Introduction

1.1 History of Reliability Analysis:

The word ‘Reliable’ has a great importance in human life i.e. the concept of reliability is as old as man himself. The growth and development of reliability theory has strong relation with quality control. The utilization of statistical methods in quality control has been suggested by Shewharts (1931) and Dodge and Roming (1929).

The science of Reliability is new and still growing. During in the first world war, reliability was measured as the number of accidents per hour of the flight time, while in world war second, a group headed by rocket. The V-1 missiles was developed by Engineer Wrenher Von Braun in Germany after the war, the law of Reliability of the components, which are connected in series was derived by a mathematician named ‘Rebert Lausser’ as a consultant for analyzing the missile system. According to this Law, the reliability of a system consisting of a large number of components is equal to the product of the reliabilities of the individual components that made up the system. If the system comprises a large number of components connected in series, the system reliability may be rather law.

In the field of engineering and technology, the development are continued there after throughout the whole world. The estimation of reliability has become a demand in the context of modern technology with automation the
need of complicated control and safety of systems became steadily more pressing for researchers. The growing need of the concepts of reliability in the field of Statistics, Mathematics and Engineering sciences has made it a prominent topic of research. Investigations in reliability analysis began in the 1950 and the growth of the theory began to gain momentum after a decade. The first major committee on reliability was set up by the Department of Defence, USA in 1950. Now almost all countries have started to take interest in the applications of reliability principles.

The reliability analysis has a wide scope. It’s the most important used in risk and safety analysis to evaluate the availability and applicability of safety systems. Reliability analysis is useful for environmental protection. Many industries have realized that most of the pollution caused by their plants is due to production irregularities, which is the most important factor in reducing pollution. Many industries like aerospace, automobile and aviation have adopted reliability principles in design process. Reliability has a wide field of application in quality and reliability management also, Since it is considered as a quality characteristic. Nowadays, the applications of reliability principles have made remarkable progress in many firms.

1.2 Some Basics Concepts Related Reliability:

Here we will discuss some basic concepts used in reliability Analysis such as failure, failure time, survival time, reliability functions, hazard rate,
mean time to failure (MTTF), mean time between two failure (MTBF), censored sampling etc.

1.2.1 Failures:

Let us consider a system or a unit under some sort of stress. It may be a steel beam under a load, a fuse inserted into a circuit or an electric device put into service. The steel beam may crack or break, the fuse may burn out or the electronic device may fail to function, all these undesirable states are defined as ‘failure’. A failure is the partial or total loss or change in the property of unit or system in such a way that its functioning is partially or completely stopped. In the reliability analysis, failure means that the systems is incapable of performing its required function.

The penalties of failure are paid by people in terms of money, time and even life itself. A failure in a single unit of a system may cause the complete breakdown in the industrial plant. A failure in the network of railways may cause the delay of trains. A failure in the breaks system of a metro train in Japan was the cause for its accident with another train and resulted in deaths of hundreds of people. A failure occurred in the Union Carbide Plant (UCP) at Bhopal, resulted in break leakage of Methyl ISO Cyanide (MIC) and became the cause of the death of thousands of people. A failure in the Columbia Space Shuttle of NASA caused the death of seven aeronauts, including Kalpana Chawla of India.
1.2.2 Lifetime / Failure time / Survival time:

Life time is the time until the failure of the unit occurs. i.e. it is the length of failure free time. Life time is often expressed in hours of operation. Mathematically life time is merely non-negative valued random variable, Survival time and failure time are the alternate terms for life time that are frequently used in Reliability theory.

1.2.3 Reliability and Reliability Function:

Reliability is the ability of a system or component to perform its required functions under stated conditions for a specified period of time. It is often reported in terms of a probability. Mathematically, if $X$ denotes the random variable representing, the life time or the time of failure of a components. The probability that the component survives untill some time $t$ is called the reliability. The reliability function at time $t$ is defined as:

$$ R(t) = P(X > t) $$

$$ = 1 - P(X \leq t) $$

$$ = 1 - F(t) $$

Where $F(t)$ is distribution function of $X$ at specified time $t$, known as the failure distribution and sometimes referred as the unreliability function.

Let $f(x)$ be the pdf of $X$, the reliability function at time $t$ is
\[ R(t) = \int_t^\infty f(x) \, dx \]

The components is normally but not always assumed to be working properly at time \( t = 0 \) i.e. \( R(0) = 1 \), and no component can work forever without failure i.e. \( \lim_{t \to \infty} R(t) = 0 \). It means that \( R(t) \) is monotonic non-increasing function of \( t \).

1.2.4 Hazard – rate:

The instantaneous failure rate of a component at time \( t \) is called hazard rate. It is denoted by \( h(t) \) i.e.

\[
h(t) = \lim_{x \to 0} \frac{F(t + x) - F(t)}{xR(t)}
\]

\[
= \lim_{x \to 0} \frac{F(t) - F(t + x)}{x} \frac{1}{R(t)}
\]

\[
= \frac{dF(t)}{dt} \cdot \frac{1}{R(t)} \quad \left( \because \frac{dF(t)}{dt} = f(t) \right)
\]

\[
h(t) = \frac{f(t)}{R(t)} \quad (1.1)
\]

Where the function \( f(t) \) represents probability density function (pdf).

Integrating equation (1.1), we get

\[
\int_0^t h(x) \, dx = \int_0^t \frac{f(x)}{R(x)} \, dx
\]
Since \( f(x) = \frac{dF(x)}{dx} \)

\[
f(x) = \frac{d[1 - R(x)]}{dx} = - \frac{dR(x)}{dx}
\]

\( f(x) \, dx = - dR(x) \)

Therefore

\[
\int_0^t h(x) \, dx = - \int_0^t \frac{dR(x)}{R(x)}
\]

\[
\int_0^t h(x) \, dx = -[ \log R(x) ]_0^t
\]

Therefore

\[
- \int_0^t h(x) \, dx = -[ \log R(x) ]_0^t
\]

\[
- \int_0^t h(x) \, dx = \log R(t) \quad \{ \therefore R(0) = 1 \}
\]

\( R(t) = \exp \{ - \int_0^t h(x) \, dx \} \)

This formula holds even when the distribution of the time to failure is not exponential.
1.2.5 Mean Time to failure (MTTF) :

The expected value of life time of a component is known as the mean time to failure. It is given as the mathematical expectation of the lifetime of the components. Thus if $T$ denotes the life time of a component, then mean time to failure is given by

$$MTTF = E(T)$$

$E(T)$ referred as the expected life time to the failure.

1.2.6 Mean Time between two failure (MTBF) :

Let us suppose that on failure of a component of a system or unit, it is repaired and restored to be ‘as good-as new’

Let $T$ be the duration of the functioning period, and let $D$ be system downtime for repair or replacement.

This mean time between failure is given as –

$$MTBF = E(T) + E(D) = MTTF + MTTR$$

1.2.7 Censored Sampling :

In many cases, it is neither possible nor desirable to record the failure time of all the items under test, Since life testing experiments are usually destructive in nature.
Suppose n items are kept on a test and if the experiment is terminated after a pre-assigned time, say ‘t’ Such a sampling known as ‘time-censored sampling’ or type-I censored sampling and also if the experiment is terminated when a pre-assigned number of items, say \( r < n \) have failed such a sampling is known as ‘failure censored sampling’ or ‘type – II sampling’.

For type I censored sampling the length of the experiment is fixed, while the number of observations obtained before time t is a random variable with type II censored sampling the number of observations is fixed but the length of the experiment is random.

1.2.8 Loss Function:

Suppose \( \hat{\theta} \) is an estimator of a parameter \( \theta \) then a loss function, denoted by \( L(\hat{\theta}, \theta) \) is a real-valued function such that

\[
L(\hat{\theta}, \theta) \geq 0 \quad \text{for every} \quad \hat{\theta} \\
L(\hat{\theta}, \theta) = 0 \quad \text{When} \quad \hat{\theta} = \theta
\]

The expected value of loss function in known as risk function.

1.2.9 Squared- Error Loss Function (SELF):

If a parameter \( \theta \) is estimated by \( \hat{\theta} \), the squared-error Loss function (SELF) is given by –

\[
S.L.(\hat{\theta}, \theta) = E(\hat{\theta} - \theta)^2
\]
1.2.9 Absolute Error Loss Function:

If a parameter $\theta$ is estimated by $\hat{\theta}$, the absolute error loss function is given by

$$
A.L.(\hat{\theta}, \theta) = | \hat{\theta} - \theta |
$$

1.2.10 (Linear in Exponential) Loss Function (LINEX):

A symmetric loss function assumes that over estimation and under estimation are equally serious. However in some estimation problems such an assumption may be inappropriate. In the estimation of reliability, over estimation is usually more serious than the underestimation. For such situation when over estimation and underestimation are not equally serious. If a parameter $\theta$ is estimated by $\hat{\theta}$, the Linex loss function as suggested by Varian (1975) is given as

$$
L.L(\hat{\theta}, \theta) = b \exp \{ a(\hat{\theta} - \theta) \} - a(\hat{\theta} - \theta) - 1; a \neq 0, b > 0
$$

1.2.11 General Entropy Loss Function (GELF):

If a parameter $\theta$ is estimated by $\hat{\theta}$, the general entropy loss function (GELF) is defined as

$$
G.L.(\hat{\theta}, \theta) = \left( \frac{\hat{\theta}}{\theta} \right)^a - a \log \left( \frac{\hat{\theta}}{\theta} \right) - 1; a \neq 0
$$
1.2.12 Bayes Estimator:

Bayes Estimator $\hat{\theta}_{\text{Bays}}$ of a parameter $\theta$ is defined as the estimator, which minimizes the posterior expected loss

$$E_{\theta} \left[ L(\hat{\theta}_{\text{Bays}}, \theta) \right] = \int L(\hat{\theta}_{\text{Bays}}, \theta) \pi(\theta | x) d\theta$$

Where $\Phi$ is the parameter space of a parameter $\theta$.

1.2.13 Shrinkage Estimator:

The concept of shrinkage estimators has been introduced by Thompson (1968, a & b). Shrinkage estimation procedure is one of the intersecting procedures in which it is assumed that the prior knowledge about the parameter is available in the form of a prior point estimate or in the form of interval which contain parameter in it. Shrinkage estimator of a parameter by giving suitable weights to the usual estimator and the prior point estimate has been suggested by Thompson.

1.2.14 Minimum Mean Square Estimator (MMSE):

Minimum Mean Square Estimator is an estimator that minimizes the mean square error of the estimator.
1.2.15 Lindley Approximation:

In many situations, Bayes estimators are obtained as a ratio of two integral expressions and cannot be expressed in a closed form. However, these estimator can be numerically approximated using complex computer programming.

An asymptotic approximation to the ratio of two integrals has been suggested by Lindley (1980). The basic idea behind it is to obtain Taylor series expansion of function involved in the integral about the maximum likelihood estimator.

1.3 Different Probabilistic Models Used in Reliability Analysis:

In this section, we discuss various probabilistic models which used in Reliability Analysis such as follows;

1.3.1 Exponential Distribution:

A fundamental distribution to the reliability analysis is the exponential distribution. It is widely used in lifetime models. Davis (1952) examined that the exponential distribution appears to fit most of the data related to reliability analysis. Epstein in (1958) remarks that the exponential distribution plays an important role in life testing experiments. The reason behind the applicability of this distribution is the availability of simple statistical methods and its suitability to represent the life time of many items.

The pdf of one-parameter exponential distribution is
\[ f(x; \theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right); x > 0, \theta > 0 \]

The reliability function and hazard-rate for this distribution are given by

\[ R(t) = \exp\left(-\frac{t}{\theta}\right) \]

And

\[ h(t) = \frac{1}{\theta} ; \theta > 0 \]

For two parameter exponential distribution the pdf is given by

\[ f(x; \mu, \theta) = \frac{1}{\theta} \exp\left(-\frac{x-\mu}{\theta}\right); -\infty < \mu < \infty, \theta > 0 \]

\[ = 0, \text{ otherwise} \]

Reliability and hazard rate for two-parameter exponential distribution are given by

\[ R(t) = \exp\left(-\frac{t-\mu}{\theta}\right); t > \mu \]

\[ = 1; t \leq \mu \]

And

\[ h(t) = \frac{1}{\theta} ; \theta > 0 \]
Hence, the hazard-rate for exponential distribution is constant. It is more appropriate for a situation where the failure rate appears to be more or less constant.

1.3.2 Weibull Distribution:

The weibull distribution is widely used distribution in reliability analysis. This distribution has been discovered by Swedish scientist Weibull (1951). He showed that it is useful in describing wear out failure. It has also been used as a model for vacuum tubes, ball bearings, tumors in human beings etc.

The pdf of two-parameter weibull distribution is given by-

\[ f(x; \lambda, \mu) = \frac{\mu}{\lambda} x^{\mu-1} \exp\left(-\frac{x^\mu}{\lambda}\right); \lambda, \mu > 0, x > 0 \]

Where \( \mu \) is referred to as the shape parameter and \( \lambda \) as the scale parameter of the distribution. It reduces to exponential distribution for \( \mu = 1 \) and for \( \mu = 2 \) Rayleigh distribution.

The reliability function and the hazard-rate for this distribution are given by

\[ R(t) = \exp\left(-\frac{t^\mu}{\lambda}\right) \]

and \( h(t) = \frac{\mu}{\lambda} t^{\mu-1}; \mu, \lambda > 0 \)
The Weibull distribution has increasing failure rate (IFR) for $\mu > 1$ and decreasing failure rate (DFR) for $0 < \mu < 1$ and constant failure rate for $\mu = 1$.

1.3.3 Gamma Distribution:

The gamma distribution is also sometimes used as a lifetime distribution. The pdf of gamma distribution is given by

$$f(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right); \alpha, \beta > 0, x > 0$$

Where $\alpha, \beta$ are shape and scale parameter respectively.

The reliability function and hazard-rate for gamma distribution are

$$R(t) = \frac{\Gamma(\alpha) - \Gamma_{t/\beta}(\alpha)}{\Gamma(\alpha)}$$

and

$$h(t) = \frac{t^{\alpha-1} \exp\left(-\frac{t}{\beta}\right)}{\beta^\alpha \left[\Gamma(\alpha) - \Gamma_{t/\beta}(\alpha)\right]}$$

Where $\Gamma_b(a)$ is the well known standard incomplete gamma function, given by

$$\Gamma_b(a) = \int_0^b y^{a-1} e^{-y} \, dy, a > 0$$

There is no closed expression for $R(t)$ and $h(t)$ for this distribution. However $R(t)$ and $h(t)$ have been extensively studied and tabulated. We know that for $\alpha > 1$, it has IFR and DFR for $0 < \alpha < 1$, for $\alpha = 1$ the gamma
distribution coincides with exponential distribution and yields constant hazard-rate.

The generalized gamma distribution is a three-parameter distribution with pdf is given by-

\[ f(x; \lambda, \beta, k) = \frac{\lambda \beta (\lambda x)^{k-1}}{\Gamma(k)} \exp[-(\lambda x)^\beta]; \quad x > 0, \lambda, \beta, k > 0 \]

Stacy (1962) has been introduced that this model includes many lifetime distributions as special cases.

1.3.4 Normal Distribution:

The normal distribution gives quite a good fit for the failure time data, in the context of life testing and reliability function that was suggested by Davis (1952). The support to the normal distribution is \((-\infty, \infty)\) by taking the mean \(\mu\) to be sufficiently large positive valued and standard deviation \(\sigma\) to be sufficiently small relative to \(\mu\).

The pdf of normal distribution with location parameter \(\mu\) (mean) and scale parameter \(\sigma\) (S.D.) is given by-

\[ f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2} (x - \mu)^2\right); \quad -\infty < x, \mu < \infty, \sigma > 0 \]

The reliability function and hazard-rate for this distribution are given by-

\[ R(t) = 1 - \psi \left( \frac{t - \mu}{\sigma} \right) \]
\[ h(t) = \frac{\psi\left(\frac{t-\mu}{\sigma}\right)}{\sigma \left[1 - \psi\left(\frac{t-\mu}{\sigma}\right)\right]} \]

Where \( \psi \) is pdf of standard normal variate (SNV) and \( \psi(z) \) is the cumulative distribution function (cdf) of SNV, given by

\[ \psi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \, du \]

Although we can\'t obtain \( h(t) \) for this distribution in closed form, yet it can be shown that it has IFR.

1.3.5 Log-Normal Distribution:

In the contexts of life testing and reliability problems, the log normal distribution answers a criticism sometimes raised against the use of normal distribution \((-\infty, \infty)\), as a model for failure time distribution which must range over 0 to \(\infty\). The log-normal distribution is appropriate when hazard-rate is decreasing for large value of \(t\). Goldthwaite (1961) justified its use as a failure time distribution.

The pdf of log-normal distribution is given by:

\[ f(x; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(\log x - \mu)^2\right); \quad 0 < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0 \]
The reliability function and hazard rate for the log normal distribution are given as

\[ R(t) = 1 - \psi \left( \frac{\log t - \mu}{\sigma} \right) \]

and

\[ h(t) = \frac{\psi \left( \frac{\log t - \mu}{\sigma} \right)}{\sigma t \left[ 1 - \psi \left( \frac{\log t - \mu}{\sigma} \right) \right]} \]

It should be noted that the hazard-rate initially increases over the time and then decreases as time increases, thus the log normal distribution serves as a model when failure rate is high rather than initially.

1.3.6 Inverse Gaussian Distribution:

This distribution as a life time model its applications has been proposed by Chikara and folks (1974, a) for studying its applications for studying reliability aspects where the initial failure rate is high. They also developed inferential procedure for inverse Gaussian distribution after studying its property.

The pdf of this distribution is given by-

\[ f(x; \mu, \lambda) = \frac{\lambda}{(2\lambda x^3)^{\frac{1}{2}}} \exp \left( \frac{-\lambda}{2\mu^2 x} (x - \mu)^2 \right); x, \mu, \lambda > 0 \]
Where $\lambda$ is shape parameter.

The reliability function and hazard-rate are given as

$$R(t) = \psi \left[ \left( \frac{\lambda}{t} \right)^{1/2} \left( 1 - \frac{t}{\mu} \right) \right] - e^{2\lambda/\mu} \psi \left[ -\left( \frac{\lambda}{t} \right)^{1/2} \left( 1 + \frac{t}{\mu} \right) \right]$$

Chhikara and folks (1974,a) suggested that this distribution has IFR for $t < t_m$ and DFR for $t > \frac{2\lambda}{3}$, where $t_m$ is the mode of the distribution.

1.3.7 Maxwell and Generalized Maxwell Distribution:

The Maxwell distribution as a failure model has been suggested by Tyagi and Bhattacharya (1989,a&b). The generalized form of Maxwell distribution has been suggested by Chaturvedi and Rani (1998) and known as ‘generalized Maxwell distribution’.

The pdf of the generalized Maxwell failure distribution is given by

$$f(x; \theta, k) = \frac{2x^{2k-1} \exp \left( -\frac{x^2}{\theta} \right)}{\theta^k \Gamma(k)} ; x, \theta, k > 0$$

The reliability function and hazard-rate is given by-

$$R(t) = \Gamma \left[ \frac{t^2}{\theta} \Gamma(k) \right]$$

and

$$h(t) = 2t \left[ \int_0^\infty \left( 1 + \frac{2s}{t^2} \right)^{k-1} e^{-s} \, ds \right]^{-1}$$
It is observed that this distribution has IFR.

1.3.8 Negative Binomial Distribution:

The negative binomial distribution as lifetime model has been considered by Kumar and Bhattacharya (1989). The pmf of negative binomial distribution is given by

\[ P(x;r,\theta) = \binom{r+x-1}{x} \theta^x (1-\theta)^r, \quad 0 < \theta < 1, r > 0, x = 0,1,2, \ldots \]

They studied the behaviour of the hazard-rate and suggested that it has IFR for \( r > 1 \) and DFR for \( r < 1 \). For \( r = 1 \), it leads to geometric distribution and yields constant hazard-rate.

The Bayes estimator of the reliability function for the zero-truncated negative binomial distribution has been derived by Kyria Koussis and Papadopoulos (1993) also this distribution as reliability model has been considered by Chaturvedi and Sharma (2007). The justification behind its use as a reliability model is based on the behaviour of its hazard-rate.

The zero-truncated negative binomial distribution has the probability mass function (pmf)

\[ P(x;s,\theta) = \frac{\binom{s+x-1}{x} \theta^x (1-\theta)^s}{1-(1-\theta)^s}; \quad 0 < \theta < 1, s > 0, x = 1,2, \ldots \]
The reliability function and the hazard-rate for this distribution are given as-

\[ R(t) = \sum_{x=t}^{\infty} \binom{s + x - 1}{x} \theta^x (1 - \theta)^s \]

and

\[ h(t) = \frac{\binom{s + t - 1}{t}}{\sum_{x=0}^{\infty} \binom{s + x + t - 1}{x + t} \theta^x} \]

This distribution has IFR for S<1, DFR for S>1 and constant failure rate for S=1.

1.3.9 Binomial Distribution:

The binomial distribution as a lifetime model has been introduced by Chaturvedi, Tiwari and Kumar (2007). The pmf of binomial distribution is given by

\[ P(x; r, \theta) = \binom{r}{x} \theta^x (1 - \theta)^{r-x}; 0 < \theta < 1, x = 0, 1, 2, \ldots, r. \]

They studied the behavior of the hazard-rate and showed that the binomial distribution has IFR.
1.3.10 Poisson Distribution:

Poisson distribution as a lifetime model has been considered by Chaturvedi, Tiwari and Kumar (2007). The pmf of Poisson distribution is given by

\[ P(x; \theta) = \frac{e^{-\theta} \theta^x}{x!}; \quad x = 0, 1, 2, \ldots \]

They showed that the Poisson distribution can represent lifetime model when one has IFR.

1.3.10 Families of Life time Model:

Many authors have discussed different families of lifetime models, which contain several lifetime distributions as particular cases. We have used the following two families of lifetime distributions for classical and Bayesian inferential procedure.

Let us consider a family of lifetime models originated by chaturvedi et.al (2002, 2003, a) with probability density function (pdf) given by-

\[ f(x; \delta, \theta) = \frac{g^{\delta-1}(x)g'(x)}{\theta^{\delta} \Gamma(\delta)} \exp\left( -\frac{g(x)}{\theta} \right); \quad x > 0; \quad g(x), \theta, \delta > 0 \]

Where ‘a’ is known and \( \delta \) and \( \theta \) are parameters. Here, \( g(x) \) is real valued, strictly increasing function of \( x \) with \( g(a) = 0 \) and \( g'(x) \) denotes the first derivative of \( g(x) \).

The above family, known as the generalized life distributions, includes the following life distributions useful in reliability analysis as particulars cases.
(a) for \( g(x) = x \); \( a = 0 \) and \( \delta = 1 \), we obtain the one-parameter exponential distribution.

(b) for \( g(x) = x \), \( a = 0 \) we get the gamma distribution and for \( \delta \) taking integer values, it is known as Erlong distribution.

(c) For \( g(x) = x^p \) and \( a=0 \), the family gives the generalized gamma distribution.

(d) For \( g(x) = x^p \), \( a=0 \) and \( \delta = 1 \) it leads to weibull distribution.

(e) For \( g(x) = x^p \), \( a=0 \) and \( \delta = 1 \), it represents the half-normal distribution.

(f) For \( g(x) = x^2 \), \( a=0 \) and \( \delta = 1 \), we obtain Rayleigh distribution.

(g) For \( g(x) = \frac{x^2}{2} \), \( a=0 \) and \( \delta = \frac{\alpha}{2} \), it turns out to be chi-distribution.

(h) For \( g(x) = \frac{x^2}{2} \), \( a=0 \) and \( \delta = \frac{3}{2} \), we get the Maxwell distribution and for \( g(x) = x^2 \), we obtain the generalized Maxwell distribution.

(i) For \( g(x) = \log (1 + x^b) \), \( a=0 \) and \( \delta = 1 \), we obtain Burr distribution.

(j) For \( g(x) = \log x \), \( a=1 \) and \( \delta = 1 \), it represents Pareto distribution.

The reliability function and hazard-rate for this family are given by :

\[
R(t) = \int_{g(t)/\theta}^{\infty} \frac{y^{\delta-1}}{\Gamma(\delta)} \exp(-y) \, dy, \quad \text{where} \quad y = \frac{g(x)}{\theta}
\]
and

\[ h(t) = \left[ \int_{g(t)/\theta}^{\infty} \left( \frac{1}{\theta} + \frac{s}{g(t)} \right)^{s-1} e^{-s} ds \right]^{-1} \frac{g'(t)}{\theta} \]

Where \( S = \left( y - \frac{g(t)}{\theta} \right) \)

The behaviour of hazard-rate depends upon \( g'(t) \). If \( g'(t) \) is constant (exponential distribution and Weibull distribution for \( p=1 \)), \( h(t) \) is also constant. If \( g'(t) \) is monotonically increasing in \( t \) (Weibull distribution for \( p>1 \), Rayleigh distribution and chi distribution for \( \alpha = 2 \)), \( h(t) \) is monotonically increasing in \( t \). If \( g'(t) \) is monotonically decreasing in \( t \) (Weibull distribution for \( p>1 \), Burr distribution and Pareto distribution), \( h(t) \) is monotonically decreasing in \( t \).

We consider one more family of lifetime distribution proposed by Moore and Bilikam (1978). Let the random variable \( X \) follows the distribution given by the pdf

\[ f(x; \beta, \theta) = \left( \frac{\beta}{\theta} \right) g'(x) g^{\beta-1}(x) \exp \left( -\frac{g^\beta(x)}{\theta} \right) ; x, \theta > 0 \]

This family is known as family of lifetime distribution since it includes the following probabilistic distributions useful in reliability analysis as particular cases.

(a) For \( g(x) = x \) and \( \beta = 1 \), we get exponential distribution.
For \( g(x) = x \), it yields Weibull distribution.

For \( g(x) = \log \left(1 + x^b\right) \) \( b > 0 \) and \( \beta = 1 \), we obtain Burr distribution.

For \( g(x) = \log \left(\frac{x}{a}\right) \) and \( \beta = 1 \), it leads to Pareto distribution.

For \( g(x) = x \) and \( \beta = 2 \), it gives Rayleigh distribution.

The Reliability function and hazard-rate are given by

\[
R(t) = \exp \left( -\frac{g^\beta(t)}{\theta} \right)
\]

and

\[
h(t) = \left(\frac{\beta}{\theta}\right) g'(t) g_0^{\beta-1}
\]

The behavior of hazard-rate depends upon \( g'(t) \). If \( g'(t) \) is constant \( h(t) \) is also constant (exponential and Weibull distribution for \( \beta = 1 \)), if \( g'(t) \) is monotonically decreasing in \( t \), \( h(t) \) is monotonically decreasing in \( t \) (Burr, Pareto and Weibull distribution for \( \beta > 1 \)) and if \( g'(t) \) is monotonically increasing in \( t \), \( h(t) \) is monotonically increasing in \( t \), (Raleigh and Weibull distribution for \( \beta > 1 \)).

1.4 Classical Inferential Procedures in Reliability Analysis:

The Classical inferential procedures have been introduced in the field of Reliability analysis for deriving maximum likelihood estimators (MLE’s) and
uniformly minimum variance unbiased estimators (UMUVE's) of the reliability and other parametric function.

In case of censoring from right for one-parameter exponential distribution, Epstein and Sobel (1953) derived the MLE of scale parameter. The UMVVE of the reliability function of the exponential distribution has been obtained Pugh (1963). The behavior of UMVUE of reliability function of the exponential distribution when a spurious observations may be present has been discussed by Sinha (1972). This result to the two-parameter exponential distribution has been expended by Epstein and Sobel (1954) and Epstein (1960).

The UMVUE of reliability function of two-parameter exponential distribution with complete sample was obtained by Tate (1959), Laurent (1963) and Sathe and Varde (1969). The UMVUE of reliability function for exponential, gamma, weibull distribution under type II Censoring has been derived by Basu (1964). The UMVUE of reliability function for the gamma and normal distribution has been derived by Patil and Wani (1966). The numerical approximations to the MLE's for the parameters of generalized gamma distribution has been obtained Harter (1969). The UNUVE's of reliability function of normal distribution has been derived by Feldman and fox (1968). The reliability function of the inverse Gaussian distribution was obtained by Ray and Wasan (1968) and Chhikara and Fox (1974,b). The UMVUE of the reliability function for generalized Maxwell distribution has
been considered by Chaturvedi and Rani (1998). A family of lifetime distributions has been developed by Chaturvedi and Rani (1997) and obtained UMVUE of reliability function and moments Chaturvedi and Tomer (2002, 2003, a) obtained the UMVUE of reliability function for negative binomial distribution and generalized life distributions.

Another measure of reliability under stress-strength set-up is the probability $P=P(X > Y)$, which represents the reliability of performance of an item of strength $X$ subject to stress $Y$. Owen, Craswell and Hanson (1964), Church and Harris (1970) and Dowton (1973) discussed the estimation of $P$ when $X$ and $Y$ are namely distributed. Tong (1974) and Kelly, Kelly and Schucany (1976) considered the case when $X$ and $Y$ are exponentially distributed.

The case when $X$ and $Y$ follow gamma distribution has been considered by Tong (1975) also. The various estimation procedures for $P$ in discretized data has been discussed by Simonoff, Hochberg and Reiser (1985). The MLE and UMVUE of $P$ when both $X$ and $Y$ follow gamma distribution with unequal scale and shape parameter, were considered by Constantine, Karson and Tse (1986). UMVUE of $P$ for exponential case under type I and type II censorings by using a simpler technique of deriving UMVUEs has been derived by Chaturvedi and Surinder (1999). Using the same technique Chaturvedi and Tomer (2002) obtained the MLE and UMVUE of $P$ for negative binomial distribution.
Wald (1947) has been done the pioneering work in the field of sequential analysis, who developed sequential probability ratio Test (SPRT) for testing a simple null hypothesis against a simple alternative hypothesis. He also obtained expressions for the operating characteristic (OC) and Average Sampler Number (ASN) for the proposed sequential test. Epstein and spbel (1955) considered sequential life test in exponential case to the simple null hypothesis against a simple alternative hypothesis. They derived approximate formula for OC and ASN functions. A lot of work has been done in this field by several authors such as, Epstein (1960), Woodall and Kurkijain (1962), Aroian (1976) and Baryant and Schmee (1979). Sequential test for composite hypothesis for the shape parameter of gamma distribution has been proposed by Phatarfod (1971). SPRT for inverse Gaussian distribution has been developed by Joshi and Shah (1990). SPRT for testing simple and composite hypothesis for the parameter of generalized life distribution has been proposed by Chaturvedi, Kumar and Kumar (2000).

The robustness of SPRT for different distributions have been discussed by Harter and Moore (1976), Montange and Singpurwala (1985), Chaturvedi, Kumar and Chauhan (1998) and Chaturvedi, Tiwari and Tomar (2002).

Dantzing (1940) proved the non-existence of test of student’s hypothesis having power function independent of variance for normal population. Consequently, one can not construct a confidence interval of pre assigned width and coverage probability for the mean of a normal population when
variance is unknown. To deal with this problem, Stein (1945) proposed a two-stage procedure determining the sample size as a random variable. Ruben (1961) studied properties of Stein's two-stage procedure this procedure is easy to apply since it requires only two stages. However, it has some drawbacks, firstly, it is not asymptotically efficient in Chow and Robbins (1965) viewpoint. According to Chow and Robbins (1965), a sequential procedure is asymptotically efficient, if the ratio of the average sample size to the optimal fixed sample size converges to unity. Secondly, this procedure does not utilize the second stage sample size for the purpose of the estimation of nuisance parameter. Moreover, the 'cost of ignorance' of the nuisance parameter does not remain asymptotically bounded.

These drawbacks can be removed by sequential upgrading of the observations. An important contribution to this direction was made by Starr (1966). Mukhopadhyay (1980) proposed a modified two-stage procedure in order to make Stein's two-stage procedure 'asymptotically efficient'. Anscombe (1949) provided a large sample theory for sequential estimation. Roobins (1959) considered the problem of minimum risk point estimation of the mean of normal population under absolute error loss function and linear cost of sampling. Starr and Woodroofe (1969) introduced another measure of the optimality of a sequential point estimation procedure, known as 'regret'. Regret of a sequential procedure is defined as the difference between the risks
of the sequential procedure and that of the optimal fixed sample size procedure. A sequential procedure is ‘optimal’ if its ‘regret’ is asymptotically bounded.

Woodroofe (1977) introduced the concept of ‘second order approximations’ in the area of sequential estimation. In this theory, one may be able to study the behavior of the remainder terms after the optimum position achieved by the fixed sample size procedure. Chaturvedi (1988) generalized the result of Woodroofe (1977) by obtaining the second order approximations for the regret of the sequential procedure for the minimum risk point estimation of the mean of a normal population by taking a family of loss function and a general cost function. Chaturvedi, Tiwari and Panday (1992) developed a class of sequential procedures for the point estimation of the parameters of an absolutely continuous population in the presence of an unknown scalar nuisance parameter. They also derived second-order approximations for the expected sample size and the regret of the sequential procedure.

Hall (1981, 1983) proposed there-stage and ‘accelerated’ sequential procedure. Chaturvedi, Tiwari and Panday (1993) further analyzed the problem of constructing a confidence interval of pre-assigned width and Coverage probability considered by Constanza, Hamdy and Son (1986). They utilized several multi-stage (purely sequential accelerated sequential, three stage and two-stage) estimation procedures to deal with the same estimation problem. Kumar and Chaturvedi (1993) and Chaturvedi and Rani (1999) proposed the classes of two-stage procedures to construct fixed width confidence intervals.
and point estimation Chaturvedi and Trivedi (2002) developed a class of three-stage estimation procedures taking into consideration the common distributional properties of the estimators of the parameters to be estimated under different continuous probabilistic models and those of nuisance parameters involved there in. They also considered the problem of constructing fixed size confidence regions as well as point estimation. They also presented the asymptotic properties of the proposed class. Chaturvedi and Tomer (2003, b) Considered the there-stage and accelerated sequential procedures for the mean of a normal population with known co efficient of variation.

1.5 Bayesian Inferential Procedures in Reliability Analysis.

First off all Bhattacharya (1967) introduce the Bayesian ideas in reliability analysis. He considered the Bayesian estimation of reliability function for one-parameter exponential distribution under uniform and beta priors. Bhattacharya and Kumar (1986) and Bhattacharya and Tyagi (1988) obtained Bayes estimators for the reliability function with other priors. The Bayes estimators for reliability functions of exponential and weibull distribution using uniform and Gamma priors have been obtained by Haris and Singpurwala (1968). Canfield (1970) considered an asymmetric loss function for the Bayesian estimation of the reliability function under Beta priors. Soland (1969) derived the Bayes estimation of reliability function for the weibull distribution using a discrete prior distribution for the shape parameter using Monte-carlo simulation Tsokos (1972,b) and Canvos and Tsokos (1973)
showed that Bayes estimators of reliability function in case of uniform, exponential and Gamma priors have uniformly smaller mean squared-error than minimum variance unbiased estimators (MVUE’s). Lian (1975) and Marts and Lian (1977) obtained the Bayes estimator of reliability of weibull distribution using a piecewise linear prior distribution. Canvos and Tsokos (1971) obtained Bayes estimator of reliability function for gamma distribution, restricting the scale parameter as integer valued function. Padgett and Tsokos (1977) studied the mean squared-error performance of Bayes estimator of reliability function compared to MLE for lognormal distribution Tyagi and Bhattacharya (1989, b) Considered the Bayesian estimation of the reliability function of Maxwell distribution. Chatruvedi and Rani (1998) extended the results of Tyagi and Bhattacharya (1989, a & b) for the generalized Maxwell distribution. Chaturvedi, Tiwari and Kumar (2007) obtained the Bayes estimator of the reliability function for binomial and Poisson distributions.

of 'p' for binomial and poison distributions. All these authors considered SELF, which is a symmetrical loss function.

While estimating reliability function, the use of a symmetrical loss function is inappropriate because of the recognition of the fact that overestimation is usually more serious than the underestimation. For the situations when overestimation and underestimation are not equally serious, Varian (1975) suggested LINEX (liner in exponential) loss function, which was further used by Zellner (1986). Basu and Ebrahimi (1991) derived Bayes estimators of the mean failure time, reliability function and 'P' of an exponential distribution for the complete sample case considering both the SELF and LINEX loss function. The LINEX loss function is suitable for the estimation for location parameter but not for the estimation of scale parameter and other parametric function. Calabria and Pulcin (1994) suggested the use of general entropy loss function (GELF) for the estimating these quantities.

1.6 Classical and Bayesian Reliability Estimation of Binomial and Poisson Distributions :-

In classical and Bayesian approach a lot of work has been done for estimating various parametric functions of several discrete distributions. A necessary and sufficient conditions for the existence of an unbiased estimators provided by Halmos (1946). In binomial distribution a unbiased estimators that can be investigated by Kolmogrow (1950). The expected absolute error of UMVUE of the probability of success of binomial distribution studied by

Discrete distribution has important roll in reliability theory. The negative binomial distribution as the life time model considered by Kumar and Bhattacharya (1989) and then obtained UMVUE's of the mean life and reliability functions. Another measure of reliability under stress-strength. A
setup is the probability $\Pr\{X \leq Y\}$, where $X$ is the stress variable and $Y$ is the strength variable. The consideration of Maiti (1995), the estimation of $\Pr\{X \leq Y\}$ under the assumption that $X$ and $Y$ followed geometric distributions and then Maiti (1995) derived UMVUE and Bayes estimators. Classical and Bayesian estimation procedures for the reliability function of the negative binomial distribution from different approach considered by Chaturvedi and Tomer (2002). After generalizing the results of Maiti (1995), they with the problem of estimating $\Pr(X_1 + X_2 + \ldots + X_k \leq Y)$, where the random variable $X$ and $Y$ are considered to follow negative binomial distribution.

The problems of estimating the reliability functions and $\Pr\{X_1 + X_2 + \ldots X_k \leq Y\}$ are considered and the random variables $X$'s and $Y$'s are assumed to follow binomial and passion distributions. Classical as well as Bayes estimators for these distributions are derived. In order to obtain the estimators of these parametric functions, the basic role is played by the estimators of factorial moments to the two distributions.

In order obtained Bayes estimators of parameters and various parametric functions of different distributions, the researchers have adopted conventional technique, i.e. obtaining their posterior means. In the present discussion, we consider binomial and Poisson distributions and studying behavior of their hazard-rates, we investigate the situations when these distributions can be the recommended as life-time models. We consider the problems of estimating
reliability functions and \( P = \Pr\{X_1 + X_2 + \cdots + X_k \leq Y\} \) from Bayesian viewpoint. It is worth mentioning here that, in contrary the conventional approach, only estimators of factorial moments are needed to estimate these parametric functions and no separate dealing is needed.

1.6.1. The Hazard-rates of Binomial and Poisson Distributions and Set-Up the Estimation Problems.

The random variable \( X \) is said to follow binomial distribution with parameters \( (r, \theta) \) if its probability mass function (pmf) is given by

\[
p(x; r, \theta) = \binom{r}{x} \theta^x (1 - \theta)^{r-x}; \quad 0 < \theta < 1, \quad x = 0, 1, 2, \ldots, r. \tag{1.6.1}
\]

Here we assume that \( r \) is known but \( \theta \) is unknown. The reliability function for a specified mission time, say, \( (t_0 \geq 0) \), is given by

\[
R(t_0) = P(X \geq t_0) = \sum_{x=t_0}^{r} \binom{r}{x} \theta^x (1 - \theta)^{r-x} \tag{1.6.2}
\]

From (1.6.1) and (1.6.2), the hazard-rate is

\[
h(t_0) = \frac{p(t_0; r, \theta)}{R(t_0)}
\]

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\[
\begin{align*}
&= \frac{\binom{r}{t_0} \theta^{t_0} (1-\theta)^{r-t_0}}{\binom{r}{t_0} \theta^{t_0} (1-\theta)^{r-t_0} + \binom{r}{t_0+1} \theta^{t_0+1} (1-\theta)^{r-t_0-1} + \ldots + \theta^r} \\
&= \frac{\binom{r}{t_0}}{\binom{r}{t_0} \frac{\theta}{1-\theta} + \binom{r}{t_0+1} \left( \frac{\theta}{1-\theta} \right)^2 + \ldots + \left( \frac{\theta}{1-\theta} \right)^{r-t_0}} \\
&= \frac{r-t_0}{\sum_{x=0}^{r-t_0} \binom{r}{x+t_0} \left( \frac{\theta}{1-\theta} \right)^x}^{-1} \\
&= \left[ \sum_{x=0}^{r-t_0} \binom{r}{x+t_0} \left( \frac{\theta}{1-\theta} \right)^x \right]^{-1} \\
&= (1.6.3) \\
\text{Let } U(t_0) &= \frac{\binom{r}{x+t_0} \left( \frac{\theta}{1-\theta} \right)^x}{\binom{r}{t_0}} \\
U(t_0 + 1) &= \frac{\binom{r}{x+t_0+1} \left( \frac{\theta}{1-\theta} \right)^x}{\binom{r}{t_0 + 1}}
\end{align*}
\]
\[ \frac{U(t_0+1)}{U(t_0)} = \binom{r}{x+t_0+1} \binom{r}{t_0} \binom{r}{t_0+1} \binom{r}{x+t_0} \]

so that

\[ \frac{U(t_0+1)}{U(t_0)} = \frac{x(r+1)}{(r-t_0)(x+t_0+1)} \]

\[ < 1, \text{ for all } t_0 \geq r. \]

Thus, \( U(t_0) \) is monotonically decreasing in \( t_0 \) and we conclude from (1.6.3) that binomial distribution can be taken as reliability model when we encounter increasing failure rate.

Suppose that \( \{X_i\}, i = 1, 2, \ldots, k \) be \( k \) independent random variables where \( X_i \) follows the binomial distribution (1.6.1) with parameters \( (r, \theta) \) and \( Y \) be another random variable independent of \( X_i \)'s, following binomial distribution with parameters \( (s, \beta) \). Using the additive property of binomial distribution and denoting by

\[ X^* = \sum_{i=1}^{k} X_i \text{ and } r^* = \sum_{i=1}^{k} r_i, \text{ we conclude that} \]

\[ P = Pr\{X_1 + \ldots + X_k \leq Y\}. \]
\[ = \sum_{x^* = 0}^{\infty} \sum_{y = x^*}^{\infty} p(x^*; r^*, \theta) p(y; s, \beta). \quad (1.6.4) \]

The random variable \( X \) follows Poisson distribution with parameter \( \theta \) if its pmf is

\[ p(x; \theta) = \frac{e^{-\theta} \theta^x}{x!}; \quad x = 0, 1, 2, ... \quad (1.6.5) \]

The reliability function at a specified mission time, say, \( t_0 \geq 0 \) is

\[ R(t_0) = P[t > t_0] \]

\[ R(t_0) = \sum_{x = t_0}^{\infty} \frac{e^{-\theta} \theta^x}{x!}. \quad (1.6.6) \]

from (1.6.5) and (1.6.6), the hazard-rate is:

\[ h(t_0) = \frac{P(t_0; \theta)}{R(t_0)} \]

\[ = \frac{e^{-\theta} \theta^{t_0}}{t_0!} \]

\[ = \frac{t_0!}{\sum_{x = t_0}^{\infty} e^{-\theta} \theta^x \frac{1}{x!}} \]

\[ = \frac{e^{-\theta} \theta^{t_0}}{t_0!} \frac{1}{\frac{e^{-\theta} \theta^{t_0}}{t_0!} + \frac{e^{-\theta} \theta^{t_0+1}}{(t_0 + 1)!} + \frac{e^{-\theta} \theta^{t_0+2}}{(t_0 + 2)!} + ...} \]

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\[
\begin{align*}
&= \frac{1}{1 + \frac{\theta}{t_0 + 1} + \frac{\theta^2}{(t_0 + 2)(t_0 + 1)} + \ldots} \\
&= \left[ \sum_{x=t_0}^{\infty} \frac{\theta^{x-t_0} t_0!}{x!} \right]^{-1} \\
\text{Let } \ x-t_0 = u \\
&= \left[ \sum_{u=0}^{\infty} \frac{\theta^{u} t_0!}{(t_0 + u)!} \right]^{-1} \\
&= \left[ \sum_{x=0}^{\infty} \frac{\theta^{x}}{(x + t_0)x} \right]^{-1} \\
\text{(1.6.7)}
\end{align*}
\]

Let \( u(t_0) = \frac{\theta^{x}}{(x + t_0)x} \)

Since \( \frac{u(t_0 + 1)}{u(t_0)} = \frac{x + t_0}{x + t_0 + 1} < 1 \)

we conclude that \( u(t_0) \) is monotonically decreasing in \( t_0 \) and, from (1.6.7), Poisson distribution can represent life-time model when we having increasing failure rate (IFR). Let \( \{X_i\}, i = 1, 2, \ldots, k \) are independent random variable and \( X_i \) follows Poisson distribution (1.6.5) with parameter \( \theta_i \) and \( Y \) is another random Variable independent of \( X_i \)'s, following Poisson distribution
with parameter $\beta$. Denoting by $X^* = \sum_{i=1}^{k} X_i$ and $\theta^* = \sum_{i=1}^{k} \theta_i$, from the additive property of Poisson distribution.

$$p = \sum_{x^*=0}^{\infty} \sum_{y=x^*}^{\infty} p(x^*; \theta^*) p(y; \beta).$$

(1.6.8)

Our goal is to estimate $R(t_0)$ and 'P' for binomial and Poisson distribution. In what follows, we derive classical and Bayes estimators of powers of $\theta$.

1.6.2 The U MVUE of the powers of $\theta$, $R(t_0)$ and 'P' for Binomial Distributions.

In the following result, we obtain the U MVUE of $\theta^p (p > 0)$, which comes in the expression for the $p^{th}$ factorial moment about origin.

Result1: for $p > 0$, the U MVUE of $\theta^p$ is

$$\hat{\theta}_U^p = \left( \frac{nr - p}{T - p} \right) / \left( \frac{nr}{T} \right), \text{ if } p \leq T.$$  \hspace{1cm} (1.6.9)

$$= 0, \text{ otherwise}$$

where $T = \sum_{i=1}^{n} X_i$. 
Proof: Give a random sample $\mathbf{X}=(X_1, X_2, ..., X_n)$ from (1.6.9), it can be seen that $T$ is complete and sufficient for the family of binomial distributions [see Patel, Kapaia and Owen (1976, p.157). Moreover, $T$ follows binomial distribution with parameter $(nr, \theta)$. Now we choose a function $g(T)$ such that

$$E[g(T)] = \theta^p$$

i.e. $$\sum_{t=0}^{n} g(T) \binom{nr}{t} \theta^t (1-\theta)^{nr-t} = \theta^p$$

or

$$\sum_{t=p}^{n} g(T) \binom{nr}{t} \theta^{t-p} (1-\theta)^{nr-t} = 1.$$  \hspace{1cm} (1.6.10)

Equation (1.6.10) holds if we choose $g(T) = \hat{\theta}_U^p$, as given by (1.6.9).

Hence the result.

In the following result, we provide UMVUE of $R(t_0)$ and ‘$P$’, given at (1.6.2) and (1.6.4), respectively. Given $n_i$ observations $\{x_{ij}\}, i=1, 2, ..., k; j=1, 2, ..., n_i$ from $\{X_i\}, i=1, 2, ..., k$ and $m$ observation $\{Y_j\}, j=1, 2, ..., m$ on $Y$’s, it is to see that $T_1 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}$ and $T_2 = \sum_{j=1}^{m} Y_j$ are complete and sufficient for the family of binomial distributions $p(x^*; r^*, \theta)$ and $p(y; s, \beta)$,
respectively. Furthermore, \( T_1 \) and \( T_2 \) follow binomial distributions with parameters \( \left( \sum_{i=1}^{k} n_i, r_i, \theta \right) \) and \( (m, s, \beta) \), respectively.

**Result 2:** The UMVUE of \( R(t_0) \) and 'P' are given, respectively, by \( \hat{R}_U(t_0) \) and \( \hat{P}_U \), where

\[
\hat{R}_U(t_0) = \sum_{x=t_0}^{T} \binom{r}{x} \left( \frac{(n-1)r}{T-x} \right) / \left( \frac{nr}{T} \right)
\]

and

\[
\hat{P}_U = \sum_{x=0}^{T_1} \sum_{y=x}^{T_1} \binom{r}{x} \left( \frac{s}{T_2-y} \right) \left( \sum_{i=1}^{k} \frac{(n_i-1)r_i}{T_i-x} \right) \left( \frac{(m-1)s}{T_2} \right) / \left( \sum_{i=1}^{k} n_i r_i \right) \left( \frac{ms}{T_2} \right)
\]

Proof: we can write the pmf (1.6.1) as

\[
p(x; r, \theta) = \binom{r}{x} \sum_{i=0}^{r-x} (-1)^i \binom{r-x}{i} \theta^{x+i}
\]

Using Lemma 1 of Chaturvedi and Tomer (2002) and result 1, it follows from (1.6.13) that, the UMVUE of \( p(x; r, \theta) \), at a specified point 'x', is

\[
\hat{P}_U(x; r, \theta) = \binom{r}{x} \sum_{i=0}^{r-x} (-1)^i \binom{r-x}{i} \hat{\theta}_U^{x+i}
\]
\begin{equation}
\sum_{i=0}^{r-x-1} (-1)^i \binom{r-x}{i} \binom{nr-x-i}{T-x-i} / \binom{nr}{T} \tag{1.6.14}
\end{equation}

Using a result of Feller (1960, p.62) that

\[
\sum_j (-1)^j \binom{n-j}{k} = \binom{n-a}{n-k}; n, j, k \text{ positive integers},
\]

we obtain from (1.6.14) that

\[
\hat{P}_U(x; r, \theta) = \binom{r}{x} \binom{(n-1)r}{T-x} / \binom{nr}{T}; x \leq T. \tag{1.6.15}
\]

Now from (1.6.2), we get

\[
\hat{R}_U(t_0) = \sum_{x=t_0}^{r} \hat{P}_U(x; r, \theta)
\]

Utilizing (1.6.15), we have from the above equation

\[
\hat{R}_U(t_0) = \sum_{x=t_0}^{T} \binom{r}{x} \binom{(n-1)r}{T-x} / \binom{nr}{T}
\]

and (1.6.11) follows.

From arguments similar to those used in obtaining the UMVUE of \( R(t_0) \),

it can be shown that

\[
\hat{P}_U = \sum_{x^*=0}^{r} \sum_{y=x^*}^{s} \hat{p}(x^*, r^*; \theta) \hat{p}(y, s, \beta)
\]
\[
\sum_{x=0}^{T_1} \sum_{y=x}^{T_2} \binom{r_x}{x} \binom{s_y}{y} \binom{\sum_{i=1}^{k} (n_i - 1)r_i}{T_1 - x} \binom{(m - 1)s}{T_2 - y} \\
\left( \frac{\sum_{i=1}^{k} n_i r_i}{T_1} \right) \left( \frac{m s}{T_2} \right)
\]

Hence the result (1.6.12) follows.

1.6.3 The Bayesian Estimation of the Powers of \( \theta \), \( R(t_0) \) and \( 'P' \) for Binomial Distribution

We first consider the estimation of powers of \( \theta \) under natural conjugate family of prior densities and SELF.

Give a random sample \( X = (X_1, X_2, ..., X_n) \) from (1.6.1), let \( T = \sum_{i=1}^{n} X_i \).

Denoting by \( L(\theta|x) \), the likelihood of observing \( X \), we note that

\[
L(\theta|t) \propto \theta^t (1 - \theta)^{n - t} \quad \text{(1.6.16)}
\]

Thus we consider the conjugate prior for \( \theta \) to be a beta with parameters \((\nu, \mu)\), i.e.

\[
g(\theta) \propto \theta^{\nu-1} (1 - \theta)^{\mu-1} (\nu, \mu \text{ positive integers}) \quad \text{(1.6.17)}
\]

Combining (1.6.16) and (1.6.17) via Bayes’ theorem, the posterior density of \( \theta \) comes out to be
\[ g^*(\theta | t) = k \theta^{u-1} (1 - \theta)^{n+\mu-1}, \]

where the normalizing constant \( k \) can be obtained as

\[
k^{-1} = \int_0^1 g^*(\theta | t) d\theta
\]

\[ = B(t + \nu, nr + \mu - t). \]

Hence, the posterior density of \( \theta \) is

\[ g^*(\theta | t) = \frac{\theta^{u-1} (1 - \theta)^{nr+\mu-t-1}}{B(t + \nu, nr + \mu - t)}. \quad (1.6.18) \]

In the following result, we obtain Bayes estimator of \( \theta^p (p > 0) \), which comes in the expression for the \( p \)th factorial moment about origin [see Kendall and Stuart (1958, p.122)].

**Result 3:** For \( p > 0 \), Bayes estimator of \( \theta^p \) is given by

\[
\hat{\theta}^p_{Bayes} = \frac{B(t + \nu + p, nr + \mu - t)}{B(t + \nu, nr + \mu - t)}
\]

Proof: We know that, under squared-error loss function, Bayes estimator of any parametric function is its posterior mean.

\[
\hat{\theta}^p_{Bayes} = \frac{1}{B(t + \nu, nr + \mu - t)} \int_0^1 \theta^{u+p+1} (1 - \theta)^{nr+\mu-t-1} d\theta
\]
\[
\frac{B(t + \nu + p, \ n r + \mu - t)}{B(t + \nu, \ nr + \mu - t)}
\]

and the result follows.

In the following result, we provide Bayes estimator of \( R(t_0) \) and \( \mathcal{P} \), given at (1.6.2) and (1.6.4), respectively. Given \( n_i \) observations \( \{X_{ij}\} \), \( i = 1, 2, \ldots, k \), \( j = 1, 2, \ldots, n_i \) from \( \{X_i\} \), \( i = 1, 2, \ldots, k \) and \( m \) observations \( \{Y_j\} \), \( j = 1, 2, \ldots, m \) on \( Y \)'s, let us define \( T_1 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} \) and \( T_2 = \sum_{j=1}^{m} Y_j \). In order to estimate \( \mathcal{P} \), we choose independent Beta priors for \( \theta \) and \( \beta \) with parameters \((\nu_1, \mu_1)\) and \((\nu_2, \mu_2)\), respectively.

Result 4: Bayes estimators of \( R(t_0) \) and \( \mathcal{P} \) are given, respectively, by

\[
\hat{R}_{\text{Bayes}}(t_0) \quad \text{and} \quad \hat{P}_{\text{Bayes}},
\]

where

\[
\hat{R}_{\text{Bayes}}(t_0) = \sum_{x=0}^{r} \binom{r}{x} B(t + \nu + x, (n+1)r + \mu - t) / B(t + \nu, nr + \mu - t) \quad (1.6.19)
\]

and

\[
\hat{P}_{\text{Bayes}} = \frac{\sum_{x^*=0}^{r} \sum_{y=0}^s \binom{r}{x^*} \binom{s}{y} B(t_1 + \nu_1 + x^*, \sum_{i=1}^{k} (n_i + 1)r_i + \mu_i + r^* - t_1 - t_2) - x^* B(t_2 + \nu_2 + y, (m+1)s + \mu_2 - t_2)}{B(t_1 + \nu_1, \sum_{i=1}^{k} n_i r_i + \nu_1 - t_1) B(t_2 + \nu_2, ms + \mu_2 - t_2)} \quad (1.6.20)
\]
Proof: Using Lemma 1 of Chaturvedi and Tomer (2002) (which holds good if we replace UMVUEs by Bayes estimators) and result 3, from (1.6.13) Bayes estimator of $p(x; r, \theta)$, at a specified point ‘x’, is

$$
\hat{P}_{\text{Bayes}}(x; r, \theta) = \left( \frac{r}{x} \right) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{r-x}{x+i} \hat{\theta}_{\text{Bayes}} x + i
$$

$$
= \left( \frac{r}{x} \right) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{r-x}{x+i} \frac{B(t+u+x+i, nr+\mu-t)}{B(t+u, nr+\mu-t)}. \quad (1.6.21)
$$

Utilizing Lemma 2 of "Chaturvedi and Tomer (2002), it follows from (1.6.21) that

$$
\hat{P}_{\text{Bayes}}(x; r, \theta) = \left( \frac{r}{x} \right) \frac{B(t+u+x+(n+1)r+\mu-t-x)}{B(t+u, nr+\mu-t)}. \quad (1.6.22)
$$

Now from the fact that

$$
\hat{R}_{\text{Bayes}}(t_0) = \sum_{x=t_0}^{\infty} \hat{P}_{\text{Bayes}}(x; r, \theta).
$$

Utilizing (1.6.22), we obtain

$$
\hat{R}_{\text{Bayes}}(t_0) = \sum_{x=t_0}^{\infty} \left( \frac{r}{x} \right) \frac{B(t+u+x+(n+1)r+\mu-t-x)}{B(t+u, nr+\mu-t)}
$$

and the result (1.6.19) follows:

Similarly we can obtain

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\[ \hat{P}_{\text{Boys}} = \sum_{x^* = 0}^{\hat{x}} \sum_{y=x^*}^{\hat{x}} \hat{P}_{\text{Boys}}(x^*; r^*, \theta) \hat{P}_{\text{Boys}}(y; s, \beta) \]

Utilizing (1.6.22), we obtain

\[ \hat{P}_{\text{Boys}} = \frac{\sum_{x^* = 0}^{\hat{x}} \sum_{y=x^*}^{\hat{x}} \binom{r^*}{x^*} \left( \binom{s}{y} \right) B(t_1 + \nu_1 + x^* \sum_{i=1}^{k} (n_i + 1)r_i + \mu_1 + r^* - t_1 \right.}{\left. - x^* B(t_2 + \nu_2 + y, (m + 1)s + \mu_2 - t_2 - y) \right)} \]

\[ B(t_1 + \nu_1, \sum_{i=1}^{k} n_i r_i + \nu_1 - t_1) B(t_2 + \nu_2, ms + \mu_2 - t_2) \]

and the result (1.6.20) follows.

1.6.4 The UMVUE of the powers of \( \theta, R(t_0) \) and \( P \) for Poisson Distribution:

In the following result we derive the UMVUE of \( \theta^p (p > 0) \) using the method discussed in the Section (1.6.2)

**Result 5:** For \( p > 0 \), the UMVUE of \( \theta^p \) is

\[ \hat{\theta}_n^p = \frac{T}{n^p (T - P)}, \text{ if } P \leq T \]  \hfill (1.6.23)

\[ = 0, \text{ otherwise} \]

where
\[ T = \sum_{i=1}^{n} X_i \]

Proof: Given a random sample \( X = (X_1, X_2, \ldots, X_n) \) from (1.6.5), it can be seen that \( T \) is complete and sufficient for the family of Poisson distributions [see Patel, Kapadia and Owen (1976, p.158)]. Moreover, \( T \) follows Poisson distribution with parametre \( n\theta \). Now we chose a function \( g(T) \) such that

\[ E[g(T)] = \theta^p \]

i.e.

\[ \sum_{t=0}^{\infty} g(T) \frac{e^{-n\theta} n^t \theta^t}{t!} = \theta^p \]

or

\[ e^{-n\theta} \sum_{t=p}^{\infty} g(T) \frac{n^t \theta^{t-p}}{t!} = 1. \]  

(1.6.24)

Equation (1.6.24) holds if we choose \( g(T) = \hat{\theta}_U^p \), as given by (1.6.23). Hence the result.

**Result 6:** The UMVUE of \( R(t_0) \) and '\( P' \) are given, respectively, by \( \hat{R}_U(t_0) \) and \( \hat{P}_U \), where

\[ \hat{R}_U(T_0) = \frac{1}{n_T} \sum_{x=T_0}^{T} \binom{T}{x} (n-1)^{T-x} \]

(1.6.25)
and

\[ \hat{P}_U = \left( \sum_{j=1}^{K} n_j \right) \frac{1}{m} \left( \sum_{i=1}^{T} \sum_{j=1}^{T} \left( \frac{T_1}{x^*} \right) \left( \frac{T_2}{y} \right) \left( \sum_{j=1}^{k} n_j - 1 \right)^{T_j - x^*} (m - 1)^{T_j - y} \right) \]  

(1.6.26)

Proof: we can write (1.6.5) as

\[ P(x; \theta) = \sum_{i=0}^{\infty} (-1)^i \frac{\theta^{x+i}}{i!x!} \]  

(1.6.27)

Using Lemma 1 of Chaturvedi and Tomer (2002) and result 5, it follows from (1.6.27) that, the UMVUE of \( p(x; \theta) \), at a specified point ‘x’, is

\[ \hat{p}_U(x; \theta) = \sum_{i=0}^{T-x} (-1)^i \frac{\theta^{x+i}}{i!x!} \]

\[ = \sum_{i=0}^{T-x} (-1)^i \frac{T!}{n^{i+x}x!(T-x)} \]

\[ = \left( \frac{T}{x} \right) n^{-x} \left( 1 - \frac{1}{n} \right)^{T-x} \]

(1.6.28)

Now from (1.6.6), we get

\[ \hat{R}_U(t_0) = \sum_{x = t_0}^{T} \hat{p}_U(x; \theta) \]

Utilizing (1.6.28)
\[ \hat{R}(t_0) = \frac{1}{n^T} \sum_{x=x_0}^{T_n}(n-1)^{T-x} \]

and the result (1.6.25) follows.

From arguments similar to those used to obtaining the UMVUE of \( R(t_0) \), it can be shown that

\[
\hat{P}_U = \sum_{x^* = 0}^{\infty} \sum_{y = x^*}^{\infty} \hat{p}(x^*; \theta) \hat{p}(y; \beta)
\]

\[
= \frac{1}{\left( \sum_{i=1}^{k} n_i \right) m^T} \sum_{x=x_0}^{T_1} \sum_{y=x^*}^{T_2} \left( \begin{array}{c} T_1 \end{array} x \right) \left( \begin{array}{c} T_2 \end{array} y \right) \left( \sum_{i=1}^{k} n_i - 1 \right)^{r_i - x^*} (m - 1)^{r_2 - y}
\]

Hence the Result.

1.6.5 The Bayesian Estimation of the powers of \( \theta \), \( R(t_0) \) and ‘P’ for Poisson Distribution:

Denoting by \( T = \sum_{i=1}^{n} X_i \), the likelihood of observing a random sample \( \mathbf{X} = (X_1, X_2, ..., X_n) \) from (1.6.4) is

\[ L(\theta | t) \propto e^{-n\theta} \theta^t. \] (1.6.29)

We consider the conjugate prior for \( \theta \) to be gamma with parameters \((\alpha, \lambda)\), i.e.
\[ g(\theta) \propto \theta^{a-1}e^{-\lambda \theta} \quad (\alpha \text{ positive integer}). \quad (1.6.30) \]

From (1.6.29) and (1.6.30), the posterior density of \( \theta \) is

\[ g^*(\theta | t) = \frac{(n + \lambda)^{t+\alpha}}{\Gamma(t + \alpha)} \theta^{t+\alpha-1}e^{-(n+\lambda)\theta}. \]

In the following result, we mention Bayes estimator of \( \theta^p (p > 0) \), which takes place in the expression for the \( p \)th factorial moment about origin [see Johnson and Kotz (1969, p.91)].

**Result 7:** For \( p > 0 \), Bayes estimator of \( \theta^p \) is

\[ \hat{\theta}_{\text{Bayes}}^p = \left\{ \frac{\Gamma(t + \alpha + p)}{\Gamma(t + \alpha)} \right\} (n + \lambda)^{-p}. \]

Proof of the result is similar to those of result 3.

In the following result, we provide Bayes estimators of \( R(t_0) \) and ‘P’. In order to estimate ‘P’, we consider independent priors for \( \theta^* \) and \( \beta \) to be gamma with parameters \((\alpha_1, \lambda_1)\) and \((\alpha_2, \lambda_2)\), respectively.

**Result 8:** Bayes esimaors of \( R(t_0) \) and ‘P’ are given, respectively, by \( \hat{R}_{\text{Bayes}}(t_0) \) and \( \hat{P}_{\text{Bayes}} \), where,

\[ \hat{R}_{\text{Bayes}}(t_0) = \left\{ \frac{(n + \lambda)}{(n + \lambda + 1)} \right\}^{t+\alpha} \sum_{x=t_0}^{\infty} \binom{t + \alpha + x - 1}{x} (n + \lambda + 1)^{-x} \quad (1.6.31) \]
\[
\hat{P}_{\text{Bays}} = \sum_{x^*=0}^{\infty} \sum_{y=x^*}^{\infty} \frac{(t_1 + \alpha_i + x^* - 1)}{x^*} \frac{(t_2 + \alpha_2 + y - 1)}{y} \left( \sum_{i=1}^{k} n_i + \lambda \right)^{t_1 + \alpha_i} (m + \lambda_2)^{t_2 + \alpha_2 + y} \\
\left( \sum_{i=1}^{k} n_i + \lambda_i + 1 \right) (m + \lambda_2 + 1)^{t_2 + \alpha_2 + y}
\]

(1.6.32)

Proof: We can write (1.6.5) as

\[
p(x; \theta) = \sum_{i=0}^{\infty} (-1)^i \frac{\theta^{i+x}}{i! x!},
\]

which on using result 3 gives that Bayes estimator of \( p(x; \theta) \), at a specified point ‘x’, is

\[
\hat{P}_{\text{Bays}}(x; \theta) = \frac{1}{x!} \sum_{i=0}^{\infty} (-1)^i \frac{\hat{\theta}_{\text{Bays}}^{i+x}}{i!}
\]

\[
= \frac{1}{x!} \sum_{i=0}^{\infty} (-1)^i \frac{(t + \alpha + i + x - 1)!}{i!(t + \alpha - 1)! (n + \lambda)^{i+x}}
\]

\[
= \frac{1}{(n + \lambda)^x} \left( \sum_{i=0}^{\infty} (-1)^i (t + \alpha + i + x - 1) \frac{1}{i} \frac{1}{(n + \lambda)} \right)
\]

\[
= \binom{t + \alpha + x - 1}{x} \frac{(n + \lambda)^{t+\alpha}}{(n + \lambda + 1)^{t+\alpha+x}}.
\]

(1.6.33)

Results (1.6.31) and (1.6.32) follow, respectively, from (1.6.7) and (1.6.9) on using (1.6.33).