CHAPTER 4
SOME GENERALIZED SEQUENCE SPACES OF INTERVAL NUMBERS

Some Generalized Sequence Spaces of Interval Numbers

The concept of generalized sequence spaces of two dimensional interval vectors has been introduced by Z. Zararsiz, M. Şengönül [145]. Also different classes of generalized difference convergent sequence spaces has been studied by Z. Zararsiz, M. Şengönül [145].

In this chapter we introduce some generalized null, convergent and bounded sequence spaces of two dimensional interval vectors and studied some inclusion relations related to these spaces. Also we introduced some difference sequence spaces of interval numbers and studied some algebraic and topological properties related to these spaces. We investigate some inclusion relations related to these spaces.

4.1 Some Sequence Spaces of Generalized Two Dimensional Interval Vectors

In this section we have introduced distance on the space of all two dimensional interval vectors and seen that it is complete. Furthermore, we introduced the null, convergent and bounded sequence spaces of two dimensional interval vectors. Also some inclusion relations are studied.

A two dimensional interval vector [145] is an ordered 2-tuple of intervals, $\widetilde{x} = (\bar{x}_1, \bar{x}_2) = ([x_{1l},\, x_{1r}]\,,\ [x_{2l},\, x_{2r}])$. The absolute value (magnitude or interval norm) of $\widetilde{x}$ is defined by $|\widetilde{x}| = max\,\{|x_{1l}|\,, |x_{1r}|\,, |x_{2l}|\,, |x_{2r}|\}$. If the absolute value of each element of $\widetilde{x}$ is zero, then $\widetilde{x}$ is called zero vector and is denoted by $\widetilde{\theta} = ([0,0]\ [0,0])\,.$

Let $R(I_2)$ be the set of all two dimensional interval vectors. The scalar multiplication and addition of two vectors in $R(I_2)$ are defined as follows:

$$\alpha\widetilde{x} = (\alpha\bar{x}_1,\, \alpha\bar{x}_2) = \begin{cases} ([\alpha x_{1l,} \alpha x_{1r}]\,,\ [\alpha x_{2l,} \alpha x_{2r}])\,, & if\ \alpha\ \geq\ 0 \\ ([\alpha x_{1r,} \alpha x_{1l}]\,,\ [\alpha x_{2r,} \alpha x_{2l}])\,, & if\ \alpha\ <\ 0 \end{cases}$$

$$\widetilde{x} + \widetilde{y} = (\bar{x}_1, \bar{x}_2) + (\bar{y}_1 + \bar{y}_2)$$

$$= (\bar{x}_1 + \bar{y}_1, \bar{x}_2 + \bar{y}_2) = ([x_{1l} + y_{1l}, x_{1r} + y_{1r}]\,, [x_{2l} + y_{2l}, x_{2r} + y_{2r}])$$

The distance of two vectors in $R(I_2)$ can be defined as

$$d\,(\widetilde{x},\, \widetilde{y}) = max\,\{|x_{1l}\ -\ y_{1l}|\,,\, |x_{1r}\ - y_{1r}|\,,\, |x_{2l}\ -\ y_{2l}|\,,\, |x_{2r}\ -\ y_{2r}|\}$$

where $\widetilde{x} = (\bar{x}_1,\, \bar{x}_2)\,,\quad \widetilde{y} = (\bar{y}_1,\, \bar{y}_2) \in R(I_2)\,.$

Theorem 4.1.1: The set of all two dimensional interval vectors $R(I_2)$ forms a metric space with respect to the metric d defined above.

Proof: It can be established using standard technique.

Let us define transformation f from N to $R(I_2)$ by $k \to f\,(k) = (\widetilde{x}_k)$

$= (\bar{x}_{1k}, \bar{x}_{2k})$. Then $(\widetilde{x}_k)$ is called the sequence of two dimensional interval vectors.

Definition 4.1.1: A sequence $(\widetilde{x}_k)$ of $(R(I_2), d)$ is said to be convergent to $\widetilde{x}_0$ of $R(I_2)$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $d(\widetilde{x}_k, \widetilde{x}_0) <\varepsilon$ for all $k \geq n_0$, and we denote it by writing $lim_k \widetilde{x}_k = \widetilde{x}_0$.

Thus, $lim_k \widetilde{x}_k = \widetilde{x}_0 \Leftrightarrow lim_k \bar{x}_{1k} = \bar{x}_{10}$ and $lim_k \bar{x}_{2k} = \bar{x}_{20}$.

Definition 4.1.2: Let λ^{i_2} be a sequence space of two dimensional interval vectors. Then λ^{i_2} is called normal or solid if $\tilde{y} = (\tilde{y}_k) \in \lambda^{i_2}$ whenever $\|\tilde{y}_k\| \leq \|\tilde{x}_k\|$ for all $k \in N$ and $\tilde{x} = (\tilde{x}_k) \in \lambda^{i_2}$.

Theorem 4.1.2: The space $(R(I_2), d)$ is a complete metric space.

Proof: Let $(\widetilde{x}_k)$ be any Cauchy sequence of $(R(I_2), d)$, then there exists a $n_0 \in N$ such that

$$d(\widetilde{x}_k, \widetilde{x}_m) =$$

$$max\{|x_{1lk} - x_{1lm}|, |x_{1rk} - x_{1rm}|, |x_{2lk} - x_{2lm}|, |x_{2rk} - x_{2rm}|\} < \epsilon \quad \text{..................(4.1.1)}$$

for all $k, m \geq n_0$. From this inequality, we can write that

$$max\{|x_{1lk} - x_{1lm}|, |x_{1rk} - x_{1rm}|\} < \varepsilon$$

and $max\{|x_{2lk} - x_{2lm}|, |x_{2rk} - x_{2rm}|\} < \varepsilon$

This last line shows us, the sequence $(\widetilde{x}_{lk})$ and $(\widetilde{x}_{rk})$ are Cauchy sequence in $(R(I), d)$. But $(R(I), d)$ is complete (see [109]). Hence we can write $lim_k \, \widetilde{x}_{lk} = \widetilde{x}_{l0}$ and $lim_k \, \widetilde{x}_{rk} = \widetilde{x}_{r0}$, in other word, $lim_k \, \widetilde{x}_k = \widetilde{x}_0$.

If we take the limit for $m \to \infty$ in (4.1.1), then we get $d\left(\widetilde{x}_k, \widetilde{x}_0\right) < \varepsilon$ for all $k \geq n_0$. This shows that $\widetilde{x}_0 \in \left(R\left(I_2\right), d\right)$. This step is end of the proof.

Some sequence spaces of two dimensional interval vectors:

Let ω^{i_2} denote the set of all sequences of two dimensional interval vectors of $R(I_2)$. ω^{i_2} is regarded as a quasivector space. Now the sequence spaces of null, convergent and bounded sequences of two dimensional interval vectors are given as follows:

$$c_0^{i_2} = \left\{\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2} : lim_{k\to\infty}\widetilde{x}_k = \widetilde{\theta},\ where\,\widetilde{\theta} = ([0,\,0],\,[0,\,0])\right\}$$

$$c^{i_2} = \{\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2} : lim_{k\to\infty}\widetilde{x}_k = \widetilde{x}_0,\ for\,some\,\widetilde{x}_0\}$$

$$\ell_\infty^{i_2} = \{\widetilde{x} = (\widetilde{x}_k) \in \omega^{i_2} : sup_k\ \{|x_{1lk}|,\,|x_{1rk}|,\,|x_{2lk}|,\,|x_{2rk}|\} < \infty\}$$

It is obvious that the spaces $c_0^{i_2}$, c^{i_2} and $\ell_\infty^{i_2}$ are subspaces of ω^{i_2}.

Theorem 4.1.3: $c_0^{i_2} \subseteq c^{i_2} \subseteq \ell_\infty^{i_2}$ and the inclusion is strict.

Proof: If we take any $\widetilde{x} \in c_0^{i_2}$ then we see that $\widetilde{x} \in c^{i_2}$ since $\bar{d}\left(\widetilde{x}_k, \widetilde{\theta}\right) = sup_k\,max\left\{|x_{1lk} - 0|,\,|x_{1rk} - 0|,\,|x_{2lk} - 0|,\,|x_{2rk} - 0|\right\} < \varepsilon$. Now, let $\widetilde{y} = (\widetilde{y}_k) = \left(\left[\frac{1}{k},\,1 + \frac{1}{k}\right],\,\left[1 - \frac{1}{k},\,2 + \frac{1}{k}\right]\right)$, $k \in N$, then $(\widetilde{y}_k) \in c^{i_2}$ but $(\widetilde{y}_k) \notin c_0^{i_2}$.

Example 4.1.1: Let $\widetilde{x}_k = \left(\left[\frac{1}{k},\,1 + \frac{1}{k}\right],\,\left[1 - \frac{1}{k},\,2 + \frac{1}{k}\right]\right)$, $k \in N$, then $(\widetilde{y}_k) \in c^{i_2}$ but $(\widetilde{y}_k) \notin c_0^{i_2}$.

Let $\widetilde{x}_k = \left(\left[(-1)^k,\,2 + \frac{1}{k}\right],\,\left[1 - \frac{1}{k},\,2 + \frac{1}{k}\right]\right)$, $k \in N$, then $(\widetilde{x}_k) \in \ell_\infty^{i_2}$ but $(\widetilde{x}_k) \notin c^{i_2}$.

Theorem 4.1.4: The spaces $c_0^{i_2}$, c^{i_2} and $\ell_\infty^{i_2}$ are complete metric spaces with the metric $\tilde{d}$ defined by

$$\tilde{d}\,(\widetilde{x},\,\widetilde{y}) = sup_k\, max\,\{|x_{1lk} - y_{1lk}|\,,\,|x_{1rk} - y_{1rk}|\,,\,|x_{2lk} - y_{2lk}|\,,\,|x_{2rk} - y_{2rk}|\}$$

where $\widetilde{x} = (\widetilde{x}_k)\,,\,\widetilde{y} = (\widetilde{y}_k) \in c_0^{i_2}$ or c^{i_2} or $\ell_\infty^{i_2}$.

Proof: It can be established using standard technique.

Theorem 4.1.5: The spaces $c_0^{i_2}$, c^{i_2} and $\ell_\infty^{i_2}$ are normed interval spaces with the norm

$$\|\widetilde{x}\| = sup_k\, max\,\{|x_{1lk}|\,,\,|x_{1rk}|\,,\,|x_{2lk}|\,,\,|x_{2rk}|\}\,.$$

where $\widetilde{x} = (\widetilde{x}_k) \in c^{i_2}$ or c^{i_2} or $\ell_\infty^{i_2}$.

Proof: We consider for the case of $c_0^{i_2}$ only. The remaining can be proved in similar manner.

$N_{1.}$ It is obvious that Since $\parallel \widetilde{x} \parallel \geq 0$ and $\parallel \widetilde{x} \parallel = \widetilde{\theta}$ iff $\widetilde{x} = \widetilde{\theta}$.

N_2. $\parallel \widetilde{x} + \widetilde{y} \parallel =$

$$sup_k\, max\,\{|\, x_{1lk} + y_{1lk}\,|,\,|\, x_{1rk} + y_{1rk}\,|,\,|x_{2lk} + y_{2lk}|\,,\,|x_{2rk} + y_{2rk}|\}$$

$$\leq sup_k max\,\{|\, x_{1lk}\,| + |\, y_{1lk}\,|,\,|\, x_{1rk}\,| + |\, y_{1rk}\,|,\,|x_{2lk}| + |y_{2lk}|\,,\,|x_{2rk}| + |y_{2rk}|\}$$

$$\leq sup_k\, max\,\{|\, x_{1lk}\,|,\,|\, x_{1rk}\,|,\,|x_{2lk}|\,,\,|x_{2rk}|\} +$$

$sup_k\, max \{| y_{1lk} |, | y_{1rk} |, |y_{2lk}| , |y_{2rk}|\}$

$= \| \widetilde{x} \| + \| \widetilde{y} \|$

N_3. $\| \alpha\widetilde{x} \| = sup_k\, max \{| \alpha x_{1lk} |, | \alpha x_{1rk} |, |\alpha x_{2lk}| , |\alpha x_{2rk}|\}$

$= |\alpha|\, sup_k\, max \{| x_{1lk} |, | x_{1rk} |, |x_{2lk}| , |x_{2rk}|\}$

$= | \alpha |\| \widetilde{x} \|$

Hence, $\| \widetilde{x} \|$ is a norm on $c_0^{i_2}$.

Theorem 4.1.6: The spaces $c_0^{i_2}$ and $c^{i_2}$are solid and monotone.

Proof: We consider only $c_0^{i_2}$. For c^{i_2} it can be established similarly.

Let $\widetilde{x} = (\widetilde{x}_k) \in c_0^{i_2}$ and $\widetilde{y} = (\widetilde{y}_k)$ be such that $\|\widetilde{y}_k\| \leq \|\widetilde{x}_k\|$, for all $k \in N$. Then we have, $\widetilde{d}\left(\widetilde{y}_k, \widetilde{\theta}\right) \leq \widetilde{d}\left(\widetilde{x}_k, \widetilde{\theta}\right)$, that is

$\{| y_{1lk} - 0 |, |y_{2lk} - 0| , | y_{1rk} - 0 |, |y_{2rk} - 0|\}$

$\leq \{| x_{1lk} - 0 |, |x_{2lk} - 0| , | x_{1rk} - 0 |, |x_{2rk} - 0|\}$.

Thus we have,

$y_{1lk} \leq x_{1lk}, y_{2lk} \leq x_{2lk}$ and $y_{1rk} \leq x_{1rk}$, $y_{2rk} \leq x_{2rk}$ i.e., $\widetilde{y} \leq \widetilde{x}$.

It is clear that $\widetilde{y} = (\widetilde{y}_k) \in c_0^{i_2}$. Therefore $c_0^{i_2}$ is solid. A solid sequence space is always monotone and so $c_0^{i_2}$ is monotone.

4.2 Some Difference Sequence Spaces of Interval Numbers

In this section we have introduced some new difference sequence spaces of interval numbers and studied some algebraic and topological properties. We have also investigated some inclusion relations related to these spaces.

We define the difference operator on the sequences of interval numbers by

$$\Delta\overline{x} = (\Delta\bar{x}_k) = ([\Delta x_{kl}, \Delta x_{kr}]).$$

We introduce the classes of difference sequences of null, convergent and bounded interval numbers. We denote the difference sequences of null, convergent, bounded sequences of interval numbers by $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ respectively, defined by

$c_0^i(\Delta) = \{\overline{x} = (\overline{x}_k) \in \omega^i : lim_k \Delta\overline{x}_k = \theta$, where $\theta = [0,0]\}$.

$c^i(\Delta) =\{\overline{x} = (\overline{x}_k) \in \omega^i : lim_k \Delta\overline{x}_k = \overline{x}_0$ for some $\bar{x}_0\}$.

$\ell_\infty^i(\Delta) = \overline{x} = (\overline{x}_k) \in \omega^i : sup_k\{|\Delta x_{kl}|, |\Delta x_{kr}| < \infty\}$.

It can be easily verified that the spaces $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are subsets of the space ω^i. Besides, for all $(\overline{x}_k), (\overline{y}_k) \in c_0^i(\Delta)\,(or\; c^i(\Delta), \ell_\infty^i(\Delta))$ the distance $\bar{d}_\Delta$ is defined by

$\bar{d}_\Delta((\overline{x}_k), (\overline{y}_k)) = sup_k max\{|\Delta x_{kl} - \Delta y_{kl}|, |\Delta x_{kr} - \Delta y_{kr}|\}$..................(4.2.1)

which satisfies all the axioms of metric. Thus $(c_0^i(\Delta), \bar{d}_\Delta)$, $(c^i(\Delta), \bar{d}_\Delta)$, and $(\ell_\infty^i(\Delta),\, \bar{d}_\Delta)$ are metric spaces.

Theorem 4.2.1: $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are complete metric spaces with the metric defined by (4.2.1).

Proof: We proof the result for the class $c_0^i(\Delta)$. The rest can be established similarly.

Let $(\overline{x}_n^k) = (\overline{x}_n^1, \overline{x}_n^2, \overline{x}_n^3, \dots) \in c_0^i(\Delta)$ for each n, then $lim_{k\to\infty}\overline{x}_n^k = \theta$ for each $n \in N$. Let $(\overline{x}_n\ )$ be a Cauchy sequence. Then for each $\varepsilon > 0$, there exists a $k_0 \in N$ such that $\bar{d}_\Delta(\overline{x}_n, \overline{x}_m) < \varepsilon$, whenever $n, m \geq k_0$. Hence we have $sup_{n,m}\{max|\Delta x_{nl}^k - \Delta x_{ml}^k, \Delta x_{nr}^k - \Delta x_{mr}^k|\} < \varepsilon$. Thus we have $|\Delta x_{nl}^k - \Delta x_{ml}^k| < \varepsilon$ and $|\Delta x_{nr}^k - \Delta x_{mr}^k| < \varepsilon$. This means that $(\Delta\overline{x}_n^k)$ is a Cauchy sequence in $R(I)$. Since $R(I)$ is a complete, $(\Delta\overline{x}_n^k\ )$ is convergent i.e $lim_{nl\to\infty}\Delta x_{nl}^k = 0$ and $lim_{nr\to\infty}\Delta x_{nr}^k = 0$.

Now, $|\Delta x_{nl}^k - 0| < \varepsilon$ and $|\Delta x_{nr}^k - 0| < \varepsilon$, taking $m\to\infty$ gives

$sup_n max\{|\Delta x_{nl}^k - 0|, |\Delta x_{nr}^k - 0|\} < \varepsilon$ i.e., $\bar{d}_\Delta(\overline{x}_n, \theta) < \varepsilon$.

This implies that $(\overline{x}_n)$ is a convergent sequence and converge to $\theta \in c_0^i(\Delta)$.

The norm function on the sequences of interval numbers can be extended to the difference sequence spaces of interval numbers. Suppose that $\lambda^i(\Delta)$ is a subset of ω^i where $\lambda^i(\Delta) = c_0^i(\Delta)$ or $c^i(\Delta)$ or $\ell_\infty^i(\Delta)$.

Definition 4.2.1: A norm on $\lambda^i(\Delta)$ is a non-negetive function $\|.\|_{\lambda^i} = \lambda^i(\Delta) \to R^+ \cup \{0\}$ that satisfies the following properties: $\forall \overline{x}, \overline{y} \in \lambda^i(\Delta)$ and $\forall \alpha \in$ R,

N_1. $\|\overline{x}\|_{\lambda^i} > 0, \forall\, \overline{x} \in \lambda^i(\Delta) - \{\theta\}$

N_2.$\|\overline{x}\|_{\lambda^i} = 0 \Leftrightarrow \overline{x} = \theta$

N_3. $\|\overline{x} + \overline{y}\|_{\lambda^i} \leq \|\overline{x}\|_{\lambda^i} + \|\overline{y}\|_{\lambda^i}$

N_4. $\|\alpha\overline{x}\|_{\lambda^i} = |\alpha|\ \|\overline{x}\|_{\lambda^i}$

Theorem 4.2.2: The spaces $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are normed spaces with the norm

$$\|\overline{x}\|_\Delta = max(|x_l^1|, |x_u^1|) + sup_k max\{|\Delta x_l^k|, |\Delta x_r^k|\}.$$

Proof: Let $\lambda^i(\Delta) = c_0^i(\Delta)$ or $c^i(\Delta)$ or $\ell_\infty^i(\Delta)$ and $\overline{x}$, $\overline{y} \in \lambda^i(\Delta)$.

N_1. $\|\overline{x}\|_{\lambda^i} = sup_k max\ \{|\Delta x_l^k|, |\Delta x_r^k|\}$.

It can be easily verified that $\|\overline{x}\|_{\lambda^i} > 0$ for $\overline{x} \in \lambda^i(\Delta) - \{\theta\}$

N_2. $\|\overline{x}\|_{\lambda^i} = 0 \Leftrightarrow max(|x_l^1|, |x_r^1|) + sup_k max\{|\Delta x_l^k|, |\Delta x_r^k|\} = 0 \Leftrightarrow \overline{x} = \theta$

N_3. $\|\overline{x}+\overline{y}\|_{\lambda^i} = max(|x_l^1+y_l^1|, |x_r^1+y_r^1|) + sup_k max\{|\Delta(x_l^k+y_l^k)|, |\Delta(x_r^k+y_r^k)|\}$

$$\leq max(|x_l^1| + |y_l^1|, |x_r^1| + |y_r^1|) + sup_k max\ \{|\Delta x_l^k| + |\Delta y_l^k|, |\Delta x_r^k| + |\Delta y_r^k|\}$$

$$= max(|x_l^1|, |x_r^1|) + sup_k max\{(|\Delta x_l^k|, |\Delta x_r^k|)\}$$

$$+ max(|y_l^1|, |y_r^1|) + sup_k max\{(|\Delta y_l^k|, |\Delta y_r^k|)\}$$

$= \|\overline{x}\|_{\lambda^i} + \|\overline{y}\|_{\lambda^i}$

N_4. $\|\alpha\overline{x}\|_{\lambda^i} = max(|\alpha x_l^1|, |\alpha x_r^1|) + sup_k max\{|\alpha\Delta x_l^k|, |\alpha\Delta x_r^k|\}$

$= max(|\alpha||x_l^1|, |\alpha||x_r^1|) + sup_k max\{|\alpha||\Delta x_l^k|, |\alpha||\Delta x_r^k|\}$

$= |\alpha| max(|x_l^1|, |x_r^1|) + |\alpha| sup_k max\{|\Delta x_l^k|, |\Delta x_r^k|\}$

$= |\alpha|\|\overline{x}\|_{\lambda^i}$

So $\|\overline{x}\|_{\lambda^i}$ is a norm on $\lambda^i(\Delta)$.

Result 4.2.1: The spaces $c_0^i(\Delta), c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are neither solid nor monotone.

Proof: This follows from the following example.

Example 4.2.1: Consider the sequence $\bar{x}_k = [k, k]$, for all $k \in N$. Then $(\bar{x}_k) \in c^i(\Delta)$. Consider the sequence $(\bar{y}_k)$ defined by $\bar{y}_k = (-1)^k[k, k]$, for all $k \in N$. Then $lim_k \Delta\bar{y}_k$ does not exist. We have $| \bar{y}_k | \leq | \bar{x}_k |$, for all $k \in N$. Hence $c^i(\Delta)$ is not monotone and hence is not solid. Similarly, for the others.

Theorem 4.2.3: The spaces $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are sequence algebra.

Proof: We prove that $c_0^i(\Delta)$ is a sequence algebra. Let $(\overline{x}_k)$, $(\overline{y}_k) \in c_0^i(\Delta)$. Then $lim_k \Delta\overline{x}_k = \theta$ and $lim_k \Delta\overline{y}_k = \theta$. Then we have $lim_k \Delta(\overline{x}_k\overline{y}_k) = \theta$.

Thus $(\overline{x}_k\overline{y}_k) \in c_0^i(\Delta)$ is a sequence algebra. For the space $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$, the results can be proved similarly.

Result 4.2.2: The spaces $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ are not convergence free in general.

Proof: Here we provide an example to establish the result.

Example 4.2.2: Let $\overline{x}_k = [k, k+1] \in c^i(\Delta)$ and $\overline{y}_k = [(-1)^k, 2 + \frac{1}{K}]$ for all $(k \in N)$. Then $(\overline{x}_k) \in c^i(\Delta)$ but $(\overline{y}_k) \notin (c^i(\Delta))$. Hence, the space $c^i(\Delta)$ is not convergence free. Similarly, it can be proved that $c_0^i(\Delta)$ is not convergence free.

Theorem 4.2.4: The inclusions $c_0^i(\Delta) \subset c^i(\Delta) \subset \ell_\infty^i(\Delta)$ hold and are strict.

Proof: The first inclusion follows from the definitions of the above classes of sequences.

The inclusion is strict follows from the following example.

Example 4.2.3: It is obvious that $\overline{y} = ([n, n+1]) \notin c_0^i(\Delta)$ but $\overline{y} \in c^i(\Delta)$, since y_{kl} and y_{kr} are both divergent sequences and $\overline{y}(\Delta) = (\Delta\bar{y}_k) = ([k - k - 2, k + 1 - k - 1]) = ([-2, 0])$. Therefore, $lim_{k\to\infty} y_{kl}(\Delta) = -2$ and $lim_{k\to\infty} y_{kr}(\Delta) = 0$.

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