CHAPTER 6

SEPARATION AXIOMS THROUGH $\alpha^*\text{- OPEN SET}$

In this chapter, we introduce $\alpha^*\text{-}T_i$ spaces ($i=0,1,2$) using $\alpha^*$-open set and investigate their properties. We give characterizations for these spaces. We study the relationship among themselves and with known separation axioms.

6.1. $\alpha^*\text{-}T_0$ SPACES

Definition 6.1.1: A topological space $X$ is said to be $\alpha^*\text{-}T_0$ space if for each pair of distinct points $x,y$ of $X$, there exists an $\alpha^*$-open set containing one point but not the other.

Theorem 6.1.2: Every $T_0$ space is a $\alpha^*\text{-}T_0$ space.

Proof: Let $X$ be a $T_0$ space. Let $x,y$ be two distinct points in $X$. Since $X$ is $T_0$ space, there exists an open set $M$ in $X$ such that $x \in M$, $y \not\in M$. Since, every open set is $\alpha^*$-open, $M$ is $\alpha^*$-open in $X$. Thus, for any two distinct points $x,y$ in $X$ there exists an $\alpha^*$-open $M$ such that $x \in M$, $y \not\in M$. Hence $X$ is a $\alpha^*\text{-}T_0$ space.

Remark 6.1.3: The following example supports that the converse of the above theorem is not true in general.

Example 6.1.4: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a,b\}, X\}$. In this space, $\alpha^* O (X, \tau) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$. Clearly $X$ is $\alpha^*\text{-}T_0$ but not $T_0$ space.

Remark 6.1.5: Clearly, every $\alpha\text{-}T_0$ space is a $\alpha^*\text{-}T_0$ space. Since, every $\alpha$ open set is $\alpha^*$-open but the converse is not true.
Example 6.1.6: Let \( X = \{a, b, c\}, \tau = \{\phi, \{a,b\}, X\} \). In this space
\[ \alpha^* \mathcal{O}(X, \tau) = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\} \]
and
\[ \alpha \mathcal{O}(X, \tau) = \{\phi, \{a,b\}, X\}. \]
Clearly \( X \) is \( \alpha^* \)-T\(_0\) but not \( \alpha^* \)-T\(_0\) space.

Theorem 6.1.7: A space \( X \) is an \( \alpha^* \)-T\(_0\) space if and only if \( \alpha^* \)-closures of distinct points are distinct.

**Proof:** Let \( x,y \in X \) with \( x \neq y \) and \( X \) be an \( \alpha^* \)-T\(_0\) space. We shall show that
\[ \alpha^* \text{cl}(\{x\}) \neq \alpha^* \text{cl}(\{y\}). \]
Since \( X \) is \( \alpha^* \)-T\(_0\), there exists an \( \alpha^* \)-open \( M \) such that \( x \in M, y \in M \). Also, \( x \notin X-M \) and \( y \in X-M \), where \( X-M \) is \( \alpha^* \)-closed set in \( X \). Since \( \alpha^* \text{cl}(\{y\}) \) is the intersection of all \( \alpha^* \)-closed sets which contain \( y \).
Hence \( y \in \alpha^* \text{cl}(\{y\}) \) but \( x \notin \alpha^* \text{cl}(\{y\}) \) as \( x \notin X-M \). Therefore,
\[ \alpha^* \text{cl}(\{x\}) \neq \alpha^* \text{cl}(\{y\}). \]
Conversely, suppose that for any pair of distinct points \( x,y \in X \),
\[ \alpha^* \text{cl}(\{x\}) \neq \alpha^* \text{cl}(\{y\}). \]
Then, there exists at least one point \( z \in X \) such that \( z \in \alpha^* \text{cl}(\{x\}) \) but \( z \notin \alpha^* \text{cl}(\{y\}). \)
We claim that \( x \notin \alpha^* \text{cl}(\{y\}). \) If \( x \in \alpha^* \text{cl}(\{y\}) \) then \( \alpha^* \text{cl}(\{x\}) \subseteq \alpha^* \text{cl}(\{y\}). \)
So, \( z \in \alpha^* \text{cl}(\{x\}) \) which is a contradiction.
Now, \( x \notin \alpha^* \text{cl}(\{y\}) \) implies \( x \in X - \alpha^* \text{cl}(\{y\}) \) which is a \( \alpha^* \)-open set in \( X \) containing \( x \) but not \( y \). Hence, \( X \) is \( \alpha^* \)-T\(_0\) space.

Theorem 6.1.8: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijection, \( \alpha^* \)-open map and \( X \) is \( \alpha^* \)-T\(_0\) space, then \( Y \) is also \( \alpha^* \)-T\(_0\) space.

**Proof:** Let \( y_1, y_2 \in Y \) with \( y_1 \neq y_2 \). Since \( f \) is a bijection, there exists \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Since, \( X \) is \( \alpha^* \)-T\(_0\), there exists a \( \alpha^* \)-open set \( M \) in \( X \) such that \( x_1 \in M, x_2 \notin M \).
Since, \( f \) is \( \alpha^* \)-open map, \( f(M) \) is...
a $\alpha^*$-open set in $Y$. Now, we have $x_1 \in M \Rightarrow f(x_1) \in f(M) \Rightarrow y_1 \in f(M)$ and $x_2 \notin M \Rightarrow f(x_2) \notin f(M) \Rightarrow y_2 \notin f(M)$. Hence, for any two distinct points $y_1, y_2 \in Y$, there exists $\alpha^*$-open set $f(M)$ in $Y$ such that $y_1 \in f(M)$ and $y_2 \notin f(M)$. Hence $Y$ is a $\alpha^*$-$T_0$ space.

**Theorem 6.1.9**: Let $f: (X, \mathcal{T}) \to (Y, \sigma)$ be a bijection, $\alpha^*$-open map and $X$ is $T_0$ space, then $Y$ is also $\alpha^*$-$T_0$ space.

**Proof**: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is a bijection, there exists $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since, $X$ is $T_0$, there exists an open set $M$ in $X$ such that $x_1 \in M$, $x_2 \notin M$. Since, $f$ is $\alpha^*$-open map, $f(M)$ is a $\alpha^*$-open set in $Y$. Now, we have $x_1 \in M \Rightarrow f(x_1) \in f(M) \Rightarrow y_1 \in f(M)$ and $x_2 \notin M \Rightarrow f(x_2) \notin f(M) \Rightarrow y_2 \notin f(M)$. Hence, for any two distinct points $y_1, y_2 \in Y$, there exists $\alpha^*$-open set $f(M)$ in $Y$ such that $y_1 \in f(M)$ and $y_2 \notin f(M)$. Hence $Y$ is a $\alpha^*$-$T_0$ space.

**Theorem 6.1.10**: Let $f: (X, \mathcal{T}) \to (Y, \sigma)$ be a bijection, $\alpha^*$-irresolute and $Y$ is $\alpha^*$-$T_0$ space, then $X$ is also $\alpha^*$-$T_0$ space.

**Proof**: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is a bijection, there exists $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2 \Rightarrow x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since, $Y$ is $\alpha^*$-$T_0$, there exists an $\alpha^*$-open set $M$ in $Y$ such that $y_1 \in M$, $y_2 \notin M$. Since, $f$ is $\alpha^*$-irresolute map, $f^{-1}(M)$ is a $\alpha^*$-open set in $X$. Now, we have $y_1 \in M \Rightarrow f^{-1}(y_1) \in f^{-1}(M) \Rightarrow x_1 \in f^{-1}(M)$ and $y_2 \notin M \Rightarrow f^{-1}(y_2) \notin f^{-1}(M) \Rightarrow x_2 \notin f^{-1}(M)$. Hence, for any two distinct points $x_1, x_2 \in X$, there exists
\( \alpha \)-open set \( f^{-1}(M) \) in \( X \) such that \( x_1 \in f^{-1}(M) \) and \( x_2 \notin f^{-1}(M) \). Hence \( X \) is a \( \alpha \)-\( T_0 \) space.

**Theorem 6.1.11:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijection, \( \alpha \)-continuous and \( Y \) is T\(_0\) space, then \( X \) is also \( \alpha \)-\( T_0 \) space.

**Proof:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a \( \alpha \)-continuous map and \( Y \) is T\(_0\) space. Let \( x_1, x_2 \in X \), with \( x_1 \neq x_2 \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since, \( f \) is one to one, \( y_1 \neq y_2 \). Since, \( Y \) is T\(_0\), there exists an open set \( M \) in \( Y \) containing \( y_1 \) or \( y_2 \) but not the other. Since, \( f \) is \( \alpha \)-continuous, \( f^{-1}(M) \) is a \( \alpha \)-open set containing one of \( x_1 \) and \( x_2 \) but not the other. Thus, \( X \) is \( \alpha \)-\( T_0 \) space.
6.1 $\alpha^*\cdot T_1$ SPACE

**Definition 6.2.1:** A topological space $X$ is said to be $\alpha^*\cdot T_1$ space if for each pair of distinct points $x, y$ of $X$, there exists a pair of $\alpha^*$-open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

**Remark 6.2.2:** Every $\alpha^*\cdot T_1$ space is $\alpha^*\cdot T_0$ space but the converse is not true.

**Example 6.2.3:** Let $X = Y = \{a, b, c\}$, $\mathcal{T} = \{\phi, \{a\}, \{ab\}, X\}$. In this space $\alpha^*O(X, \mathcal{T}) = \{\phi, \{a\}, \{b\}, \{ab\}, \{ac\}, X\}$. It is $\alpha^*\cdot T_0$ but not $\alpha^*\cdot T_1$ because for the pair of distinct points $a$ and $c$, there is no $\alpha^*$-open sets containing $c$ but not $a$.

**Theorem 6.2.4:** For a topological space $X$ the following are equivalent.

(i) $X$ is $\alpha^*\cdot T_1$ space.

(ii) for every $x \in X$, $\{x\}$ is $\alpha^*$-closed in $X$.

(iii) Each subset of $X$ is the intersection of $\alpha^*$-open sets containing it.

(iv) The intersection of all $\alpha^*$-open sets in $X$ containing the points $x$ is $\{x\}$.

**Proof:**

(i) $\Rightarrow$ (ii):

Suppose that $X$ is $\alpha^*\cdot T_1$ space. Let $x \in X$. Then for every $y \neq x$, there exists a $\alpha^*$-open sets $U$ in $X$ containing $y$ but not $x$. $U \cap \{x\} \neq \phi$. Therefore, $x \in U$ a contradiction. Thus, $x$ is $\alpha^*$-closed.

(ii) $\Rightarrow$ (iii):

Let $A \subseteq X$. Then, for each $x \in X \setminus A$, $\{x\}$ is $\alpha^*$-closed in $X$ and hence, $X \setminus \{x\}$ is $\alpha^*$-open. Clearly, $A \subseteq X \setminus \{x\}$ for each $x \in X \setminus A$. Therefore,
A \subseteq \cap \{ X\{x\}: x \in X\A \}. On the other hand, if y \not\in A, then y \in X\setminus A and y \not\in X\setminus \{y\}. This implies y \not\in \cap \{ X\{x\}: x \in X\A \}. Hence, \cap \{ X\{x\}: x \in X\A \} \subseteq A.

Therefore, A = \cap \{ X\{x\}: x \in X\A \} which proves (iii)

(iii) \Rightarrow (iv):

Taking A = \{x\}, by (iii) A = \{x\} = \cap \{ U: U is \alpha^*\text{-open and } x \in U \}. This proves (iv)

(iv) \Rightarrow (i):

Let x, y \in X with y \neq x. Then y \not\in \{x\} = \cap \{ U: U is \alpha^*\text{-open and } x \in U \}. Hence, there exists a \alpha^*\text{-open set } U\text{ containing } x\text{ but not } y. Similarly, there exists a \alpha^*\text{-open set } V\text{ containing } y\text{ but not } x. Thus, X is \alpha^*\text{-T}_1\text{ space.}

**Theorem 6.2.5:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijection

(i) If \( f \) is a \( \alpha^*\text{-continuous} \) and \( Y \) is \( \alpha^*\text{-T}_1 \), then \( X \) is \( \alpha^*\text{-T}_1 \).

(ii) If \( f \) is a \( \alpha^*\text{-irresolute} \) and \( Y \) is \( \alpha^*\text{-T}_1 \), then \( X \) is \( \alpha^*\text{-T}_1 \).

(iii) If \( f \) is \( \alpha^*\text{-open} \) and \( X \) is \( \alpha^*\text{-T}_1 \), then \( Y \) is \( \alpha^*\text{-T}_1 \).

**Proof:**

(i) Suppose \( f \) is \( \alpha^*\text{-continuous} \) bijection and \( Y \) is \( \alpha^*\text{-T}_1 \). Let \( x_1, x_2 \in X \), with \( x_1 \neq x_2 \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since, \( f \) is one to one, \( y_1 \neq y_2 \). Since, \( Y \) is \( \alpha^*\text{-T}_1 \), there exists an open set \( U \) and \( V \) in \( Y \) such that \( y_1 \in U \) but \( y_2 \not\in U \) and \( y_2 \in V \) but \( y_1 \not\in V \). Since, \( f \) is bijection, \( x_1 \in f^{-1}(U) \) but \( x_2 \not\in f^{-1}(U) \) and \( x_1 \not\in f^{-1}(V) \) but \( x_2 \in f^{-1}(V) \). Since, \( f \) is \( \alpha^*\text{-continuous} \), \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \alpha^*\text{-open sets in } X \). Thus, \( X \) is \( \alpha^*\text{-T}_1 \).

(ii) Suppose \( f \) is \( \alpha^*\text{-irresolute} \) bijection and \( Y \) is \( \alpha^*\text{-T}_1 \). Let \( x_1, x_2 \in X \), with \( x_1 \neq x_2 \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since, \( f \) is one to one, \( y_1 \neq y_2 \). Since, \( Y \) is
\( \alpha^*-T_1 \), there exists an \( \alpha^*- \) open set \( U \) and \( V \) in \( Y \) such that \( y_1 \in U \) but \( y_2 \notin U \) and \( y_2 \in V \) but \( y_1 \notin V \). Since, \( f \) is bijection, \( x_1 \in f^{-1}(U) \) but \( x_2 \notin f^{-1}(U) \) and \( x_1 \notin f^{-1}(V) \) but \( x_2 \in f^{-1}(V) \). Since, \( f \) is \( \alpha^*- \) irresolute, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \alpha^*- \) open sets in \( X \). Thus, \( X \) is \( \alpha^*-T_1 \).

(iii) Suppose \( f \) is a \( \alpha^*- \) open bijection and \( X \) is \( T_1 \). Let \( y_1 \neq y_2 \in Y \). Since, \( f \) is a bijection, there exists \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \) with \( x_1 \neq x_2 \).

Since, \( X \) is \( T_1 \), there exists open sets \( U \) and \( V \) in \( X \) such that \( x_1 \in U \) but \( x_2 \notin U \) and \( x_2 \in V \) but \( x_1 \notin V \). Since, \( f \) is \( \alpha^*- \) open, \( f(U) \) and \( f(V) \) are \( \alpha^*- \) open sets in \( Y \) such that \( y_1 = f(x_1) \in f(U) \) and \( y_2 = f(x_2) \in f(V) \). Since, \( f \) is a bijection, \( y_2 = f(x_2) \notin f(U) \) and \( y_1 = f(x_1) \notin f(V) \). Thus, \( Y \) is \( \alpha^*-T_1 \).

**Theorem 6.2.6:** A topological space \( (X, \tau) \) is \( \alpha^*-T_1 \) if and only if the singletons are \( \alpha^*- \) closed.

**Proof:** Let \( (X, \tau) \) be \( \alpha^*-T_1 \) and \( x \) be any point of \( X \). Suppose \( y \in \{x\}^c \) then \( x \neq y \) and so there exists a \( \alpha^*- \) open \( U \) such that \( y \in U \) but \( x \notin U \). Consequently, \( y \in U \subseteq \{x\}^c \) that is \( \{x\}^c = \cup \{ U : y \in \{x\}^c \} \) which is \( \alpha^*- \) open. Hence, \( \{x\} \) is \( \alpha^*- \) closed.

Conversely, Let \( x,y \) be two distinct points of \( X \). Then \( y \in \{x\}^c \) and \( \{x\}^c \) is \( \alpha^*- \) open set containing \( y \) but not \( x \). Similarly, \( \{y\}^c \) is a \( \alpha^*- \) open set containing \( x \) but not \( y \). Hence, \( X \) is \( \alpha^*-T_1 \).
6.3. \( \alpha \ * - T_2 \) SPACES

**Definition 6.3.1:** A space \( X \) is said to be \( \alpha \ * - T_2 \) Spaces, if for each pair of distinct points \( x, y \) of \( X \), there exists disjoint \( \alpha \ * - \) open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

**Theorem 6.3.2:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijection

(i) If \( f \) is a \( \alpha \ * - \) continuous and \( Y \) is \( T_2 \), then \( X \) is \( \alpha \ * - T_2 \).

(ii) If \( f \) is a \( \alpha \ * - \) irresolute and \( Y \) is \( \alpha \ * - T_2 \), then \( X \) is \( \alpha \ * - T_2 \).

(iii) If \( f \) is \( \alpha \ * - \) open and \( X \) is \( T_2 \), then \( Y \) is \( \alpha \ * - T_2 \).

**Proof:**

(i) Suppose \( f \) is \( \alpha \ * - \) continuous bijection and \( Y \) is \( T_2 \). Let \( x_1, x_2 \in X \), with \( x_1 \neq x_2 \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since, \( f \) is one to one, \( y_1 \neq y_2 \). Since, \( Y \) is \( T_2 \), there exists an open set \( U \) and \( V \) containing \( y_1 \) and \( y_2 \) respectively. Since, \( f \) is \( \alpha \ * - \) continuous bijection, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( \alpha \ * - \) open sets in \( X \) containing \( x_1 \) and \( x_2 \) respectively. Thus \( X \) is \( \alpha \ * - T_2 \).

(ii) Proof is similar to (i)

(iii) Suppose \( f \) is \( \alpha \ * - \) open bijection and \( X \) is \( T_2 \). Let \( y_1 \neq y_2 \in Y \). Since \( f \) is bijection, there exist \( x_1, x_2 \) in \( X \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \) with \( x_1 \neq x_2 \). Since \( X \) is \( T_2 \), there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( x_1 \in U \) and \( x_2 \in V \). Since, \( f \) is \( \alpha \ * - \) open in \( Y \) such that \( y_1 = f(x_1) \in f(U) \) and \( y_2 = f(x_2) \in f(V) \). Since, \( f \) is a bijection, \( y_2 = f(x_2) \notin f(U) \) and \( y_1 = f(x_1) \notin f(V) \). Thus, \( Y \) is \( \alpha \ * - T_2 \).

**Theorem 6.3.4:** The following statements are equivalent for a topological space \( (X, \tau) \).
(i) $X$ is $\alpha^* T_2$.

(ii) Let $x \in X$ for each $x \neq y$, there exist a $\alpha^*$-open $U$ such that $x \in U$ and $y \notin \alpha^* \text{cl}(U)$

(iii) For each $x \in X$, $\cap \{ \alpha^* \text{cl}(U) : U \in \alpha^* O(X) \text{ and } x \in U \} = \{x\}$

Proof:

(i) $\Rightarrow$ (ii):

Suppose $X$ is a $\alpha^*$-T$_2$. Let $x \in X$ and $y \in X$ with $x \neq y$. Then there exist disjoint $\alpha^*$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Since $V$ is $\alpha^*$-open, $X \setminus V$ is $\alpha^*$-closed and $U \subseteq X \setminus V$. This implies that $\alpha^* \text{cl}(U) \subseteq X \setminus V$. Since, $y \notin X \setminus V$, $y \notin \alpha^* \text{cl}(U)$.

(ii) $\Rightarrow$ (iii):

If $x \neq y$ then there exist a $\alpha^*$-open set $U$ such that $x \in U$ and $y \notin \alpha^* \text{cl}(U)$.
Hence $y \notin \cap \{ \alpha^* \text{cl}(U) : U \in \alpha^* O(X) \text{ and } x \in U \}$. This proves (iii).

(iii) $\Rightarrow$ (i):

Let $x \neq y$ in $X$. Then $y \notin \cap \{ \alpha^* \text{cl}(U) : U \in \alpha^* O(X) \text{ and } x \in U \}$. This implies that there exist a $\alpha^*$-open set $U$ such that $x \in U$ and $y \notin \alpha^* \text{cl}(U)$. Then $V = X \setminus \alpha^* \text{cl}(U)$ is $\alpha^*$-open and $y \in V$. Now, $U \cap V = U \cap (X \setminus \alpha^* \text{cl}(U)) \subseteq U \cap (X \setminus V) = \emptyset$. This proves (i)