APPLICATIONS OF
MULTIPLE
HYPERGEOMETRIC
FUNCTION OF
SRIVASTAVA-DAOUS AND
THE SRIVASTAVA
POLYNOMIALS OF
SEVERAL VARIABLES IN A
PROBLEM INVOLVING
LAPLACE EQUATION
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1. Introduction. Appell’s functions and the functions related
to them have many applications in Mathematical Physics ([13],[14],[15],
Srivastava, Gupta and Goyal [27] have discussed a problem on heat
conduction in a finite bar involving $H$-function of two variables of
Srivastava and Panda ([24],[25],[26]). Chandel and Gupta [4] have
discussed this problem involving generalized multiple hypergeometric
function of Srivastava and Daoust ([19],[20],[21]; also see Srivastava and
Karlsson [23], p.37, eqns (2.1) to (2.3)); Srivastava and Manocha [22,
p.64 eqns. (18) to (20)) Chandel-Bhargava [1] and Chandel-Gupta [5]
have discussed the problem on cooling of a heated cylinder. Chandel and
Yadava [2] have evaluated certain integrals involving multiple hypergeometric
function of Srivastava and Daoust ([19],[20],[21]) and their applications
have been given in solving a problem on heat conduction. Chandel and
Tiwari [6] also employed multiple hypergeometric function of Srivastava
and Daoust ([19],[20],[21]) in two boundary value problems. Chaurasia
and Patni [9] have discussed a problem on heat conduction involving the
product of multivariable $H$-function of Srivastava and Panda ([24],[25],[26])
and two generalized polynomials of Srivastava [17], while Chaurasia and
Gupta [10] have discussed a solution of partial differential equation of heat
conduction in a rod under Robin condition.
Recently, Chandel and Sengar [7] have discussed two boundary value problems on heat conduction involving the product of multivariable \( H \)-function of Srivastava-Panda ([24],[25],[26]) and several generalized polynomials of Srivastava [17] and their special cases have been discussed. Further Chandel and Sengar [8] have also discussed a problem on heat conduction under Robin condition involving the product of above multivariable \( H \)-function ([24],[25],[26]) and several generalized polynomials of Srivastava [17].

In continuation of the above study, the present Chapter is motivated by the frequent requirement of various properties of special functions which play a vital role in the study of potential theory, heat conduction and other allied problems in Quantum Mechanics. We first evaluate a finite integral involving the product of multiple hypergeometric function of Srivastava and Daoust ([19],[20],[21]; also see Srivastava and Karlsson [23, p.37, eqns. (2.1) to (2.3)]; Srivastava and Manocha [22, p.64, eqns.(18) to (20)] and Srivastava's polynomials of several variables [18] and make its applications to solve a problem involving Laplace equation. In end of the Chapter various special cases have also be discussed.

### 10.2 Main Integral

In this section, we evaluate the following integral very useful in our further investigations:

\[
\left( 10.2.1 \right) \int_0^a \cos^{m-1}(\pi x/a) \cos(p \pi x/a) S^{A;B;\ldots;B^{(n)}}_{C;D;\ldots;D^{(n)}} \left( \begin{array}{c} (a) : \theta, \ldots, \theta^{(n)} \\ \psi, \ldots, \psi^{(n)} \end{array} \right)
\]

\[
\left[ \left( \delta^{(n)} \right) : \phi^{(n)} \right]
\]

\[
\left[ \left( \delta' \right) : \delta' \right]
\]

\[
S^{M_1,\ldots,M_r}_{N_1,\ldots,N_r} \left[ y_1 \cos^{2p_1}(\pi x/a), \ldots, y_r \cos^{2p_r}(\pi x/a) \right] dx
\]

\[
= \frac{a}{\sqrt{\pi 2^p}} \sum_{n=0}^{M_1} \sum_{s=0}^{M_2} (-N_1)_{M_1} \ldots (-N_r)_{M_r} A(N_1, s_1, \ldots; N_r, s_r) \frac{y_1^{s_1}}{s_1!} \ldots \frac{y_r^{s_r}}{s_r!}
\]
\[ S_{C^{+2D;...;D^{(e)}}}^{A^{+2B;...;B^{(e)}}} \left( \left[ (a): \theta^{(a)} \right] \left[ m + 2p, s_1 + ... + 2p, s_r : 2\sigma_1, ..., 2\sigma_n \right] \right) \]

\[ \left[ \left( c : \psi^{(a)} \right] \left[ m - p + 2p, s_1 + ... + 2p, s_r : 2\sigma_1, ..., 2\sigma_n \right] \right) \]

\[ \left( \left[ \left( m - \frac{p}{2} \right) + p, s_1 + ... + p, s_r : \sigma_1, ..., \sigma_n \right] : \left[ \left( b' \right) : \phi' \right] ; ..., \left[ \left( b^{(ae)} \right) : \phi^{(ae)} \right] \right) \]

\[ \left( \left[ \left( m + p + 1 \right) \right] + \left( \left( m + p + 1 \right) \right) + p, s_1 + ... + p, s_r : \sigma_1, ..., \sigma_n \right] : \left[ \left( d' \right) : \delta' \right] ; ..., \left[ \left( d^{(ae)} \right) : \delta^{(ae)} \right] \right) \]

provided that \( m, p, \sigma_1, ..., \sigma_n, \rho_1, ..., \rho_n \) are positive integers such that \( m > p \) and \( m - p \) is a positive integer; \( S_{N_1, ..., N_r}^{M_1, ..., M_r} (x_1, ..., x_r) \) are Srivastava's polynomials of several variables [18] defined by

\[ S_{N_1, ..., N_r}^{M_1, ..., M_r} (x_1, ..., x_r) = \sum_{s_1=0}^{N_1} \cdots \sum_{s_r=0}^{N_r} \left( -N_1 \right)_{M_1} \cdots \left( -N_r \right)_{M_r} A[N_1, s_1; ..., N_r, s_r] \frac{x_1^{s_1}}{s_1!} \cdots \frac{x_r^{s_r}}{s_r!} \]

where \( M_i, N_i \) (\( i = 1, ..., r \)) are arbitrary positive integers and the coefficients \( A[N_1, s_1; ..., N_r, s_r] \) are arbitrary parameters real or complex, while \( S_{C^{+2D;...;D^{(e)}}}^{A^{+2B;...;B^{(e)}}} \) is generalized multiple hypergeometric function of Srivastava and Daoust ([19],[20],[21])

\[ 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B} \phi_j^{(i)} > 0, i = 1, ..., n. \]

**Proof.** Making an appeal to the integral

\[ \int_0^\infty \cos^n x/a \cos pnx/a \, dx = \frac{\Gamma(m+1)\Gamma\left(\frac{m-p+1}{2}\right)}{\sqrt{\pi} 2^p \Gamma(m-p+1)\Gamma\left(\frac{m+p+2}{2}\right)} \]

where \( m, p \) are positive integers such that \( m > p \) and \( m - p \) is an even integer, shows that

left hand side of (10.2.1)
\[\sum_{r_1=0}^{N_r} \ldots \sum_{r_s=0}^{N_s} (-N_1)_{M,\beta} \cdots (-N_r)_{M,\gamma, \ldots} \frac{Y_{r_1}}{s_{r_1}} \cdots \frac{Y_{r_s}}{s_{r_s}} \]

\[= \sum_{n=0}^{N_1} \ldots \sum_{n=0}^{N_s} \frac{\prod_{j=1}^{A} \Gamma(a_j + r_1 \theta_j + \ldots + r_s \theta_j^{(n)}) \prod_{j=1}^{B} \Gamma(b_j + r_1 \phi_j + \ldots + r_s \phi_j^{(n)})}{\prod_{j=1}^{C} \Gamma(c_j + r_1 \psi_j + \ldots + r_s \psi_j^{(n)}) \prod_{j=1}^{D} \Gamma(d_j + r_1 \delta_j + \ldots + r_s \delta_j^{(n)})} \]

\[\frac{z_1^n}{r_1!} \ldots \frac{z_s^n}{r_s!} \int \cos px/a \cos(\pi x/a)^{m-1+2\sigma_1r_1+\ldots+2\sigma_{m-1}r_{m-1}} dx \]

= right hand side of (10.2.1.).

**10. 3 Main Problem.** In this Section, we shall find the steady state temperature \(u(x,y)\) in a rectangular plate with following boundary conditions when no heat escapes from the lateral surface of the plate:

\[
\begin{array}{c}
\begin{array}{c}
\text{Insulated} \\
(0,0) \quad u(x,0)=0 \\
(0,b)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(0,0) \quad u(x,0)=0 \\
(a,b)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Insulated} \\
(0,0) \quad u(x,0)=0 \\
(a,b)
\end{array}
\end{array}
\]

(10.3.1) \[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < a, \quad 0 < y < b,\]

with boundary conditions

(10.3.2) \[\left( \frac{\partial u}{\partial x} \right)_{x=0} = 0, \quad \left( \frac{\partial u}{\partial x} \right)_{x=a} = 0,\]

(10.3.3) \[u(x,0)=0, \quad 0 < x < a,\]

(10.3.4) \[u(x,b)=f(x), \quad 0 < x < a,\]

Now we consider the problem of determining \(u(x,y)\), when

(10.3.5) \[u(x,b)=f(x) = \cos^{-1}(\pi x/a) [A_{A', \ldots, A''} R_{C', \ldots, C''}] \left( \frac{(e) \cdot \theta', \ldots, \theta^{(n)}}{\psi', \ldots, \psi^{(n)}} \right).\]
\[ \begin{align*}
&\left[ \theta' \right]: \phi', \ldots; \left[ \theta^{(n)} \right]: \phi^{(n)}; \\
&\left[ (a') \right]: \delta', \ldots; \left[ (a^{(n)} \right): \delta^{(n)}: z_1 \cos^{2\sigma_1}(\pi x/a), \ldots, z_n \cos^{2\sigma_n}(\pi x/a) \] \\
&\mathcal{S}_{N_1, \ldots, N_r}^{M_1, \ldots, M_r} \left[ y_1 \cos^{2\rho_1}(\pi x/a), \ldots, y_r \cos^{2\rho_r}(\pi x/a) \right].
\end{align*} \]

**10.4. Solution of the problem.** In view of Zill [30, p. 468 (10.4.3)], the solution of the problem can be written as

\[ u(x, y) = A_0 y + \sum_{p=1}^{\infty} A_p \sinh \frac{p\pi y}{a} \cos \frac{p\pi x}{a}. \]

Now our aim is to find \( A_0 \) and \( A_p \) \( (p=1,2,3,\ldots) \).

For \( y=b \)

\[ u(x, b) = f(x) = A_0 b + \sum_{p=1}^{\infty} A_p \sinh \frac{pb\pi}{a} \cos \frac{p\pi x}{a}, \]

which is half range expansion of \( f(x) \) in a cosine series. If we make the identifications \( A_0 b = a_0/2 \) and \( A_p \sinh \frac{pb\pi}{a} = a_n, \) \( n=1,2,3,\ldots \), then making an appeal to Zill [30, p.449 (10.2.2)], we obtain

\[ (10.4.3) \quad A_0 = \frac{1}{ab} \int_0^b f(x) dx, \]

\[ (10.4.4) \quad A_p = \frac{2}{a \sinh \frac{pb\pi}{a}} \int_0^b f(x) \cos \frac{p\pi x}{a} dx. \]

Making the use of (10.2.1) for \( p=0 \), we derive from (10.4.3), that

\[ (10.4.5) \quad A_0 = \frac{1}{b \sqrt{\pi}} \sum_{s_1=0}^{[N_1/M_1]} \sum_{s_2=0}^{[N_2/M_2]} (-1)^{s_1} (-1)^{s_2} A[N_1, s_1; \ldots; N_r, s_r] \frac{y_1^{s_1}}{s_1!} \ldots \frac{y_r^{s_r}}{s_r!}, \]

\[ \mathcal{S}_{C^{+1}:D^{+1}: \ldots: G^{(n)}}^{A_0 + B_0 + C_0 + \ldots + G^{(n)}} \left[ (a): \theta', \ldots, \theta^{(n)} \right]; \left[ m/2 + \rho_1 s_1 + \ldots + \rho_r s_r : \sigma_1, \ldots, \sigma_n \right]; \]

\[ \left[ (c): \psi', \ldots, \psi^{(n)} \right]; \left[ (m+1)/2 + \rho_1 s_1 + \ldots + \rho_r s_r : \sigma_1, \ldots, \sigma_n \right]; \]

\[ \left[ (d'): \delta', \ldots; \left[ (d^{(n)}): \delta^{(n)} \right] : z_1, \ldots, z_n \right], \]
provided that all conditions of (10.2.1) are satisfied, and

$$(10.4.6) \quad A_p = \frac{1}{2^{p-1} \sqrt{\pi \text{Sinh } \pi b/a}} \sum_{s_1=0}^{[N_1/M_1]} \sum_{s_2=0}^{[N_2/M_2]} \sum_{s_r=0}^{(-N_r)_{M_r,s_r}} A[N_1,s_1;...;N_r,s_r] \frac{y_1^{s_1}}{s_1!} ... \frac{y_r^{s_r}}{s_r!}$$

valid if all conditions of (10.2.1) are satisfied.

Now substituting the values of $A_q$ and $A_p$ from (10.4.5) and (10.4.6) respectively in (10.4.1); we derive the following required solution of the problem:

$$(10.4.7) \quad u(x,y) = \frac{y}{b \sqrt{\pi}} \sum_{s_1=0}^{[N_1/M_1]} \sum_{s_2=0}^{[N_2/M_2]} \sum_{s_r=0}^{(-N_r)_{M_r,s_r}} F[N_1,s_1;...;N_r,s_r] \frac{y_1^{s_1}}{s_1!} ... \frac{y_r^{s_r}}{s_r!}$$

valid if all conditions of (10.2.1) are satisfied.
\[ A[N_1, s_1; \ldots; N_r, s_r] \frac{y_1^{y_1}}{s_1!} \ldots \frac{y_r^{y_r}}{s_r!} \]

\[ S_{C \oplus D \ldots D}^{A \oplus B \ldots B} \left( \{a\} : \theta' \ldots, \theta^{(n)} \right) \left[ m + 2p, s_1 + \ldots + 2p, s_r : 2\sigma_1, \ldots, 2\sigma_n \right], \]

\[ S_{C \oplus D \ldots D}^{A \oplus B \ldots B} \left( \{c\} : \psi', \ldots, \psi^{(n)} \right) \left[ m + 2p, s_1 + \ldots + 2p, s_r : 2\sigma_1, \ldots, 2\sigma_n \right], \]

\[ \left[ \begin{array}{c} m - \frac{p}{2} + p, s_1 + \ldots + p, s_r : \sigma_1, \ldots, \sigma_n \end{array} \right] \left[ \begin{array}{c} (b') : \phi' \ldots ; (b^{(n)} : \phi^{(n)} \end{array} \right], \]

\[ \left[ \begin{array}{c} m + \frac{p}{2} + 1 + p, s_1 + \ldots + p, s_r : \sigma_1, \ldots, \sigma_n \end{array} \right] \left[ \begin{array}{c} (d') : \delta' \ldots ; (d^{(n)} : \delta^{(n)} \end{array} \right], \]

provided that all conditions of (10.2.1) hold true.

**10.5. Expansion Formula.** For \( y = b \), the relation (10.4.7) gives the following expansion formula by an appeal to (10.3.5):

\[ (10.5.1) \quad \cos^{m-1}(\pi x / a) S_{C \oplus D \ldots D}^{A \oplus B \ldots B} \left( \{a\} : \theta' \ldots, \theta^{(n)} \right), \]

\[ \left[ \begin{array}{c} (b') : \phi' \ldots ; (b^{(n)} : \phi^{(n)} \end{array} \right], \]

\[ \left[ \begin{array}{c} (d') : \delta' \ldots ; (d^{(n)} : \delta^{(n)} \end{array} \right], \]

\[ S_{N_1 \ldots N_r}^{M_1 \ldots M_r} \left[ y_1 \cos^{2n}(\pi x / a) \ldots, y_r \cos^{2n}(\pi x / a) \right] \]

\[ = \frac{1}{\sqrt{\pi}} \sum_{s_1=0}^{N_1} \ldots \sum_{s_r=0}^{N_r} (-N_1)_{M_1} \ldots (-N_r)_{M_r} A[N_1, s_1; \ldots; N_r, s_r] \frac{y_1^{y_1}}{s_1!} \ldots \frac{y_r^{y_r}}{s_r!} \]

\[ S_{C \oplus D \ldots D}^{A \oplus B \ldots B} \left( \{a\} : \theta' \ldots, \theta^{(n)} \right) \left[ \begin{array}{c} m / 2 + p, s_1 + \ldots + p, s_r : \sigma_1, \ldots, \sigma_n \end{array} \right], \]

\[ S_{C \oplus D \ldots D}^{A \oplus B \ldots B} \left( \{c\} : \psi', \ldots, \psi^{(n)} \right) \left[ \begin{array}{c} m / 2 + p, s_1 + \ldots + p, s_r : \sigma_1, \ldots, \sigma_n \end{array} \right], \]

\[ \left[ \begin{array}{c} (b') : \phi' \ldots ; (b^{(n)} : \phi^{(n)} \end{array} \right], \]

\[ \left[ \begin{array}{c} (d') : \delta' \ldots ; (d^{(n)} : \delta^{(n)} \end{array} \right], \]

\[ + \frac{1}{\sqrt{\pi}} \sum_{p=1}^{n} \frac{1}{2^{p-1}} \cos \left( \frac{\pi px}{a} \right) \sum_{s_1=0}^{N_1} \ldots \sum_{s_r=0}^{N_r} (-N_1)_{M_1} \ldots (-N_r)_{M_r}. \]
\[
A[N_1, s_1; \ldots; N_r, s_r] \frac{y^{r_1}_{s_1}}{s_1!} \ldots \frac{y^{r_r}_{s_r}}{s_r!}
\]

\[
S_{C^2+D^2}^{A:B; \ldots; D(n)}(\frac{(a): \theta', \ldots, \theta^{(n)}}{(c): \psi', \ldots, \psi^{(n)}}; [m + 2\rho, s_{\theta_1} + \ldots + 2\rho, s_{\theta_n}, : 2\sigma_1, \ldots, 2\sigma_n],
\]

\[
[m - p + 2\rho, s_{\psi_1} + \ldots + 2\rho, s_{\psi_n}, : 2\sigma_1, \ldots, 2\sigma_n],
\]

\[
([m - p]/2 + \rho, s_{\theta_1} + \ldots + \rho, s_{\theta_n}, : \sigma_1, \ldots, \sigma_n]: \left[\begin{array}{c}
(b): \phi'\ldots; (b^{(n)}): \phi^{(n)}\end{array}\right]
\]

\[
([m + p + 1]/2 + \rho, s_{\psi_1} + \ldots + \rho, s_{\psi_n}, : \sigma_1, \ldots, \sigma_n]: \left[\begin{array}{c}
(d'): \delta'\ldots; (d^{(n)}): \delta^{(n)}\end{array}\right] z_1, \ldots, z_n,
\]

valid if all conditions of (10.2.1) are satisfied.

Finally specialize the parameters and arguments of \( S_{C^2+D^2}^{A:B; \ldots; D(n)} \) and \( S_{N_1, \ldots, N_r}^{M_1, \ldots, M_r} \) we can derive the results for various special functions of different variables.

**REFERENCES**


