APPENDIX A

DEFINITION OF SPECIAL FUNCTIONS

1. \( \text{erf}(x) = \left( \frac{2}{\sqrt{\pi}} \right) \int_0^x \exp(-t^2) \, dt \)

2. \( \text{erfc}(x) = \left( \frac{2}{\sqrt{\pi}} \right) \int_x^\infty \exp(-t^2) \, dt \)

3. \( \gamma(a) = \int_0^\infty \exp(-t) \, t^{a-1} \, dt \quad a > 0 \)

4. \( \gamma(a, x) = \int_x^\infty \exp(-t) \, t^{a-1} \, dt \quad a > 0 \)

5. \( \Gamma(a, x) = \int_0^x \exp(-t) \, t^{a-1} \, dt \quad a > 0 \)

6. \( B(m, n) = \int_0^1 (1-x)^{m-1} x^{n-1} \, dx \quad m, n > 0 \)

7. \( J_{\nu}(x) = \frac{1}{\pi} \int_0^{\pi} (1 - t^2)^{-\frac{\nu}{2}} \cos(\nu t) \, dt \)

8. \( I_{\nu}(x) = \frac{1}{\pi} \int_0^{\pi} (1 - t^2)^{-\frac{\nu}{2}} \cosh(\nu t) \, dt \)

8a. \( I_{\nu}(x) = \frac{1}{\pi} \int_0^{\pi} \exp(x \cos(\theta)) \sin^2(\theta) \, d\theta \)

9. \( K_{\nu}(x) = (1/2)(x/2)^\nu \int_0^{\infty} \exp(-t - (x^2/4t)) \, dt/t^{\nu+1} \)

9a. \( K_{\nu}(x) = \int_0^{\infty} \exp(-x \cosh(t)) \cosh(\nu t) \, dt \)

10. \( K(k) = \int_0^{\pi/2} \frac{1 - k^2 \sin^2(\theta)}{\sqrt{1 - k^2 \sin^2(\theta)}} \, d\theta \)

11. \( K(x) = \int_C (1 - x^2 \sin^2 \theta)^{1/2} d\theta \)

12. \( \delta(x) = 0 \text{ if } x \neq 0 \) and \( \int_{-\infty}^{\infty} \delta(x)dx = 1 \)

13. \[ p_q^{F_q} (a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!} \]

where \( (e)_n = (e + 1)(e + 2) \cdots (e + n - 1) \), \( (e)_0 = 1 \)

\( b_j \neq 0 \) or a negative integer, for any \( j \)

If \( e \) is not a negative integer, then we can write \( (e)_n = \Gamma(e + n)/\Gamma(e) \)

14. \( \,_1F_1 (a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!} \)

15. \( \,_2F_1 (a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \)

16. \( M_{k, \gamma} (x) = x^{-k} e^{-x/2} \Gamma(1/2 + k, 2 - 1); x \)

17. \( W_{k, \gamma} (x) = \frac{1}{\sqrt{2\pi}} e^{1/2} \frac{x^{1/2}}{\left(1 + \frac{1}{2} x \right)^{1/2}} \Gamma(1/2 + k) \left(1 + \frac{1}{2} x \right)^{-1/2} \)

18. \( D_{\gamma} (x) = x^{1/2} e^{-x/2} \Gamma(1/2, x) \)

19. \( P_{\gamma}(x) = \frac{1}{2} \frac{1 - x}{2} \)

For the above definitions see Erdelyi (1954a, 1954b), Sneddon (1961) and Watson (1952).
### APPENDIX - B

#### TABLE OF INTEGRALS OF THE LAPLACE TRANSFORM TYPE

<table>
<thead>
<tr>
<th>n(t)</th>
<th>g(p) = \int_0^\infty \exp(-pt)f(t)dt</th>
</tr>
</thead>
</table>

1. \((t^2 + 2at)^{n-1/2}\)     \((n+1/2)^{1/2}(2a/p)^n\exp(ap)K_n(ap)\)  
   \(|\text{Arg } a| < \text{Re } n < 1\)  
   \(\text{ke } p > 0\)

2. \((2bt-t^2)^{n-1/2}\) for  
   \(0 < t < 2b\)  
   \(0 \text{ for } t > 2b\)  
   \(\text{Re } n > -1/2\)

3. \(0 \text{ for } 0 < t \leq b\)  
   \((\pi a)^{1/2}\exp(ap^2)\text{erf}(a/p)\) for \(b < t < \infty\)
   \(\text{Re } a > 0\)

4. \(\sinh(2a^{1/2}t)^{1/2}\)  
   \(\pi^{1/2}\exp(ap)\)  
   \(\text{ke } p > 0\)

5. \(t^{-1/2}\sinh(2\alpha t)\)  
   \(\pi \exp(\alpha/p)\text{erf}(\alpha/\sqrt{p})\)  
   \(\text{ke } p > 0\)
\begin{align*}
f(t) & \quad g(p) = \int \exp(-pt)f(t)dt \\
6. \quad t^{\gamma-1} \cosh \left[ (2at)^{1/2} \right] & \\
\gamma > 0 & \\
7. \quad \text{erf}(a^{1/2}, t^{1/2}) & \quad a^{1/2} p^{-1/2} \left( p + a \right)^{-1/2} \\
8. \quad t^{m-1/2} I_{2n}(2, at) & \\
\text{Re} \ (m + n) > -1/2 & \quad \text{Re} \ p > 0 \\
\end{align*}

Notes to this Table - For equations (1) to (8) see pages 138, 138, 146, 165, 166, 166, 176 and 197 respectively of Erdelyi (1954b)
### APPENDIX - G

#### TABLE OF INTEGRALS OF THE MEHYX TRANSFORM TYPE

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( g(s) = \int_0^1 f(x)x^{s-1}dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1-x)^n(1+ax)^n)</td>
<td>(B(n+1,s)_2_F_1(-n,s; n+s+1; -a))</td>
</tr>
<tr>
<td>for ( 0 &lt; x &lt; 1 )</td>
<td>( \Re n &gt; -1 )</td>
</tr>
<tr>
<td>( 0 ) for ( 1 &lt; x &lt; \infty )</td>
<td>( \Re s = 0 )</td>
</tr>
<tr>
<td>( \Arg (1+a) )</td>
<td></td>
</tr>
</tbody>
</table>

1. \( \exp(-ax^h) \) | \( h^{-1}x^{-a/h} - (a/h) \) |
| \( \Re a > 0, \ h > 0 \) | \( \Re s = 0 \) |

2. \( \exp(-ax^n - bx^h) \) | \( (2/h)(b/a)^x K_{a/h}(2,ab) \) |
| \( h > 0, \ Re \ a > 0, \ Re \ b > 0 \) | |

3. \( K_n(ax) \) | \( a^{-s}e^{-2\pi i(s-n)/2} \_\Gamma(s-n)/2 \_\Gamma(s+n)/2 \) |
| \( \Re a > 0 \) | \( \Re s = \Re n \) |

4. \( \exp(-ax)K_n(ax) \) | |
| \( \Re a > 0 \) | |

5. \( \) | |
<table>
<thead>
<tr>
<th>f(x)</th>
<th>( g(s) = \int f(x)x^{s-1}dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. ( K_m(ax)K_n(bx) )</td>
<td>( 2^{s-3}a^{-s-n}b^n \left( \frac{s}{2} \right)^{(s+m+n)/2} )</td>
</tr>
<tr>
<td>&amp;</td>
<td>( \cdot \left( \frac{s-m+n}{2} \right)^{(s-m-n)/2} )</td>
</tr>
<tr>
<td>&amp;</td>
<td>( \cdot \left( \frac{s-m-n}{2} \right)^{2^{s-1}(s+m+n)/2} )</td>
</tr>
<tr>
<td>&amp;</td>
<td>( \left( \frac{s-m+n}{2} \right) \left( 1 - \frac{b^2}{a^2} \right) )</td>
</tr>
</tbody>
</table>

Note to this Table: For equations (1) to (6) see the pages 310, 313, 313, 331, 331, and 334 of Erdelyi (1954a).
(a) The Gamma and Beta Functions

1. \( \Gamma(n + 1) = n \Gamma(n) \)

2. \( \Gamma(n + 1) = n! \) if \( n \) is a positive integer

3. \( \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi} \)

4. \( B(m, n) = \Gamma(m) \Gamma(n) / \Gamma(m + n) \)

5. The duplication formula
   \[ \Gamma(1/2) \Gamma(2n) = 2^{2n-1} n! \Gamma(n) \Gamma(n + 1) \]

6. \( \Gamma(n) = \lim_{k \to \infty} k^n B(n, k) \)

7. \( \gamma(1, x) = 1 - \exp(-x) \)

8. \( \gamma(n + 1, x) = n \gamma(n, x) - x^n \exp(-x) \) for \( n \geq 0 \)

(b) The Error Function

9. \( \text{erf}(0) = 0, \quad \text{erf}(\infty) = 1 \)

10. \( \text{erfc}(x) = 1 - \text{erf}(x) \)
erf(x) is an odd function. That is erf(sx) = s erf(x) where s = ± 1

\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt

\frac{d}{dy} \left[ \text{erf}(by^k) \right] = b(\sqrt{y})^{-k} \exp(-b^2 y)

\int_0^\infty x^{n-1} \exp(b^2 x^2) \text{erfc}(ax) dx

= \frac{1}{\sqrt{\pi}} \frac{a^2}{b^2} \left[ \frac{n/2}{(n+1)/2} \right] \frac{1}{\sqrt{b^2/a^2}} \left[ \frac{n/2}{(n+1)/2} \right]

\text{Re } n > 0, \quad 0, \quad \text{Re } b^2 > \text{Re } a^2

\text{(c) The Bessel Function}

I_n(x) = \exp(x)(2x)^{-\frac{n}{2}} \left[ 1 - \frac{x^2}{2} \right] \left[ \frac{x}{2} \right]^{-\frac{n}{2}} \exp(-x) \left[ 1 - \frac{2x}{3} \right] \exp(-x) \left[ 1 - \frac{2x}{3} \right]

\text{for large } x

K_n(x) = (\pi/2)x^{-\frac{n}{2}} \exp(-x) \left[ 1 - \frac{2x}{3} \right] \exp(-x) \left[ 1 - \frac{2x}{3} \right]

\text{for large } x

\text{(d) Miscellaneous Results}

K(0) = \pi/2, \quad E(0) = \pi/2, \quad K(1) = 1

\lim_{x \to -1} (1 - x)K \left( \frac{2x}{(1 + x)^k} \right) = 0

\int_0^1 f(x)(x - z)dx = f(z)
\[ \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \lim_{x \to \infty} x^n \exp(-x) = 0 \]

\[ \sinh(x + iy) = \sinh(x) \cos y + i \cosh(x) \sin y \]

\[ \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \tag{2a} \]

\[ \int \sinh(at^b) \exp(-t) t^{-1} dt \]

\[ \exp(-x^2 - a^2 / x^2) dx = (\pi/2) \exp(-2a) \]

where \( \beta_{mn} = 1 \) if \( i = n \)

\[ 0 \] if \( i \neq n \)

Let \( F_e[f(x)] \) denote the exponential Fourier transform of \( f(x) \). If

\[ f(x) = \begin{cases} (a^2 - x^2)^{-1/2} & \text{for } 0 < x < a \\ 0 & \text{for } a < x \]

then

\[ F_e[f(x)] = 2^{-1/2} \int_{-\infty}^{\infty} \left( \chi^2 + \frac{1}{\chi^2} \right)^{-1/2} \left( e^{-\chi^2} - e^{-\chi^2/a^2} \right) d\chi \]

\[ = g(R) \]

where \( R \) is the variable in the Fourier space.
Let $P_c [\mathcal{G}(x)]$ denote the cosine Fourier transform of $\mathcal{G}(x)$. Then

$$P_c [J_\nu (x)J_{-\nu} (x)] = \frac{(1/2)P_\nu (R^2/2)}{(R^2/2 - 1)}$$

for $0 \leq R < 2$

$$= 0$$

for $R = 2$

**Notes to this Appendix**

For equations (1) to (5) see p.11 of Sneddon (1961). For (6) see p.31 of Rainville (1960). For (8) see p.14 of Jahnke et al (1960). Equation (11) follows from (E.6). For (12) and (14) see p.295 and p.306 of Erdelyi (1954b). For (15) and (16) see p.182 and p.183 of Dwight (1957). For (17) use the equation in p.62 of Jahnke et al (1960). For (22) and (23) see p.14 of Dwight (1957). Equation (24) is obtained by using (23) in the integral in (24) and by interchanging the order of integration and summation and then carrying out the integration. For (25) see p.275 of Hodgman et al (1959). For (26) see p.63 of Feller (1960). For (27) and (28) see p.11 and p.46 of Erdelyi (1954a). Equations (7), (9), (10), (13), (17) and (13) to (21) follow from the definition of these functions.
APPENDIX - B

SOME USEFUL PROPERTIES AND RESULTS INVOLVING
HYPERGEOMETRIC FUNCTIONS

(a) Properties of Hypergeometric Function

1. Any permutation of numerator parameters within themselves (similarly, the denominator parameters within themselves) leave the hypergeometric function unchanged. For example \( {}_2F_1(a, b; c; x) = {}_2F_1(b, a; c; x) \).

2. Contraction Property: If a pair of numerator and denominator parameters are identical in \( {}_pF_q \), then the function \( {}_pF_q \) can be abbreviated (by omitting this pair of parameters) and written as an \( {}_{p-1}F_{q-1} \) function. For example \( {}_3F_2(a, b, c; d, a; x) = {}_2F_1(b, c; d; x) \).

3. If any numerator parameter is a negative integer the hypergeometric series terminates and becomes a polynomial whose degree is the magnitude of the negative integer. If any numerator parameter is zero, the \( {}_pF_q \) function becomes unity. For example, \( {}_2F_1(-1, b; c; x) = 1 \) when \( 1 - bx/c \) and \( {}_2F_2(a, 0; b, c; x) = 1 \).
4. If the value of the argument of \( \frac{p}{q} \) is zero, then the \( \frac{p}{q} \) function becomes unit. For example, 
\[ 2^F_1(a, b; c; 0) = 1. \]

(b) Some Useful \( _1F_1 \) Functions

5. \[ _1F_1(a; a; x) = \exp(x) \]

6. \[ _1F_1(a + 1; a; x) = (1 + x/a)\exp(x) \]

7. \[ _1F_1(1/2; 3/2; -x^2) = (-x/\sqrt{2})\exp(x) \cdot \text{erf}(x) \]

8. \[ _1F_1(1; 3/2; x^2) = (-2x)\exp(x^2)\exp(x) \cdot \text{erf}(x) \]

9. \[ _1F_1(n + 1/2; 2n + 1; x) = 2^n\exp(x/2)x^{-n} I_n(x/2) \]

10. \[ _1F_1(n + 3/2; 2n + 2; x) = 2^n\exp(x/2)x^{-n} \cdot I_n(x/2) + I_{n+1}(x/2) \]

11. \[ _1F_1(3/2; 2; x) = \exp(x/2) I_0(x/2) + I_1(x/2) \]

12. \[ _1F_1(1/2; 1; x) = \exp(x/2) I_0(x/2) \]

(c) Some Useful \( _2F_1 \) Functions

13. \[ _2F_1(a, b; c; x) = (1 - x)^{-a} \]

14. \[ _2F_1(1, 1; 2; x) = -x\log_e(1 - x) \]

15. \[ _2F_1(1/2, 1; 3/2; x^2) = (2x)^{-1}\log_e((1+x)/(1-x)) \]

16. \[ _2F_1(1/2, 1/2; 3/2; x^2) = (1/x)\sin^{-1}(x) \]

17. \[ _2F_1(1/2, 1; 3/2; -x^2) = (1/x)\tan^{-1}(x) \]
18 \[ 2F_1 \left( 1/2, 1/2; 1; x^2 \right) = (2/\cdots) x \] 19 \[ 2F_1 \left( -1/2, 1/2; 1; x^2 \right) = (2/\cdots) x \] 20 \[ 2F_1 \left( 1/2, 1/2; 1; \frac{1-x}{2} \right) = F_{-1} (x) \]

(d) Polynomial Representation of Some Hypergeometric Functions

21 \[ \begin{align*} \mbox{1F}_1 (-n/2; 1; -x) &= \quad 1 + x \quad \text{for n = 2} \\
&= 1 + 2x + \left( \frac{1}{2} \right) x^2 \quad \text{for n = 4} \\
&= 1 + 3x + \left( \frac{3}{2} \right) x^2 + \left( \frac{1}{6} \right) x^3 \quad \text{for x = 6} \\
&= 1 + 4x + 3x^2 + \left( \frac{2}{3} \right) x^3 + \left( \frac{1}{24} \right) x^4 \quad \text{for x = 8} \end{align*} \]

22 \[ \begin{align*} \mbox{1F}_1 (-n/2; 1/2; -x) &= \quad 1 + 2x \quad \text{for n = 2} \\
&= 1 + 4x + \left( \frac{4}{3} \right) x^2 \quad \text{for n = 4} \\
&= 1 + 6x + 4x^2 + \left( \frac{8}{15} \right) x^3 \quad \text{for n = 6} \\
&= 1 + 8x + 6x^2 + \left( \frac{32}{15} \right) x^3 + \left( \frac{16}{105} \right) x^4 \quad \text{for n = 8} \end{align*} \]

23 \[ \begin{align*} \mbox{2F}_1 (-n/2, 1/2; 1; -x) &= \quad 1 + (1/2)x \quad \text{for n = 2} \\
&= 1 + x + \left( \frac{3}{8} \right) x^2 \quad \text{for n = 4} \\
&= 1 + \left( \frac{3}{2} \right)x + \left( \frac{9}{8} \right)x^2 + \left( \frac{5}{16} \right)x^3 \quad \text{for n = 6} \\
&= 1 + 2x + \left( \frac{3}{4} \right)x^2 + \left( \frac{5}{4} \right)x^3 + \left( \frac{35}{126} \right)x^4 \quad \text{for n = 8} \end{align*} \]
\[ 2^F_2\left(-\frac{n}{2}, \frac{1}{2}; 1, 1; -x\right) = \]
\[
1 + \left(\frac{1}{2}\right)x \quad \text{for } n = 2 \\
1 + x + \left(\frac{3}{16}\right)x^2 \quad \text{for } n = 4 \\
1 + \left(\frac{3}{2}\right)x + \left(\frac{9}{16}\right)x^2 + \left(\frac{5}{96}\right)x^3 \quad \text{for } n = 6 \\
1 + 2x + \left(\frac{9}{8}\right)x^2 + \left(\frac{5}{24}\right)x^3 + \left(\frac{35}{3072}\right)x^4 \\
\quad \text{for } n = 8
\]

(e) Some Important Integrals and Formulas

25  \[ \text{Kummer's Relation} \]
\[ _1F_1(a; b; x) = \exp(x) \quad _1F_1(b-a; b; -x) \]

26  \[ D_y(x) = \left(\frac{1}{x} \right) _1F_1\left(\frac{1}{2}; \frac{1}{2}; \frac{x}{2}\right) \]

27  \[ D_y(x) + B_x(-x) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right) \cdot \]
\[ _1F_1\left(-\frac{1}{2}; \frac{1}{2}; \frac{x^2}{2}\right) \]

28  \[ _2F_1(a, b; c; x) = (1-x)^{-b} \quad _2F_1(a, b; c; \frac{x}{x-1}) \]

28a  \[ = (1-x)^{c-a-b} \quad _2F_1(a, b; c; x) \]

29  \[ \int x^{n-1}(1-x)^{m-1} \quad \text{p}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; k) \]
\[ = B(1, m) \quad \text{p}_q+1(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 1 + m; k) \]
\[ \int_0^\infty \binom{p}{q} (a_i; \ldots, a_p; b_1, \ldots, b_q; \infty) \exp(-ax)x^{m-1}dx = \Gamma(m)\alpha^{m-1} \binom{p+1}{q} (a, \ldots, a_p, m; b_1, \ldots, b_q; \frac{1}{\alpha}) \]

**(f) Some Recurrence Relations for \( {}_1F_1 \) and \( {}_2F_1 \)**

\[ {}_1F_1(a + 1; b; x) - {}_1F_1(a; b; x) = \frac{x}{b} {}_1F_1(a + 1; b + 1; x) \]

\[ (b - a) {}_1F_1(a - 1; b; x) = (a - b + x) {}_1F_1(a; b; x) \]

\[ b(b - 1) {}_1F_1(a; b - 1; x) - b(b - 1 + x) {}_1F_1(a; b; x) = 0 \]

\[ b {}_1F_1(a; b; x) - b {}_1F_1(a - 1; b; x) = x {}_1F_1(a; b + 1; x) \]

\[ (c - a - b + (b - a)(1 - x)) {}_2F_1(a, b; c; x) - a(1 - x) {}_2F_1(a + 1, b; c; x) = (c - a) \]

\[ (c - b - 1) {}_2F_1(a, b; c; x) = (c - a - b - 1) \]

\[ c {}_2F_1(a, b + 1; c; x) \]

\[ {}_2F_1(a + 1, b; c; x) \]

\[ {}_2F_1(a, b; c; x) \]

\[ {}_2F_1(a + 1, b; c; x) \]

\[ {}_2F_1(a, b + 1; c; x) \]
37 \[
\left(1-b\right)x/c_{2}^F_{1}(a, b; c + 1; x) = 2^F_{1}(a-1, b-1; c; x)
\]
\[
- 2^F_{1}(a, b - 1; c; x)
\]

38 \[
2^F_{1}(a, b; c; x) = 2^F_{1}(a + 1, b - 1; c; x) +
\]
\[
\left(a-b+1\right)x/c_{2}^F_{1}(a+1, b; c + 1; x)
\]

39 \[
2a - c + (b - a)x \quad 2^F_{1}(a, b; c; x) =
\]
\[
a(1-x) \quad 2^F_{1}(a + 1, b; c; x) - (c-a) -
\]
\[
\cdot 2^F_{1}(a - 1, b; c; x)
\]

(R) Miscellaneous Results

40 \[
2^F_{1}(1/2, 2; 3/2; -x) =
\]

41 \[
2^F_{1}(1, 1/2; 2; x) = (2/x) - (1-x)^{1/2}
\]

42 \[
2^F_{1}(1, 3/2; 2; x) = (2/x) \cdot (1-x)^{-1/2}
\]

43 \[
1^F_{1}(1; 2; x) = (1/x) \cdot \exp(x) - 1
\]

44 \[
1^F_{1}(-1/2; 1/2; -x) = \exp(-x) + (1x)^{1/2} \cdot \text{erf}(x^{1/2})
\]

45 \[
1^F_{1}(n + 1/2; 1; x) =
\]

(1) \[
\exp(x/2)I_0(x/2) (= T, say) \text{ for } n = 0
\]

(11) \[
(1 + x)T + xV \text{ for } n = 1 \text{ where } V = \exp(x/2)I_1(x/2)
\]
(iii) \[ (1/3) \left[ (2x^2 + 6x + 3)^n + (2x^2 + 4x) \right] \quad \text{for } n = 2 \]

(iv) \[ (1/15) \left[ (4x^3 + 28x^2 + 45x + 15)^n + (4x^3 + 24x^2 + 23x) \right] \quad \text{for } n = 3 \]

(v) \[ (1/105) \left[ (6x^4 + 104x^3 + 376x^2 + 420x + 105)^n + (6x^4 + 96x^3 + 234x^2 + 176x) \right] \quad \text{for } n = 4 \]

46. \[ \begin{array}{c} 1 \times (1/2; 1; x) = \\
(1) \quad 1 + (2x)^{1/2} \exp(x) \text{erf}(x^{1/2}) = 1 + 1, \text{ say} \quad \text{for } n = 1 \\
(11) \quad 1 + x + (3/2 + x) \quad \text{for } n = 2 \\
(iii) \quad (1/4)(2x^2 + 9x + 3)^n + (1/4)(4x^2 + 20x + 15)^n \quad \text{for } n = 3 \\
(iv) \quad (1/24)(4x^3 + 40x^2 + 97x + 24)^n + (1/4)(6x^3 + 84x^2 \\
\quad + 210x + 105)^n \quad \text{for } n = 4 \\
\end{array} \]

47. \[ \begin{array}{c} 2 \times (n/2, 1/2; 1; x) = \\
(1) \quad \frac{(2n)!}{(2^n n!)^2} \text{ Y, say} \quad \text{for } n = 1 \\
(ii) \quad \frac{(2/\pi)^{(1-x)^n}}{1-x} \quad = \text{ z/(1-x), say} \quad \text{for } n = 3 \\
(iii) \quad \frac{1}{3(1-x)^3} \quad \text{for } n = 5 \\
(iv) \quad \frac{1}{15(1-x)^5} \quad \text{for } n = 7 \\
\end{array} \]
for \( n = 9 \)

\[
\frac{\Gamma(a)}{105(a-x)} \left[ \frac{\Gamma(b)}{\Gamma(b-a)} \right] \left( 1 + o\left( x^{-1}\right) \right)
\]

for large \( x \)

Notes to Appendix E

Equations (1) to (4) follow from the definition of hypergeometric function. For (5) to (7) see p.46 of Sneddon (1961). For (8) see p.295 of Erdelyi (1954b). For (9) and (10) see p.30 and p.31 of Slater (1960). Equations (11) and (12) follow from (9) and (10). For (13) to (19) see .42 of Sneddon (1961). For (20) see p.370 of Erdelyi (1954a). Equations (21) to (24) follow from the definition of hypergeometric functions. For (25) see p.38 of Sneddon (1961). Equation (26) follows from the use of (A-16), (A-17) and (A-18). Equation (27) follows from (26) since the first term in the right and side of (26) is an odd function and the second term an even function of \( x \). For (28) see p.24 of Sneddon (1961).
For (29) and (30) see p. 47 and p. 48 of Sneddon (1961).

N.B. The term $B(1, m)$ in equation (26) of this Appendix is wrongly printed as $3(1, m)$ in Sneddon (1961). This is evident because (26) should become the definition for $B(1, m)$ for $k = 0$.

For (31) see p. 46 of Sneddon (1961). For (32) to (34) see p. 19 of Slater (1960).

For (35) see p. 34, for (36) and (37) see p. 43 and for (38) see p. 44 of Sneddon (1961). For (39) see p. 71 of Rainville (1960). To obtain (40), use (36), (13) and (17). To obtain (41) use (31), (13) and the properties of the hypergeometric function. To obtain (42) use (37), (13) and the properties of the hypergeometric function. To obtain (43) use (31), (5) and the properties of the hypergeometric function. To obtain (44) use (31), (5) and (7). To obtain 45(i) use (11); to obtain 45(ii) use (11) and (12); to obtain 45(iii) use (4) and 45(i) and 45(ii); and to obtain 45(iv) and 45(v) use (32) and (45(i) 45(ii) and 45(iii). To obtain 46(i) use (29) and (44); to obtain 46(ii), 46(iii) and 46(iv) use 46(i) and (32).

To obtain 47(i) and 47(ii) use (35), (11) and (17); and to obtain 47(iii), 47(iv) and 47(v) use 47(i), 47(ii) and (39). For (42) see p. 50 of Slater (1960).
APPENDIX - Y

THE PRINCIPLE OF VARIABLE TRANSFORMATION

Let $P_i(x_i)$ be the PDF's of the random variables $x_i$ ($i = 1, 2$). Let the joint PDF of $x_1$ and $x_2$, namely $P_x(x_1, x_2)$ be defined over some region $R_x$ of the $(x_1, x_2)$ plane. Let $y_1$ and $y_2$ be single valued functions of $x_1$ and $x_2$ such that the $x$'s can be recovered uniquely if the $y$'s are given. Then the joint PDF of $y_1$ and $y_2$, namely $P_y(y_1, y_2)$, defined in the region $R_y$ of the $(y_1, y_2)$ plane will be given by (Wadsworth and Bryan, 1960)

$$P_y(y_1, y_2) = J P_x(x_1(y_1, y_2), x_2(y_1, y_2))$$

(1)

where $x_i(y_1, y_2)$ represents the functional expression of the $x$'s in terms of the $y$'s and $J$ is the Jacobian of the $x$-system with respect to the $y$-system. That is

$$J = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}$$

(2)
The region $R_y$ corresponding to $R_x$ could be obtained from the range of variations of $x_1$ and $x_2$ and the functional relationship of the $x$'s and $y$'s. The marginal (or absolute) PDF's of $y_1$ and $y_2$ can be obtained from (1) by the usual method. It may be noted here that if $x_1$ and $x_2$ are independent random variables, $P_x(x_1, x_2) = P_1(x_1)P_2(x_2)$ so that (1) can be conveniently written as

$$P_y(y_1, y_2) = |J| P_1 x_1(y_1, y_2) P_2 x_2(y_1, y_2)$$

(3)

We have the following useful deduction for the one-dimensional case: If $y$ is a monotonic function of $x$ such that the $x$ can be uniquely recovered when $y$ is given, then

$$P_y(y) = P_x(x(y))$$

(4)

Two Useful Examples

If $x_1$ and $x_2$ are two independent random variables such that $P_i(x_i)$ is defined in the region $0 < x_i < 1, i = 1, 2$ and if $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ then, since $|J| = 1/2$, the joint PDF of $y_1$ and $y_2$ will
be given by (see (3) above)

$$P_y(y_1, y_2) = \begin{cases} \frac{1}{2} \cdot \frac{y_1 + y_2}{R} \cdot \frac{y_1 - y_2}{y_2} & \text{in } A_1 \\ 0 & \text{elsewhere} \end{cases} \quad (5)$$

where $A_1$ is the area bounded by the lines $y_1 + y_2 = 0$
and $y_1 - y_2 = 0$ in the $(y_1, y_2)$ plane (see Figure (10-1)).

For the same conditions on $x_1$ and $x_2$, if

$$s_1 = x_1 x_2 \text{ and } s_2 = x_1 / x_2 \text{ then, since } J = (2s_2)^{-1},$$

the joint PDF of $s_1$ and $s_2$ will be given by (see (3) above)

$$P_z(s_1, s_2) = \begin{cases} \frac{1}{2} \cdot \frac{s_1 + s_2}{R} \cdot \frac{s_1 - s_2}{s_2} & \text{in } A_2 \\ 0 & \text{elsewhere} \end{cases} \quad (6)$$

where $A_2$ is the area bounded by the lines $s_1 = 0$
and $s_2 = 0$ in the $(s_1, s_2)$ plane (see Figure (10-2)).
APPENDIX C

DERIVATION OF THE PDF OF THE NORMALISED STRUCTURE AMPLITUDE FOR A CUBIC SYMMETRIC CRYSTAL WITH FOUR ATOMS IN THE UNIT CELL

We consider a centrosymmetric crystal containing two independent pairs of atoms of the same type in the unit cell. The structure factor equation for a reflection \( H \) can be written as

\[
P(H) = 2f \cos \theta_1 + \cos \theta_2 \quad (1)
\]

where \( \theta_j = 2\pi i j \), \( j = 1, 2 \). Since

\[
\langle |F|^2 \rangle = \frac{1}{2f}
\]

the normalised structure factor will be given by

\[
y = \frac{P(H)}{2f} = \cos \theta_1 + \cos \theta_2
\]

\[
= x_1 + x_2, \text{ say } -1 \cdot x_1 \cdot 1
\]

\[
\quad -2 \cdot y \cdot 2 \quad (2)
\]

Since \( \theta_1 \) is uniformly distributed, the PDF of \( x_1 \) will be given by

\[
p(x_1) = \frac{1}{2} \quad (3)
\]
To obtain the PDF of \( y \) we use the following theorem from the theory of probability. If \( x_1 \) and \( x_2 \) are independent random variables with \( P_1(x_1) \) and \( P_2(x_2) \) as their respective PDF's and if \( x_3 = x_1 + x_2 \) and if \( P_3(x_3) \) is the PDF of \( x_3 \), then the Fourier transform of \( P_3(x_3) \) is the product of the Fourier transforms of \( P_1(x_1) \) and \( P_2(x_2) \). That is,

\[
P_e \left[ P_3(x_3) \right] = P_e \left[ P_1(x_1) \right] \cdot P_e \left[ P_2(x_2) \right]
\]

(4)

where \( P_e \) is to denote the exponential Fourier transform.

From (D-27) we obtain

\[
P_e \left[ P_2(x_2) \right] = \mathcal{F}_{\mathbb{R}} \left[ P_2(x_2) \right]
\]

(5)

where we have used the property that \( P_1(x_1) \) is an even function of \( x_1 \). In (5), \( x \) represents the variable in the Fourier space. Let \( P(y) \) denote the PDF of \( y \). Since \( x_1 \) and \( x_2 \) are independent random variables, we obtain from equations (2) to (5) that

\[
P_e \left[ P(y) \right] = 2 \mathcal{F}_{\mathbb{R}}(R)
\]

(6)
Taking the inverse Fourier transform, (6) gives

\[ I(y) = \frac{1}{2\pi} \int \frac{e^{-iyx}}{x} \, dx = \frac{1}{\pi} \int \frac{\cos(yx)}{x} \, dx \]

(7)

where \((2\pi)^{-1}\) is the usual factor occurring in such inverse transformations. Making use of (5-28) in (7) we obtain

\[ P(y) = \frac{1}{2\pi} \int \frac{e^{iyx}}{x} \, dx = \frac{1}{\pi} \int \frac{\sin(yx)}{x} \, dx \]

where we have used (5-29). The PDF of \(y = \sqrt{y}\)

is therefore given by

\[ P(y) = (1/y) \, _2F_1(1/2, 1/2; 1; 1 - y^2/4), 0 < y < 2 \]

(8)

To prove that the PDF of \(y\) in (8) is in the normalised form we must show that

\[ I = \int_0^2 \frac{\sin(yx)}{x} \, dx = 1 \]

(9)
Making the substitution \(1 - y^2/4 = x\) in (9) we obtain

\[
I = \frac{1}{\pi} \int_{0}^{1} (1-x)^{-\frac{1}{2}} \, _2F_1(1/2, 1/2; 1/1; x) \, dx
\]

\[
= \frac{1}{\pi} \mathbf{B}(1, 1/2) \, _3F_2(1/2, 1/2, 1/1; 1, 3/2; 1)
\]

\[
= \frac{2}{\pi} \, _2F_1(1/2, 1/2; 3/2; 1)
\]

\[
= \frac{2}{\pi} \sin^{-1}(1) = 1
\]

as required. To obtain (10) we have used (E-29) and then (E-2) and (E-16).
EXPLANATION OF INTERMEDIATE STEPS IN THE
MATHEMATICAL SIMPLIFICATIONS

Chapter 2

(2a) To obtain (2-14) from (2-13) change
the variable of integration to \( z = y_Q^2 - t^2 \) and then
use (A-3).

(2b) To obtain (2-16b), use (7-19) in
(2-16a). To obtain (2-17b) first make the substitution
\( y_p^2 = 2/(w + 1) \) in (2-17a) and then use (9-1). To obtain
(2-16b), use (C-3) in (2-19a). To obtain (2-19b)
use (D-25) in (2-19a).

(2c) To obtain (2-20b), use (C-2) in (2-20a).
To obtain (2-21b), first make the substitution \( y = x^2/4 \) in
(2-21a) and then use (C-5). To obtain (2-22b), use
(C-4) in (2-22a). To obtain (2-23b), use (C-2) in
(2-23a).
Chapter 3

(3a) To obtain (3-10) make the substitution
\[ y_N^2 = u \] in (3-9) and then use (B-4).

(3b) To obtain (3-21), make the substitution
\[ y_N = \sigma z \sqrt{\frac{\mu}{\lambda}} \] in (3-20) and then use (D-24) and (D-5).

(3c) To obtain (3-25), carry out the integration in (3-24) first over \( u \) by using (B-24) and then interchange the order of summation and integration. Then express \( \exp(-x) \) in terms of the \( \text{I}_1 \) function by using (E-5) and then use (E-23) to evaluate the integral and then use the contraction property given in (E-2). Then use (E-25), (E-4) and (E-5) for further simplifications.

(3d) To obtain (3-29) first carry out the integration over \( v \) by using (B-5). Then express the resulting \( \text{erf}(x) \) in terms of the \( \text{I}_1 \) function by using (E-8). Then carry out the integration over \( u \) by using (E-30). Finally use (E-1) and (E-15).

(3e) To obtain (3-33) first carry out the integration over \( v \) by using (B-4) and then carry out the integration over \( u \) by using (C-2).
Chapter 4

(4a) To obtain (4-11), write the integral in (4-10) as the sum of two integrals (since the integrand is a sum of two terms) and then make the substitution \( y = v^2 \) in the first one whose integrand involves \( v \exp(-v^2) \). To evaluate the second integral whose integrand involves \( \exp(-v^2) \) use the definition of \( \text{erf}(x) \) and (D-11).

(4b) To evaluate the integral in (4-35), first express \( \text{erf}(x) \) in terms of \( y \) function by using (6-7) and then carry out the integrations by using (E-3). For further simplifications use (5-40), (4-13) and (4-14).

Chapter 6

(6a) To arrive at equation (6-11) first split \( \int_0^\infty f(\alpha v)dv = \int_0^\infty f(v)dv - \int_0^\infty f(v)v \), then use the definition of error function and the property that the error function is an odd function of \( x \), namely
\[
\text{erf}(sx) = s \text{erf}(x), \quad s = \pm 1 \text{ (see (5-11))},
\]
and the property \( \text{erf}(\infty) = 1 \) (see (4-9)).
(6b) To arrive at (6-14), first make the substitution \( y_p^2 = 2v \) in the integral in (6-14), then substitute \( \text{erf}(x) \) for \( \text{erf}(x) \) by using (3-7). Then carry out the integration by using (3-13).

(6c) To obtain (6-17) make the substitution \( y_p = 2 \cos \alpha \) in (6-16) and then use (8-12).

(6d) To evaluate the integral in (6-14), first make the substitution \( y_r^2 = 2v \), then express \( \text{erf}(x) \) in terms of \( \text{erf}(x) \) function by using (3-7) and finally carry out the integration by using (3-10). Then use (3-17).

Chapter 7

(7a) To evaluate the integral in equation (7-2), first make the substitution \( y_3^2 = 2 \alpha \) and then carry out the integration by using (3-6). For further simplification make use of the relation (3-26) and the Kummer's relation in (3-25).

(7b) To obtain (7-10), make use of the duplication formula given in (3-5) for the value of \( \tan \theta/(1/2) \) and then use this result in (7-9).
(7c) To arrive at (7-14a) first make the substitution \( y_P^2 = 2x \) in the integral in (7-14) and then carry out the integration by using (E-29). For further simplification use the contraction property in (E-2). To arrive at (7-14b) use the Kummer's relation (E-25) in (7-14a).

(7d) To evaluate the integral in (7-15a), first make the substitution \( y_1^2 = 2x \) in the integral and then carry out the integration by using (E-30). For further simplification use (7-13) and the relation \( \sigma_1^2 + \sigma_2^2 = 1 \).

(7e) To obtain (7-19), first make the substitution \( y_N^2 = \sigma_1^{-2} x \) in (7-17) and then carry out the integration by using (8-9). For further simplification make use of (A-16) and then the Kummer's relation (E-25).

(7f) To obtain (7-23), first make the substitution \( y_P^2 = 2x \) in (7-22) and then carry out the integration using (5-23).

(7g) To obtain (7-25), first make the substitution \( y_P^2 = 2x \) in (7-24) and then carry out the integration by using (5-30). To obtain (7-25b) use (E-28) in (7-25a).
(7h) To obtain (7-27), first make the substitution \( y_p^2 = x \) in (7-26) and then carry out the integration by using (2-30). For further simplification use (E-13) and the property that \( 3^{1/3} \cdot 3^{1/3} = 1 \).

**Chapter 8**

(8a) **To Prove** (7-27): Since \( g_1 \to 1 \) as \( k \to \infty \), (8-26) can be written, by using the binomial theorem and the notation \( r = (-1/2) \) as

\[
I = \lim_{k \to \infty} \left( \frac{\sum_{n=0}^{\infty} \binom{n}{r} \left( \frac{1}{k} \right)^n}{n!} \right)
\]

Splitting the summation over \( 1 \) into two summations, (1) can be written as

\[
I = \lim_{k \to \infty} \sum_{r=0}^{\infty} \left( \frac{\sum_{n=0}^{\infty} \binom{n}{r} \left( \frac{1}{k} \right)^n}{n!} \right)
\]

since the second summation which involves powers of \( 1/k \) would vanish in the limiting process. Equation (1) can be written as

\[
I = \lim_{k \to \infty} \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{\infty} \binom{n}{r} \left( \frac{1}{k} \right)^n}{n!}
\]

\[
= \lim_{k \to \infty} \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{\infty} \left( \frac{1}{k} \right)^n}{n!}
\]

\[
= \lim_{k \to \infty} \sum_{r=0}^{\infty} \frac{1}{3 \cdot 5 \cdot 7 \cdots n}
\]

(3)
where we have used \((I-26)\) and the value \((-1/2)\) for \(r\).

Equation (3) can be written as

\[
I = \frac{(2n-1)!}{2 \cdot 4 \cdot 6 \cdots (2n-2)} = \frac{(2n-1)!}{2^{n-1} n!} = \frac{1}{2^{n-1}} \cdot \frac{n!}{n!} = \frac{1}{2^{n-1}}
\]

as required. To obtain (4) we have used \((1-5)\).

**Chapter 10**

(10a) To simplify the integral in \((10-27a)\) make use of the substitution \(x_0 = x_0^*\). To arrive at \((10-28b)\) and \((10-28c)\) use the substitution \(x_0 = yx_0^*\) in \((10-27b)\) and \((10-27c)\) respectively. The integration of \((10-27d)\) is straightforward.

(10b) To arrive at \((10-30a)\), use \((6-3)\) in \((10-29a)\). To arrive at \((10-30b)\) and \((10-30c)\), use the substitution \(x_0 = x_0^* + 2y\) in \((10-29b)\) and \((10-29c)\) respectively. The integration of \((10-29d)\) is straightforward.

(10c) To arrive at \((10-37a)\) and \((10-37d)\), use \((6-3)\) in \((10-36a)\) and \((10-36d)\) respectively. To arrive at \((10-37b)\) and \((10-37c)\), use the substitutions \(x_p x_q = 4y\) and \(2x_p x_q = y^2\) in \((10-36b)\) and \((10-36c)\) respectively.
(10d) The integration of (10-3a) is straightforward. To arrive at (10-38b), use the substitution \( x_p = 4x_q y \) in (10-38b). To arrive at (10-39c), first use the substitution \( 2x_p x_q = y^2 \) in (10-39c) and then carry out the integration by using (C-6) and for further simplification use (B-14). To arrive at (10-39d), use (C-2) in (10-3-d).

(10e) Making use of (B-10), equation (48a) can be written as

\[
\langle x_d \rangle = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \left( 1 + \frac{1}{x} \right) dx
\]

\[
= 2\sqrt{\frac{2}{\pi}} \ln \left( \frac{\sqrt{2}}{\sqrt{\pi}} \right) + \frac{2}{\sqrt{\pi}}
\]

A study of equation (10-2a) shows that the second term in equation (1) is nothing but \( x_s \). For the one atom case and hence its value is \( 2/\sqrt{\pi} \). We can therefore write (1) as

\[
\langle x_d \rangle = \frac{2\sqrt{2}}{\sqrt{\pi}} \ln \left( \frac{\sqrt{2}}{\sqrt{\pi}} \right) + \frac{2}{\sqrt{\pi}}
\]

\[
= \left( 2\sqrt{\frac{2}{\pi}} \right) (-2 - 1)
\]