CHAPTER 5
STATISTICALLY CONVERGENT SEQUENCES OF INTERVAL NUMBERS

Statistically Convergent Sequences of Interval Numbers

In this chapter we have introduced some statistically convergent sequence spaces of interval numbers and some statistically convergent difference sequence spaces of interval numbers. Also we have introduced some statistically convergent sequence spaces of two dimensional interval vectors. We have also studied some of their algebraic and topological properties related to these spaces.

5.1 Some Statistically Convergent Sequence Spaces of Interval Numbers

The concept of statistical convergence of interval numbers has been introduced by Esi [7].

Definition 5.1.1: [7] A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent statistically to the interval number $\overline{x}_0$ if for every $\epsilon > 0$, $lim_n \frac{1}{n}|\{k \leq n : d(\overline{x}_k, \overline{x}_0) \geq \epsilon\}| = 0$, denote it by writing $stat - lim_k \overline{x}_k = \overline{x}_0$.

In this section we introduced some statistically null and statistically convergent sequence spaces of interval numbers. We studied different algebraic and topological properties of these spaces and also investigated inclusion relations related to these spaces.

For any sequence of interval numbers $\overline{x} = (\overline{x}_k) \in \omega^i$, the statistically null and statistically convergent sequence spaces of interval numbers $c_0^{S(i)}$ and $c^{S(i)}$ respectively are as follows:

$c_0^{S(i)} = \{\overline{x} = (\overline{x}_k) \in \omega^i : lim_{n\to\infty}\frac{1}{n}|\{k \le n : d(\overline{x}_k, \theta) \ge \varepsilon\}| = 0\}$, *where* $\theta = [0,\ 0]$.

$c^{S(i)} = \{\overline{x} = (\overline{x}_k) \in \omega^i : lim_{n\to\infty}\frac{1}{n}|\{k \le n : d(\overline{x}_k, \overline{x}_0) \ge \varepsilon\}| = 0\}$.

Theorem 5.1.1: $(c_0^{S(i)} \bigcap \ell_\infty^i,\ \overline{d})$, $(c^{S(i)} \bigcap \ell_\infty^i,\ \overline{d})$ are complete metric spaces with the metric defined by

$$\overline{d}((\overline{x}_k), (\overline{y}_k)) = sup_k max\{|x_{lk} - y_{lk}|, |x_{rk} - y_{rk}|\}.$$

Proof: It can be established using standard technique.

Theorem 5.1.2: $c_0^{S(i)} \bigcap \ell_\infty^i$ and $c^{S(i)} \bigcap \ell_\infty^i$ are normed interval spaces with the norm

$$\|\overline{x}\| = sup_k\, max\ \{|x_{lk}|, |x_{rk}|\}.$$

Proof: Let $\mu^i = c_0^{S(i)} \bigcap \ell_\infty^i$ (or $c^{S(i)} \bigcap \ell_\infty^i$) and $\overline{x}$, $\overline{y} \in \mu^i$.

N_1. $\|\overline{x}\|_{\mu^i} = sup_k\, max\ \{|x_{lk}|, |x_{rk}|\}$,

then we have $\|\overline{x}\|_{\mu^i} > 0$, $\forall\ \overline{x} \in \mu^i$ - $\{\theta\}$.

N_2. $\|\overline{x}\|_{\mu^i} = 0 \Leftrightarrow sup_k\, max\{|x_{lk}|, |x_{rk}|\} = 0 \Leftrightarrow \overline{x} = \theta$.

N_3. $\|\overline{x}+\overline{y}\|_{\mu^i} = sup_k\, max\ \{|x_{lk}+y_{lk}|, |x_{rk}+y_{rk}|\}$

$\leq sup_k\, max\ \{|x_{lk}|+|y_{lk}|, |x_{rk}|+|y_{rk}|\}$

$= sup_k\, max\ \{(|x_{lk}|, |x_{rk}|)\} + (|y_{lk}|, |y_{rk}|)$

$\leq sup_k\, max\ \{(|x_{lk}|, |x_{rk}|)\}\ +\ sup_k max\ \{(|y_{lk}|, |y_{rk}|)\}$

$= \|\overline{x}\|_{\mu^i}\ +\ \|\overline{y}\|_{\mu^i}$.

N_4. $\|\alpha\overline{x}\|_{\mu^i} = sup_k\, max\ \{|\alpha x_{lk}|, |\alpha x_{rk}|\}$

$= |\alpha|\ sup_k\, max\ \{|x_{lk}|, |x_{rk}|\}$

$= |\alpha|\|\overline{x}\|_{\mu^i}$.

Hence, $\|\overline{x}\|_{\mu^i}$ is a norm on μ^i .

Theorem 5.1.3: The spaces $c_0^{S(i)} \bigcap \ell_\infty^i$ and $c^{S(i)} \bigcap \ell_\infty^i$ are solid.

Proof: We consider only $c_0^{S(i)} \bigcap \ell_\infty^i$. Now, let $\|\overline{y}_k\| \leq \|\overline{x}_k\|$, for all (k $\in$ N) and for some $\overline{x} \in c_0^{S(i)} \bigcap \ell_\infty^i$. Then we have, $\overline{d}(\overline{y}_k, \theta) \leq \overline{d}(\overline{x}_k, \theta)$, that is $\{|y_{lk}-0|, |y_{rk}-0|\} \leq \{|x_{lk}-0|, |x_{rk}-0|\}$. Thus we have $y_{lk} \leq x_{lk}$ and $y_{rk} \leq x_{rk}$, i.e., $\overline{y} \leq \overline{x}$.

So, clearly $\overline{y} \in c_0^{S(i)} \bigcap \ell_\infty^i$. Hence $c_0^{S(i)} \bigcap \ell_\infty^i$ is solid.

Theorem 5.1.4: The spaces $c_0^{S(i)} \bigcap \ell_\infty^i$ and $c^{S(i)} \bigcap \ell_\infty^i$ are sequence algebra.

Proof: We prove that $c_0^{S(i)} \bigcap \ell_\infty^i$ is a sequence algebra.

Let $(\overline{x}_k)$, $(\overline{y}_k) \in c_0^{S(i)} \bigcap \ell_\infty^i$.

Then, $stat - lim_k\ \overline{x}_k = \theta$ and $stat - lim_k\ \overline{y}_k = \theta$

Then we have, $stat - lim_k\ (\overline{x}_k\ \overline{y}_k\ ) = \theta$.

Thus $(\overline{x}_k\ \overline{y}_k) \in c_0^{S(i)} \bigcap \ell_\infty^i$. Hence $c_0^{S(i)} \bigcap \ell_\infty^i$ is a sequence algebra.

For the space $c^{S(i)} \bigcap \ell_\infty^i$, the result can be proved similarly.

Theorem 5.1.5: The inclusions $c_0^{S(i)} \bigcap \ell_\infty^i \subset c^{S(i)} \bigcap \ell_\infty^i \subset \ell_\infty^i$ hold and are strict.

Proof: Clearly $c_0^{S(i)} \subset c^{S(i)}$. So, $c_0^{S(i)} \bigcap \ell_\infty^i \subset c^{S(i)} \bigcap \ell_\infty^i$.

Now to prove $c^{S(i)} \bigcap \ell_\infty^i \subset \ell_\infty^i$ we are taking an example.

Example 5.1.1: Let, $\overline{x} = (\overline{x}_k) = \begin{cases} [-3,\ 7]\,, & k\,is\,odd \\ [3,\ -7]\,, & otherwise; \end{cases}$

It has been shown that the above sequence of interval numbers is bounded but not statistically convergent.

5.2 Some Statistically Convergent Difference Sequence Spaces of Interval Numbers

In this section we introduced some statistically convergent difference sequence spaces of interval numbers and studied some of their algebraic and topological properties and also investigated some inclusion relations related to these spaces.

For any sequence of interval numbers $\overline{x} = (\overline{x}_k) \in \omega^i$ the difference sequence of interval numbers are denoted by $\triangle\overline{x} = (\triangle\overline{x}_k) = (\overline{x}_k - \overline{x}_{k+1})$ and the difference sequence spaces of statistically null, statistically convergent and bounded sequences of interval numbers are denoted by $c_0^{S(i)}(\triangle)$, $c^{S(i)}(\triangle)$ and $\ell_\infty^i(\triangle)$ respectively and defined as follows:

$$c_0^{S(i)}(\triangle) = \{\overline{x} = (\overline{x}_k) \in \omega^i : stat - lim_k \triangle\overline{x}_k = \theta\}, \text{where } \theta = [0, 0].$$

$$c^{S(i)}(\triangle) = \{\overline{x} = (\overline{x}_k) \in \omega^i : stat - lim_k \triangle\overline{x}_k = \overline{x}_0\}.$$

$$\ell_\infty^i(\triangle) = \{\overline{x} = (\overline{x}_k) \in \omega^i : sup_k \, max\left(|\triangle x_{lk}|, |\triangle x_{rk}|\right) < \infty\}.$$

and the sequence spaces $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are defined as follows:

$$m_0^{S(i)}(\triangle) = c_0^{S(i)}(\triangle) \cap \ell_\infty^i(\triangle)$$

and $m^{S(i)}(\triangle) = c^{S(i)}(\triangle) \cap \ell_\infty^i(\triangle)$.

Theorem 5.2.1: $\left(m_0^{S(i)}(\Delta), \overline{d}_\Delta\right)$, $\left(m^{S(i)}(\Delta), \overline{d}_\Delta\right)$ are complete metric spaces with the metric defined by

$$\overline{d}_{\Delta}\left((\overline{x}_k),\,(\overline{y}_k)\right) = sup_k\, max\left\{\left|\triangle x_{lk}\,-\,\Delta y_{lk}\right|,\,\left|\triangle x_{rk}\,-\,\Delta y_{rk}\right|\right\}.$$

Proof: It can be established using standard technique.

Theorem 5.2.2: $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are normed interval spaces with the norm

$$\left\|\overline{x}\right\|_{\Delta} = max\left(\left|x_{l1}\right|,\,\left|x_{r1}\right|\right) + sup_k\, max\left\{\left|\triangle x_{lk}\right|,\,\left|\triangle x_{rk}\right|\right\}.$$

Proof: Let $\lambda^i(\Delta) = m_0^{S(i)}(\triangle)$ $(or\, m^{S(i)}(\triangle))$ and $\overline{x},\,\overline{y} \in \lambda^i$

$N_{1.}$ Since $\left\|\overline{x}\right\|_{\lambda^i} = max\left(\left|x_{l1}\right|,\,\left|x_{r1}\right|\right) + sup_k\, max\left\{\left|\triangle x_{lk}\right|,\,\left|\triangle x_{rk}\right|\right\}$

We easily see that $\|\,\overline{x}\,\|_{\lambda^i} > 0,\,\forall\,\overline{x} \in \lambda^i(\Delta)$ - $\{\theta\}$.

N_2. $\|\,\overline{x}\,\|_{\lambda^i} = 0 \Leftrightarrow max\left(\left|x_{l1}\right|,\,\left|x_{r1}\right|\right) + sup_k\, max\left\{\left|\,\Delta x_{lk}\,\right|,\,\left|\,\Delta x_{rk}\,\right|\right\} = 0 \Leftrightarrow \overline{x} = \theta$, where $\theta = [0,\,0]$.

N_3.$\|\,\overline{x}+\overline{y}\,\|_{\lambda^i} =$

$$max\left(\left|x_{l1}\,+\,y_{l1}\right|,\,\left|x_{r1}\,+\,y_{r1}\right|\right) + sup_k\, max\left\{\left|\,\Delta\left(x_{lk}\,+\,y_{lk}\right)\,\right|,\,\left|\,\Delta\left(x_{rk}\,+\,y_{rk}\right)\,\right|\right\}$$

$$\leq max\left(\left|x_{l1}\right|\,+\,\left|y_{l1}\right|,\,\left|x_{r1}\right|\,+\,\left|y_{r1}\right|\right)$$

$$+\, sup_k\, max\left\{\left|\,\Delta x_{lk}\,\right| + \left|\,\Delta y_{lk}\,\right|,\,\left|\,\Delta x_{rk}\,\right| + \left|\,\Delta y_{rk}\,\right|\right\}$$

$$\leq max\left(\left|x_{l1}\right|,\,\left|x_{r1}\right|\right) + sup_k\, max\left\{\left|\,\Delta x_{lk}\,\right|,\,\left|\,\Delta x_{rk}\,\right|\right\}$$

$$+ max\left(\left|y_{l1}\right|,\,\left|y_{r1}\right|\right) + sup_k\, max\left\{\left|\,\Delta y_{lk}\,\right|,\,\left|\,\Delta y_{rk}\,\right|\right\}$$

$= \parallel \overline{x} \parallel_{\lambda^i} + \parallel \overline{y} \parallel_{\lambda^i}$

N_4. $\parallel \alpha\overline{x} \parallel_{\lambda^i} = max\left(|\alpha x_{l1}|, |\alpha x_{r1}|\right) + sup_k\, max\left\{| \alpha\Delta x_{lk} |, | \alpha\Delta x_{rk} |\right\}$

$= max\left(|\alpha| |x_{l1}|, |\alpha| |x_{r1}|\right) + sup_k\, max\left\{|\alpha| |\Delta x_{lk}|, |\alpha| |\Delta x_{rk}|\right\}$

$= |\alpha|\, max\left(|x_{l1}|, |x_{r1}|\right) + | \alpha |\, sup_k\, max\left\{| \Delta x_{lk} |, | \Delta x_{rk} |\right\}$

$= | \alpha | \parallel \overline{x} \parallel_{\lambda^i}$

Hence, $\parallel \overline{x} \parallel_{\lambda^i}$ is a norm on λ^i.

Theorem 5.2.3: The spaces $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are solid.

Proof: We consider only $m_0^{S(i)}(\triangle)$. For $m^{S(i)}(\triangle)$ it can be established following similar technique.

Now, let $\|\overline{y}_k\| \leq \|\overline{x}_k\|$, for all $k \in N$ and for some $\overline{x} \in m_0^{S(i)}(\triangle)$. Then we have, $\overline{d}_\Delta\ (\overline{y}_k, \theta) \leq \overline{d}_\Delta\ (\overline{x}_k, \theta)$, that is $\{| \Delta y_{lk} - 0 |, | \Delta y_{rk} - 0 |\} \leq \{| \Delta x_{lk} - 0 |, | \Delta x_{rk} - 0 |\}$.

Thus we have $\Delta y_{lk} \leq \Delta x_{lk}$ and $\Delta y_{rk} \leq \Delta x_{rk}$, i.e., $\Delta\overline{y} \leq \Delta\overline{x}$.

So, clearly $\overline{y} \in m_0^{S(i)}(\triangle)$. Hence $m_0^{S(i)}(\triangle)$ is solid.

Result 5.2.1: The spaces $c^{S(i)}(\triangle)$ and $c_0^{S(i)}(\triangle)$ are not solid.

Proof: We consider only $c^{S(i)}(\triangle)$.

Example 5.2.1: Let $\overline{x} = (\overline{x}_k) \in c^{S(i)}(\triangle)$, where $\overline{x}_k = [k,\ k+1]$ and $k \in N$ and let $\alpha_k = \begin{cases} [1,1], & \text{for } k = 2n, \; and\, n \in N \\ 0, \text{otherwise} \end{cases}$

Then, $(\alpha_k \overline{x}_k) \notin c^{S(i)}(\triangle)$ and so $c^{S(i)}(\triangle)$ is not solid.

For the space $c_0^{S(i)}(\triangle)$ the result can be proved similarly.

Theorem 5.2.4: The spaces $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are sequence algebra.

Proof: We prove that $m_0^{S(i)}(\triangle)$ is a sequence algebra.

Let $(\overline{x}_k)$, $(\overline{y}_k) \in m_0^{S(i)}(\triangle)$.

Then, $stat - lim_k \Delta \overline{x}_k = \theta$ and $stat - lim_k \Delta \overline{y}_k = \theta$, where $\theta = [0, 0]$.

Then we have, $stat - lim_k (\Delta \overline{x}_k \Delta \overline{y}_k) = \theta$.

Thus $(\overline{x}_k \overline{y}_k) \in m_0^{S(i)}(\triangle)$. Hence $m_0^{S(i)}(\triangle)$ is a sequence algebra.

For the space $m^{S(i)}(\triangle)$, the result can be proved similarly.

Result 5.2.2: The spaces $c^{S(i)}(\triangle)$ and $c_0^{S(i)}(\triangle)$ are not convergence free.

Proof: Here, we give a counter example.

Example 5.2.2: Let, $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be two sequences of interval numbers.

Now let, $\overline{x}_k = \left[\frac{1}{k}, \; \frac{(-1)^k}{k}\right]$

and $\overline{y}_k = \left[(-1)^k,\, 2 + \frac{1}{k}\right]$ for all $k \in N$.

Then $(\overline{x}_k) \in c^{S(i)}(\triangle)$ but $(\overline{y}_k) \notin c^{S(i)}(\triangle)$.

Hence the spaces $c^{S(i)}(\triangle)$ is not convergence free in general.

Similarly, it can be shown that the space $c_0^{S(i)}(\triangle)$ is not convergence free.

Theorem 5.2.5: The inclusion $c_0^{S(i)}(\triangle) \subset c^{S(i)}(\triangle)$ holds and is strict.

Proof: If we take $\overline{x} \in c_0^{S(i)}(\triangle)$ then clearly $\overline{x} \in c^{S(i)}(\triangle)$. The inclusion is strict follows from the following example.

Example 5.2.3: Consider, the interval sequence $\overline{x} = (\overline{x}_k)$ is defined as $\overline{x}_k = [k,\, k+2]$, where $k \in N$.

Then, clearly $(\overline{x}_k) \in c^{S(i)}(\triangle)$ but $(\overline{x}_k) \notin c_0^{S(i)}(\triangle)$.

5.3 Some Statistically Convergent Sequence Spaces of Two Dimensional Interval Vectors

In this section we introduced the concept of statistical convergence of two dimensional interval vectors. We introduced some statistically null and statistically convergent sequence spaces of two dimensional interval vectors and studied some algebraic and topological properties of these spaces and also investigated some inclusion relations related to these spaces..

Throughout the section ω^{i_2}, $c_0^{i_2}$, c^{i_2} and $l_\infty^{i_2}$ denote the spaces of all, null, convergent and bounded sequence spaces of two dimensional interval vectors.

Definition 5.3.1: A sequence $\widetilde{x} = (\widetilde{x}_k) = ([x_{1lk}, x_{1rk}], [x_{2lk}, x_{2rk}])$ of two dimensional interval vectors is said to be convergent statistically to $\widetilde{x}_0 = ([x_{1l0}, x_{1r0}], [x_{2l0}, x_{2r0}])$ if for every $\varepsilon > 0$

$$\delta\left(\{n \in N : d(\tilde{x}_n, \tilde{x}_0) \geq \varepsilon\}\right) = 0$$

and we denote it by writing $stat{-}lim_{k\to\infty}\widetilde{x}_k = \widetilde{x}_0$. Thus $stat{-}lim_{k\to\infty}\widetilde{x}_k = \widetilde{x}_0 \Longleftrightarrow stat{-}lim_{k\to\infty}x_{1lk} = x_{1l0}$, $stat{-}lim_{k\to\infty}x_{1rk} = x_{1r0}$ and $stat{-}lim_{k\to\infty}x_{2lk} = x_{2l0}$, $stat{-}lim_{k\to\infty}x_{2rk} = x_{2r0}$.

Definition 5.3.2: Let λ^{i_2} be a sequence space of two dimensional interval vectors. Then λ^{i_2} is said to be sequence algebra if $(\tilde{x}_k\tilde{y}_k) \in \lambda^{i_2}$ whenever $\tilde{x} = (\tilde{x}_k) \in \lambda^{i_2}$, $\tilde{y} = (\tilde{y}_k) \in \lambda^{i_2}$ $(k \in N)$.

For any sequence of two dimensional interval vectors $\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2}$, the sequence spaces of two dimensional interval vectors $\widetilde{c}_0^{S(i_2)}$ and $\widetilde{c}^{S(i_2)}$ are defined as follows:

$\widetilde{c}_0^{S(i_2)} = \left\{\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2} : stat - lim_{k\to\infty}\widetilde{x}_k = \widetilde{\theta}\right\}$, where $\widetilde{\theta} = ([0,0]\,,\,[0,0])$.

$\widetilde{c}^{S(i_2)} = \{\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2} : stat - lim_{k\to\infty}\widetilde{x}_k = \widetilde{x}_0\}$.

We also denote $\widetilde{m}_0^{S(i_2)} = \widetilde{c}_0^{S(i_2)} \cap \ell_\infty^{i_2}$.

$$\widetilde{m}^{S(i_2)} = \widetilde{c}^{S(i_2)} \cap \ell_\infty^{i_2}.$$

Theorem 5.3.1: $\left(\widetilde{m}_0^{S(i_2)},\, \widetilde{d}\right), \left(\widetilde{m}^{S(i_2)},\, \widetilde{d}\right)$ are complete metric spaces with the metric defined by

$$\widetilde{d}((\widetilde{x}_k),\,(\widetilde{y}_k)) = sup_k\, max\,\{|x_{1lk} - y_{1lk}|\,,\,|x_{1rk} - y_{1rk}|\,,\,|x_{2lk} - y_{2lk}|\,,\,|x_{2rk} - y_{2rk}|\}.$$

where $\widetilde{x} = (\widetilde{x}_k)$, $\widetilde{y} = (\widetilde{y}_k) \in \widetilde{m}_0^{S(i_2)}$ or $\widetilde{m}^{S(i_2)}$.

Proof:. It can be established using standard technique.

Theorem 5.3.2: $\widetilde{m}_0^{S(i_2)}$ and $\widetilde{m}^{S(i_2)}$ are normed interval spaces with the norm

$$\|\widetilde{x}\| = sup_k\, max\,\{|x_{1lk}|\,,\,|x_{1rk}|\,,\,|x_{2lk}|\,,\,|x_{2rk}|\}.$$

Proof: Let $\mu^{i_2} = \widetilde{m}_0^{S(i_2)}\left(or\ \widetilde{m}^{S(i_2)}\right)$ and $\widetilde{x}, \widetilde{y} \in \mu^{i_2}$

N_1. Since $\|\,\widetilde{x}\,\|_{\mu^{i_2}} = sup_k\, max\,\{|x_{1lk}|\,,\,|x_{1rk}|\,,\,|x_{2lk}|\,,\,|x_{2rk}|\}$, then we have $\|\,\widetilde{x}\,\|_{\mu^{i_2}} > 0$, $\forall \widetilde{x} \in \|\,\widetilde{x}\,\|_{\mu^{i_2}} - \left\{\widetilde{\theta}\right\}$.

N_2. $\|\,\widetilde{x}\,\|_{\mu^{i_2}} = 0 \Leftrightarrow sup_k\, max\,\{|x_{1lk}|\,,\,|x_{1rk}|\,,\,|x_{2lk}|\,,\,|x_{2rk}|\} = 0 \Leftrightarrow \widetilde{x} = \widetilde{\theta}$.

N_3. $\parallel \widetilde{x} + \widetilde{y} \parallel_{\mu^{i_2}} =$

$$sup_k \, max \{| \, x_{1lk} + y_{1lk} \, |, | \, x_{1rk} + y_{1rk} \, |, |x_{2lk} + y_{2lk}| \, , |x_{2rk} + y_{2rk}|\}$$

$$\leq sup_k max \{| \, x_{1lk} \, | + | \, y_{1lk} \, |, | \, x_{1rk} \, | + | \, y_{1rk} \, |, |x_{2lk}| + |y_{2lk}| \, , |x_{2rk}| + |y_{2rk}|\}$$

$$\leq sup_k \, max \{| \, x_{1lk} \, |, | \, x_{1rk} \, |, |x_{2lk}| \, , |x_{2rk}|\} +$$

$$sup_k \, max \{| \, y_{1lk} \, |, | \, y_{1rk} \, |, |y_{2lk}| \, , |y_{2rk}|\}$$

$$= \parallel \widetilde{x} \parallel + \parallel \widetilde{y} \parallel$$

N_4. $\parallel \alpha\widetilde{x} \parallel_{\mu^{i_2}} = sup_k \, max \{| \, \alpha x_{1lk} \, |, | \, \alpha x_{1rk} \, |, |\alpha x_{2lk}| \, , |\alpha x_{2rk}|\}$

$$= |\alpha| \, sup_k \, max \{| \, x_{1lk} \, |, | \, x_{1rk} \, |, |x_{2lk}| \, , |x_{2rk}|\}$$

$$= | \, \alpha \, || \parallel \widetilde{x} \parallel_{\mu^{i_2}}$$

Hence, $\parallel \widetilde{x} \parallel_{\mu^{i_2}}$ is a norm on μ^{i_2} .

Theorem 5.3.3: The spaces $\widetilde{m}_0^{S(i_2)}$ and $\widetilde{m}^{S(i_2)}$ are sequence algebra.

Proof: We prove that $\widetilde{m}_0^{S(i_2)}$ is a sequence algebra.

Let $(\widetilde{x}_k)$, $(\widetilde{y}_k) \in \widetilde{m}_0^{S(i_2)}$.

Then, $stat - lim_k \widetilde{x}_k = \widetilde{\theta}$ and $stat - lim_k \widetilde{y}_k = \widetilde{\theta}$.

Then we have, $stat - lim_k \, (\widetilde{x}_k \widetilde{y}_k) = \theta$.

Thus $(\widetilde{x}_k\widetilde{y}_k) \in \widetilde{m}_0^{S(i_2)}$. Hence $\widetilde{m}_0^{S(i_2)}$ is a sequence algebra.

For the space $\widetilde{m}^{S(i_2)}$, the result can be proved similarly.

Theorem 5.3.4: The inclusion $c_0^{i_2} \subset \widetilde{c}_0^{S(i_2)}$ holds and it is strict.

Proof: If $\widetilde{x} = (\widetilde{x}_k) \in c_0^{i_2}$ then it is obvious that $\widetilde{x} = (\widetilde{x}_k) \in \widetilde{c}_0^{S(i_2)}$. To prove the inclusion is strict we give a counter example.

Example 5.3.1:

Let, $\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2}$ where $\widetilde{x}_k = \begin{cases} \left[k,\, \frac{1}{k^2}\right],\, \left[\frac{1}{\sqrt{k}},\, \frac{1}{k^3}\right], & k = n^2,\, n \in N \\ [0,\, 0]\,,\, [0,\, 0] & \text{otherwise} \end{cases}$

Then clearly $\widetilde{x} \in \widetilde{c}_0^{S(i_2)}$ but $\widetilde{x} \notin c_0^{i_2}$.

Theorem 5.3.5: The inclusion $\widetilde{c}_0^{S(i_2)} \subset \widetilde{c}^{S(i_2)}$ holds and it is strict.

Proof: If we take $\widetilde{x} \in \widetilde{c}_0^{S(i_2)}$ then it is obvious that $\widetilde{x} \in \widetilde{c}^{S(i_2)}$. The inclusion is strict follows from the following example.

Example 5.3.2: Now let, $\widetilde{x}_k = \begin{cases} \left[k,\, \frac{1}{k}\right],\, \left[1 + \sqrt{k},\, \frac{1}{k}\right], & k = n^2,\, n \in N \\ [0,\, 0]\,,\, [0,\, 0]\,, & \text{otherwise} \end{cases}$,

which converges statistically to $([0,\, 0]\,, [1,\, 0])$. Hence $\widetilde{x} \in \widetilde{c}^{S(i_2)}$ but $\widetilde{x} \notin \widetilde{c}_0^{S(i_2)}$.

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