex of $K_{1,n}$. It can either between two pendant vertices or between two vertices with one of them is pendant or between two non pendant vertices.

2.8 Theorem: The two star $G = (K_{1,m} \cup K_{1,n})$ with an extra edge in common $(K_{1,m} \cup' K_{1,n})$ is a relaxed mean graph if and only if $|m − n| \leq 5$.

Proof: Let us first assume that $|m−n| ≤ 5$.

Without loss of generality, we assume that $m ≤ n$. There are six cases viz. $n = m$, $n = m + 1$, $n = m + 2$, 

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n = m + 3, n = m + 4 and n = m + 5. In each of these cases we prove that G is a relaxed mean graph.

**Case 1:** Let n = m. The extra edge is between two pendant vertices. Consider the graph G = K_{1,m} ∪ 'K_{1,m}.

Let V = V_1 ∪ V_2 where V_1 = \{u\} ∪ \{u_i : 1 \leq i \leq m\} and V_2 = \{v\} ∪ \{v_j : 1 \leq j \leq m\} be the vertex set of the two copies of K_{1,m}. Let E = \{uu_i : 1 \leq i \leq m\} ∪ \{vv_j : 1 \leq j \leq m\} ∪ \{u_a v_b\} be the edge set of G where u_a v_b is the extra edge joining two pendant vertices of G. Then G has 2m + 1 edges and 2m + 2 vertices.

The vertex labeling f : V(G) → \{0, 1, 2, \ldots, 2m, 2m + 2\} is defined as follows:

\[f(u) = 2; \ f(v) = 2m - 1 \]
\[f(u_i) = 2i - 1 \quad \text{for} \ 1 \leq i \leq m - 1 \]
\[f(u_m) = 0 \]
\[f(v_j) = 2j + 2 \quad \text{for} \ 1 \leq j \leq m. \]

The corresponding edge labels are as follows:

The edge label of uu_m is 1 and uu_i is i + 1 for 1 ≤ i ≤ m - 1.

The edge label of vv_j is m + j + 1 for 1 ≤ j ≤ m. The labels of u_i 's are odd numbers and those of v_j 's are even.
numbers. Hence any two adjacent numbers are labels of one \( u_i \) and one \( v_j \). The extra edge is so chosen that the corresponding vertices \( u_a \) and \( v_b \) are having labels \( m \) and \( m+1 \) and hence the resulting induced edge label is \( m + 1 \). Therefore, the set of edge labels of \( G \) is \( \{1, 2, 3, \ldots, m, m + 1, m + 2, \ldots, 2m + 1\} \) and it has \( 2m + 1 \) distinct edge labels. Hence the induced edge labels of \( G \) are distinct.

**Example:**

\[ K_{1,10} \cup K_{1,10} \]

**Figure 2.8.1**
Case 2: Let $n = m + 1$.

Consider the graph $G = (K_{1,m} \cup' K_{1,m+1})$.

Let $V = V_1 \cup V_2$ where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $V_2 = \{v\} \cup \{v_j : 1 \leq j \leq m + 1\}$ be the vertex set of $K_{1,m}$ and $K_{1,(m+1)}$ respectively. Then $G$ has $2m + 3$ vertices and $2m + 2$ edges.

The vertex labeling $f : V(G) \to \{0, 1, 2, \ldots, 2m + 1, 2m+3\}$ is defined as follows:

$f(u) = 1$; $f(v) = 2m$;

$f(u_i) = 2i - 2$ for $1 \leq i \leq m$ and $f(v_j) = 2j + 1$ for $1 \leq j \leq m + 1$.

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $i$ for $1 \leq i \leq m$ and $vv_i$ is $m + i + 1$ for $1 \leq i \leq m + 1$. The labels of $u_i$'s are even numbers and those of $v_j$'s are odd numbers. Hence any two adjacent numbers are labels of one $u_i$ and one $v_j$. The extra edge is so chosen that the corresponding vertices $u_i$ and $v_j$ are having labels $m$ and $m+1$ and hence the resulting induced edge label is $m + 1$. Therefore, the resulting set of edge
labels of \( G \) is \( \{1, 2, 3, \ldots, m, m + 1, m + 2, \ldots, 2m + 2\} \) and it has \( 2m + 2 \) distinct edge labels.

Hence the induced edge labels of \( G \) are distinct.

**Example:**

![Diagram of \( K_{1,10} \cup K_{1,11} \)]

**Figure 2.8.2**

**Case 3:** Let \( n = m + 2 \).

Consider the graph \( G = (K_{1,m} \cup' K_{1,m+2}). \)

Let \( V = V_1 \cup V_2 \) where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq m\} \) and \( V_2 = \{v\} \cup \{v_j : 1 \leq j \leq m+2\} \) be the vertex set of \( K_{1,m} \) and
The vertex labeling $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2m + 2, 2m + 4\}$ is defined as follows:

- $f(u) = 0$; $f(v) = 2m + 1$;
- $f(u_i) = 2i - 1$ for $1 \leq i \leq m$ and
- $f(v_j) = 2i$ for $1 \leq i \leq m + 2$.

The corresponding edge labels are as follows:

- The edge label of $uu_i$ is $i$ for $1 \leq i \leq m$ and $vv_j$ is $m + j + 1$ for $1 \leq j \leq m + 2$. The labels of $u_i$'s are odd numbers and those of $v_j$'s are even numbers. Hence any two adjacent numbers are labels of one $u_i$ and one $v_j$. The extra edge is so chosen that the corresponding vertices $u_i$ and $v_j$ are having labels $m$ and $m + 1$ and hence the resulting induced edge label is $m + 1$. Therefore, the resulting set of edge labels of $G$ is \{1, 2, 3, \ldots, m, m + 1, m + 2, \ldots, 2m + 3\} and it has $2m + 3$ distinct edge labels. Hence the induced edge labels of $G$ are distinct.
Example:

\[ K_{1,10} \cup' K_{1,12} \]

**Figure 2.8.3**

**Case 4:** Let \( n = m + 3 \).

Consider the graph \( G = (K_{1,m} \cup' K_{1,m+3}) \).

Let \( V = V_1 \cup V_2 \) where, \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq m\} \) and \( V_2 = \{v\} \cup \{v_j : 1 \leq j \leq m+3\} \) be the vertex set of \( K_{1,m} \) and \( K_{1,(m+3)} \) respectively. \( G \) has \( 2m + 5 \) vertices and \( 2m + 4 \) edges.
The vertex labeling

\[ f : V(G) \rightarrow \{0, 1, 2, \ldots, 2m + 3, 2m + 5\} \]

is defined as follows:

\[ f(u) = 0; \quad f(v) = 2m + 2 \]

\[ f(u_i) = 2i \quad \text{for} \quad 1 \leq i \leq m \quad \text{and} \]

\[ f(v_j) = 2j - 1 \quad \text{for} \quad 1 \leq j \leq m + 3. \]

The corresponding edge labels are as follows:

The edge label of \( u_i \) is \( i \) for \( 1 \leq i \leq m \) and \( v_j \) is \( m+j+1 \) for \( 1 \leq j \leq m + 3 \). The labels of \( u_i \)'s are even numbers and those of \( v_j \)'s are odd numbers. Hence any two adjacent numbers are labels of one \( u_i \) and one \( v_j \). The extra edge is so chosen that the corresponding vertices \( u_i \) and \( v_j \) are having labels \( m \) and \( m+1 \) and hence the resulting induced edge label is \( m + 1 \). Therefore, the resulting edge labels of \( G = \{1, 2, 3, \ldots, m, m + 1, m + 2, \ldots, 2m + 3\} \) and it has \( 2m + 3 \) distinct edge labels.

Hence the induced edge labels of \( G \) are distinct.
Example:

Figure 2.8.4

Case 5:

Let $n = m + 4$.

Consider the graph $G = (K_{1, m} \cup' K_{1, m+4})$.

Let $V = V_1 \cup V_2$ where $V_1 = \{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $V_2 = \{v\} \cup \{v_j : 1 \leq j \leq m+4\}$ be the vertex set of $K_{1, m}$ and
$K_{1, (m+4)}$ respectively. $G$ has $2m + 6$ vertices and $2m + 5$ edges.

The vertex labeling

$$f : V(G) \to \{0, 1, 2, \ldots, 2m + 4, 2m + 6\}$$

is defined as follows:

$$f(u) = 1; f(v) = 2m + 3;$$
$$f(u_i) = 2i + 1 \text{ for } 1 \leq i \leq m \text{ and }$$
$$f(v_j) = 2j - 2 \text{ for } 1 \leq j \leq m + 4$$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $i + 1$ for $1 \leq i \leq m$ and $vv_j$ is $m + j + 1$ for $1 \leq j \leq m + 4$. Also, the edge label of $uv_1$ is 1. Therefore, the induced edge labels of $G$ are 1, 2, 3, $\ldots$, $m$, $m + 1$, $m + 2$, $\ldots$, $2m + 5$ and has $2m + 5$ distinct edge labels.

Hence the induced edge labels of $G$ are distinct.
Example:

Figure 2.8.5

Case 6: Let \( n = m + 5 \).

Consider the graph \( G = (K_{1,m} \cup K_{1,m+5}) \).

Let \( V = V_1 \cup V_2 \) where \( V_1 = \{u\} \cup \{u_i : 1 \leq i \leq m\} \) and \( V_2 = \{v\} \cup \{v_j : 1 \leq j \leq m + 5\} \) be the vertex set of \( K_{1,m} \) and \( K_{1,(m+5)} \) respectively. Then \( G \) has \( 2m + 7 \) vertices and \( 2m + 6 \) edges.
The vertex labeling

\[ f : V(G) \rightarrow \{0, 1, 2, \ldots, 2m + 5, 2m + 7\} \]

defined as follows:

\[ f(u) = 1; \quad f(v) = 2m + 4; \]
\[ f(u_i) = 2i + 1 \quad \text{for } 1 \leq i \leq m \]
\[ f(v_j) = 2j - 2 \quad \text{for } 1 \leq j \leq m + 2 \]
\[ f(v_k) = 2k - 3 \quad \text{for } m + 3 \leq k \leq m + 5 \]

The corresponding edge labels are as follows:

The edge label of \(u_i\) is \(i + 1\) for \(1 \leq i \leq m\); \(v_j\) is \(m + j + 1\) for \(1 \leq j \leq m + 2\) and \(v_k\) is \(m + k + 1\) for \(m + 3 \leq k \leq m + 5\).

Also, the edge label of \(v_1u\) is 1. Therefore, the required set of edge labels of \(G\) is \(\{1, 2, 3, \ldots, m, m + 1, m + 2, \ldots, 2m + 5\}\)
and has \(2m + 5\) distinct edges.

Hence the induced edge labels of \(G\) are distinct.
Example:

\[ K_{1,10} \cup K_{1,15} \]

**Figure 2.8.6**

Hence, the graph \( G \) is a relaxed mean graph if \( |m-n| \leq 5 \).

Next we prove the converse that if \( G \) is a relaxed mean graph then \( |m-n| \leq 5 \). Equivalently we show that if \( |m-n| > 5 \) then \( G \) is not a relaxed mean graph.

Suppose that \( G = K_{1,m} \cup K_{1,n} \) for \( m = n + r \) for \( r > 5 \)

is a relaxed mean graph.
Let us assume that $G = G_1 \cup' G_2$ for $G_1 = K_{1, m + r}$ and $G_2 = K_{1, m}$.

Let us now consider the simplest case ($r = 6$ and $m = 1$). Then the graph $G = K_{1, 7} \cup' K_{1, 1}$, has 10 vertices and 9 edges.

Let $V(G) = \{v_{1, j} : 0 \leq j \leq 1\} \cup \{v_{2, j} : 0 \leq j \leq 7\}$ and $E(G) = \{v_{1,0}v_{1,1}, v_{1,1}v_{2,0}, v_{2,0}v_{2,1}, v_{2,1}v_{2,j} : 1 \leq j \leq 7\}$.

Suppose $G$ is a relaxed mean graph.

Let $p = |V| = 10$ and $q = |E| = 9$.

Then there exists a function $f$ from the vertex set of $G$ to \{0, 1, ..., $q - 1$, $q + 1$\} such that the induced map $f^*$ from the edge set of $G$ to \{1, 2, ..., $q$\} defined by,

$$f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

then the edges get distinct labels from \{1, 2, ..., $q$\}.

For $i = 1, 2$ let $t_{i, j}$ be the label given to the vertex $v_{i, j}$ for $0 \leq j \leq 1$ and $v_{2, j}$ for $0 \leq j \leq 7$. 

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Let $x_{1,1}$ be the edge label of the edge $v_{1,0}v_{1,1}$ and $x_{2,j}$ be the edge label of the edge $v_{2,0}v_{2,j}$ for $1 \leq j \leq 7$.

**Case 1:** Let us first consider the case that $t_{2,0} = 10$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{10 + 2n - 1}{2} \right\rceil = n + 5 = \left\lceil \frac{10 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), (5 or 6), 7 and 10. These six labels are not sufficient to label the seven vertices $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is a not a relaxed mean graph when $t_{2,0} = 10$.

**Case 2:** Let us next consider the case that $t_{2,0} = 8$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{8 + 2n - 1}{2} \right\rceil = n + 4 = \left\lceil \frac{8 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), (5 or 6), 7 and 10. These six labels are not sufficient to label the seven vertices $v_{2,j}$ for $1 \leq j \leq 7$.  

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Therefore, $G$ is a not a relaxed mean graph when $t_{2,0} = 8$.

**Case 3:** Let us next consider the case that $t_{2,0} = 7$.

If $t_{2,j} = 2n$ and $t_{2,k} = 2n + 1$ for some $n, j$ and $k$ then

$$f^*(v_{2,0}, v_{2,j}) = \left\lfloor \frac{7 + 2n}{2} \right\rfloor = n + 4 = \left\lfloor \frac{7 + 2n + 1}{2} \right\rfloor = f^*(v_{2,0}, v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are (0 or 1), (2 or 3), (4 or 5), 6, 8 and 10. These six labels are not sufficient to label the seven vertices, $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is a not a relaxed mean graph when $t_{2,0} = 7$.

**Case 4:** Let us next consider the case that $t_{2,0} = 6$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n, j$ and $k$ then

$$f^*(v_{2,0}, v_{2,j}) = \left\lfloor \frac{6 + 2n - 1}{2} \right\rfloor = n + 3 = \left\lfloor \frac{6 + 2n}{2} \right\rfloor = f^*(v_{2,0}, v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), 5, (7 or 8) and 10. These six labels are not sufficient to label the seven vertices, $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is a not a relaxed mean graph when $t_{2,0} = 6$. 
Case 5: Let us next consider the case that $t_{2,0} = 5$.

If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some $n, j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \left\lfloor \frac{5 + 2n + 1}{2} \right\rfloor = n + 3 = \left\lfloor \frac{5 + 2n}{2} \right\rfloor = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are (0 or 1), (2 or 3), 4, (6 or 7), 8 and 10. These six labels are not sufficient to label the seven vertices $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is not a relaxed mean graph when $t_{2,0} = 5$.

Case 6: Let us next consider the case that $t_{2,0} = 4$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n, j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \left\lfloor \frac{4 + 2n - 1}{2} \right\rfloor = n + 2 = \left\lfloor \frac{4 + 2n}{2} \right\rfloor = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), 3, (5 or 6), (7 or 8) and 10. These six labels are not sufficient to label the seven vertices, $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is not a relaxed mean graph when $t_{2,0} = 4$.

Case 7: Let us next consider the case that $t_{2,0} = 3$. 
If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \frac{3 + 2n + 1}{2} = n + 2 = \frac{3 + 2n}{2} = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are $(0 \text{ or } 1)$, $2$, $(4 \text{ or } 5)$, $(6 \text{ or } 7)$, $8$ and $10$. These six labels are not sufficient to label the seven vertices, $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is no not a relaxed mean graph when $t_{2,0} = 3$.

**Case 8:** Let us next consider the case that $t_{2,0} = 2$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then

$$f^*(v_{2,0}v_{2,j}) = \frac{2 + 2n - 1}{2} = n + 1 = \frac{2 + 2n}{2} = f^*(v_{2,0}v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are $0$, $1$, $(3 \text{ or } 4)$, $(5 \text{ or } 6)$, $(7 \text{ or } 8)$ and $10$. These six labels are not sufficient to label the seven vertices, $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is no not a relaxed mean graph when $t_{2,0} = 2$.

**Case 9:** Let us next consider the case that $t_{2,0} = 1$.

If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then
This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are 0, (2 or 3), (4 or 5), (6 or 7), 8 and 10. These six labels are not sufficient to label the seven vertices $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is not a relaxed mean graph when $t_{2,0} = 1$.

**Case 10:** Let us next consider the case that $t_{2,0} = 0$.

If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some $n$, $j$ and $k$ then

$$f^*(v_{2,0} v_{2,j}) = \left[\frac{1 + 2n + 1}{2}\right] = n + 1 = \left[\frac{1 + 2n}{2}\right] = f^*(v_{2,0} v_{2,k}).$$

This is not possible as $f^*$ is a bijection.

Therefore, the possible vertex labels are (1 or 2), (3 or 4), (5 or 6), (7 or 8) and 10. These five labels are not sufficient to label the seven vertices $v_{2,j}$ for $1 \leq j \leq 7$.

Therefore, $G$ is not a relaxed mean graph when $t_{2,0} = 0$.

Therefore, $G = K_{1,7} \cup ' K_{1,1}$ is not a relaxed mean graph for any possible the values of $t_{2,0}$.

i.e., $G = (K_{1,m} \cup ' K_{1,n})$ is not a relaxed mean graph when $|m - n| = 6$. 

Similarly, we can prove that $G = K_{1,8} \cup K_{1,1}$ is not a relaxed mean graph when $|m - n| = 7$.

Hence, $G = K_{1,n} \cup K_{1,m}$ is not a relaxed mean graph if $|m - n| \geq 6$.

Hence the theorem.

Thus we have proved that $G = K_{1,n} \cup K_{1,m}$ is a relaxed mean graph iff $|m - n| \leq 5$. 