CHAPTER 3
SOME REGULAR MATRIX TRANSFORMATION

Some Regular Matrix Transformation

In this chapter we have introduced a new regular matrix F of Fibonacci numbers and introduced some new sequence spaces based on the newly defined regular matrix F. Also we have introduced the criterion for the regularity of a matrix whose elements are interval numbers and we have defined a regular matrix $\overline{F}$ of interval numbers using Fibonacci numbers. We have defined some new sequence spaces based on $\overline{F}$. We have also studied some of their algebraic and topological properties related to these spaces.

3.1 Regular Matrix Transformation on Sequences of Real Numbers

Let X, Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where n, $k \in N$. Then, A defines a matrix mapping from X into Y and we denote it by $A : X \rightarrow Y$, if for every sequence $x = (x_k) \in X$, the sequence $AX = \{A_n(x)\}_{n=1}^{\infty}$, the $A-$transform of x, is in Y; where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k, \; (n \in N).$$

By (X, Y), we denote the class of all matrices A such that $A : X \rightarrow Y$. Thus $A \in (X, Y)$ if and only if the series on the right hand side above converges for each $n \in N$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$. The matrix domain $X(A)$ of an infinite matrix A in a sequence space X is defined by

$$X(A) = \{x = (x_k) \in \omega : Ax \in X\},$$

which is a sequence space.

A sequence space X is called FK space if it is a complete linear metric space with continuous coordinates $p_n : X \to R \ (n \in N)$, where R denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $(n \in N)$. A BK space is a normed FK space, i.e., a BK space is a Banach space with continuous coordinates. The spaces c, c_0 and ℓ_∞ are BK spaces with $\|x\| = sup_k |x_k|$.

For any given sequence (x_n) a new sequence (y_n) is defined as follows:

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k$$

provided in the later case y_n has a meaning. If $lim_{n\to\infty} y_n$ exists, the limit is called the generalized value of the sequence (x_n) by the transformation. If whenever (x_n) converges, (y_n) converges to the same value, then the transformation is said to be regular.

The following lemma (known as the Toeplitz theorem) contains necessary and sufficient conditions for regularity of a matrix.

The matrix $A = (a_{nk})$ is regular if and only if it satisfies the following conditions:

(a) There exists $M > 0$ such that for every $n = 1, 2, \ldots$ the following inequality holds:

$$\sum_{k=1}^{\infty}|a_{nk}| \leq M \ ;$$

(b) For every $k = 1, 2, \ldots \ lim_{n\to\infty} a_{nk} = 0$.

(c) $lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

Let (p_k) be a sequence of positive numbers and $P_n = \sum_{k=1}^{\infty} p_k$.

Then the matrix $R^p = (r_{nk}^p)$ of the Riesz mean is given by

$$r_{nk}^p = \begin{cases} \frac{p_k}{P_n} & (1 \leq k \leq n); \\ 0, & \text{otherwise} \end{cases}$$

It is known that the Riesz matrix is a Toeplitz matrix if and only if $P_n \to \infty$ as $n \to \infty$.

Some sequence spaces based on regular matrix of Fibonacci numbers:

Fibonacci numbers are widely studied and the formulas to derive them are well-known. Such formulas include Binet's formula,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

and Cassini's formula,

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n,$$

where f_n is the n^{th} Fibonacci number.

The Fibonacci numbers are the sequence of numbers $(f_n)_{n=1}^{\infty}$ defined by the linear recurrence equations

$f_0 = 0$ and $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$; $n \geq 2$.

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are given as follows:

$\sum_{k=1}^{n} f_n = f_{n+2} - 1$; $n \geq 1$,

$\sum_{k=1}^{n} f_n^2 = f_n f_{n+1}$; $n \geq 1$,

$\sum_{k=1}^{\infty} \frac{1}{f_k}$ converges, $n \geq 1$,

In [51] Kara and Basarir defined a Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$ by

$$f_{nk} = \begin{cases} \frac{f_k^2}{f_n f_{n+1}} & (1 \leq k \leq n); \\ 0 & (k > n) \end{cases}$$

that is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 & \ldots \\ \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & \frac{9}{15} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

It is obvious that the matrix F is triangular matrix, i.e., $f_{nn} \neq 0$ for $k \leq n$ and $f_{nk} = 0$ for $k > n$ $(n = 1, 2, 3, \ldots)$. Also it follows that the method is regular.

In [51] Kara and Basarir defined the Fibonacci sequence space $X(F)$ as

$X(F) = \{x = (x_k) \in \omega : y = (y_k) \in X\}$, where $X = l_\infty, c, c_0$ and ℓ_p $(1 \leq p < \infty)$

where the sequence $y = (y_k)$ is the $F-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k = F_k(x) = \frac{1}{f_k f_{k+1}} \Sigma_{j=1}^{k} f_j^2 x_j \, for \, all \, k \in N.$$

It follows that $X(F)$ is the matrix domain of the triangle F.

In [51] Kara and Basarir investigated the matrix domain of F in the classical sequence spaces $X = \ell_\infty, c, c_0$ and ℓ_p $(1 \leq p < \infty)$.

We define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$,

$$f_{nk} = \begin{cases} \frac{f_k}{f_{n+2}-1} & (1 \leq k \leq n); \\ 0 & otherwise \end{cases}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ... \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & ... \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0 & ... \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

It is obvious that the matrix F is triangular matrix, i.e., $f_{nn} \neq 0$ for $k \leq n$ and $f_{nk} = 0$ for $k > n$ $(n = 1, 2, 3, ...)$. Also it follows that the method is regular.

Now, we introduce the following sequence spaces based on the infinite matrix F:

$$c(F) = \{x = (x_k) \in \omega : Fx \in c\}$$

$$c_0\,(F) = \{x = (x_k) \in \omega : Fx \in c_0\}$$

$$\ell_\infty\,(F) = \{x = (x_k) \in \omega : Fx \in \ell_\infty\}$$

where $Fx = \{F_n\,(x)\}_{n=1}^{\infty}$ and $F_n\,(x) = \sum_{k=1}^{\infty} f_{nk} x_k = \frac{f_k}{f_{n+2}-1} \sum_{k=1}^{\infty} f_{nk} x_k$, $(n \in N)$.

Theorem 3.1.1: The spaces $c\ (F)$, $c_0\,(F)$ and $\ell_\infty\,(F)$ are BK spaces with the same norm given by

$\parallel x \parallel_{\ X(F)} = \parallel Fx \parallel_X = sup_n |F_n\,(x)|$ where $X \in \{c,\, c_0,\, \ell_\infty\}$.

Proof: By theorem 4.3.2 of Wilanksy [12], [p.63] and as the matrix F is triangular, we have the result.

Remark 3.1.1: It can be easily seen that the absolute property does not hold on the spaces $c(F)$, $c_0(F)$, $\ell_\infty(F)$ i.e., $\parallel x \parallel_{\ X(F)} \neq \parallel |x| \parallel_{X(F)}$ for at least one sequence x in each of these spaces, where $|x| = (|x_k|)$. Thus the spaces $c\,(F)\,, c_0\,(F)$ and $\ell_\infty\,(F)$ are BK spaces of non-absolute type.

Theorem 3.1.2: The sequence spaces $c\,(F)$, $c_0\,(F)$ and $\ell_\infty\,(F)$ are norm isomorphic to the spaces c, c_0 and ℓ_∞ respectively i.e., $c\,(F) \cong c$, $c_0\,(F) \cong c_0$ and $\ell_\infty\,(F) \cong \ell_\infty$.

Proof: X denotes any of the spaces c, c_0 or ℓ_∞ and $X(F)$ be the respective one of the spaces $c\,(F)$, $c_0\,(F)$ or $\ell_\infty\,(F)$. Since the matrix F is triangular, it has a unique inverse, which is also triangular (Wilansky, 1984, proposition 1.1). Therefore the linear operator $L_F : X(F) \to X$, defined by $L_F(x) = F(x)$ for all $x \in X(F)$, is bijective and is norm preserving by above norm in theorem 3.1.1. Hence $X(F) \cong X$.

Theorem 3.1.3: The inclusions $c_0(F) \subset c(F) \subset \ell_\infty(F)$ hold and are proper.

Proof: It is clear that the inclusion $c_0(F) \subset c(F) \subset \ell_\infty(F)$ hold.

Example 3.1.1: Consider the sequence $x = (x_k)$ defined by $x_k = 1$, for all $k \in N$. Then we have for every $n \in N$,

$$F_n(x) = \frac{1}{f_{n+2-1}} \sum_{k=1}^{n} f_k = 1$$

This shows that $Fx \in c$ but not in c_0. Thus the sequence x is in $c(F)$ but not in $c_0(F)$. Hence the inclusion $c_0(F) \subset c(F)$ strictly holds.

Again, consider the sequence $x = (x_k)$ defined by

$$x_k = \frac{(-1)^k((f_{k+2}+f_{k+1}-1))}{f_k}, \text{ for all } k \in N.$$

Then we have for every $n \in N$,

$$F_n(x) = \frac{1}{f_{n+2-1}} \sum_{k=1}^{n} f_k \, x_k = (-1)^n$$

This shows that $Fx \in \ell_\infty$ but not in c. Thus the sequence x is in $\ell_\infty(F)$ but not in $c(F)$. Hence the inclusion $c(F) \subset \ell_\infty(F)$ holds and it is strict.

Theorem 3.1.4: The inclusions $c_0 \subset c_0(F)$, $c \subset c(F)$ and $\ell_\infty \subset \ell_\infty(F)$

hold and are proper.

Proof: As F is a regular matrix, so the inclusion $c_0 \subset c_0(F)$ and $c \subset c(F)$ are obvious.

Now, let $x = (x_k) \in \ell_\infty$. Then there is a constant $M > 0$ such that $|x_k| \leq M$ for all $k \in N$.

Thus for each $n \in N$,

$$|F_n(x)| \leq \sum_{k=1}^{n} f_k \; |x_k|.$$

$$\leq \frac{M}{f_{n+2}-1} \sum_{k=1}^{n} f_k = M$$

which shows that $Fx \in \ell_\infty$ i.e., $x \in \ell_\infty(F)$. Thus we conclude that $\ell_\infty \subset \ell_\infty(F)$.

Example 3.1.2: Consider the sequence $x = (x_k) = (1, 0, 1, 0, 1, 0, ...)$. Then we have for every $n \in N$,

$$F_n(x) = \frac{1}{f_{n+2}} \sum_{k=1}^{n} f_k \; x_k = \frac{1}{f_{n+2}-1}(f_1 + f_3 + ... + f_n)$$

which is convergent.

This shows that $Fx \in c$ but x is not in c. Thus the sequence x is in $c(F)$. Hence the inclusion $c \subset c(F)$ strictly holds. Similarly, we can show the other inclusions are strict.

3.2 Regular Matrix Transformation of Interval Numbers

In this section, we introduced the criterion for the regularity of a matrix whose elements are interval numbers and defined a regular matrix $\bar{F}$ of interval numbers using fibonacci numbers and introduced some new sequence spaces $c_0^i\left(\overline{F}\right)$, $c^i\left(\overline{F}\right)$, $\ell_\infty^i\left(\overline{F}\right)$ based on the newly defined regular matrix of interval numbers $\bar{F}$ and investigated some relations related to these spaces. For any given sequence $(\overline{x}_n)$ of interval numbers a new sequence $(\overline{y}_n)$ is defined as follows:

$$\overline{y}_n = \sum_{k=1}^{\infty} \overline{a}_{nk}\ \overline{x}_k = [\sum_{k=1}^{\infty} a_{lnk} x_{lk}, \sum_{k=1}^{\infty} a_{rnk} x_{rk}]$$

provided in the later case $\overline{y}_n$ has a meaning. If $lim_{n\to\infty}\overline{y}_n$ exists, the limit is called the generalized value of the sequence $\overline{x}_n$ by the transformation. If whenever $\overline{x}_n$ converges, $\overline{y}_n$ converges to the same value, then the transformation is said to be regular.

Theorem 3.2.1: The matrix $\overline{A} = (\overline{a}_{nk})$ of interval numbers is regular if and only if it satisfies the following conditions:

(a) There exists $M > 0$ such that for every $n = 1,\ 2,\ ...$ the following inequalities hold:

$\sum_{k=1}^{\infty} |\overline{a}_{nk}| \leq M$ i.e., $\sum_{k=1}^{\infty} |a_{lnk}| \leq M$ and $\sum_{k=1}^{\infty} |a_{rnk}| \leq M$.

(b) For every $k = 1,\ 2,\ ...\ lim_{n\to\infty}\overline{a}_{nk} = \theta$ where $\theta = [0, 0]$.

i.e., $lim_{n\to\infty} a_{lnk} = 0$ and $lim_{n\to\infty} a_{rnk} = 0$.

(c) $lim_{n\to\infty} \sum_{k=1}^{\infty} \overline{a}_{nk} = [1, 1]$, i.e., $lim_{n\to\infty} a_{lnk} = 1$ and $lim_{n\to\infty} a_{rnk} = 1$.

Proof: Suppose $\overline{A}$ satisfies the three conditions. Let $\overline{x} = (\overline{x}_k) = (x_{lk}, x_{rk})$ be such that $\overline{x}_k = \overline{x}_0$ as $k \to \infty$, i.e., $x_{lk} = x_{l0}$ and $x_{rk} = x_{r0}$ as $k \to \infty$. Obviously, by (a) $\overline{A}\overline{x} = \overline{y} = (\overline{y}_n)$ exists.

Now we have to show that $\overline{y}_n = \overline{x}_0$ as $n \to \infty$, i.e., $y_{ln} = x_{l0}$ and $y_{rn} = x_{r0}$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Then there exists integer N such that

$d\ (\overline{x}_0, \overline{x}_k) < \varepsilon$ for $k > N$

$|x_{l0} - x_{lk}| < \varepsilon$ and $|x_{r0} - x_{rk}| < \varepsilon$ for $k > N$

By (c), there is an integer K such that

$|1 - \sum_{k=1}^{\infty} a_{lnk}| < \varepsilon$ ....(3.2.1)

and $|1 - \sum_{k=1}^{\infty} a_{rnk}| < \varepsilon$ for $n > K$....................(3.2.2)

We consider the 1st case only

Now using (a), we have for $n > N$,

$|x_{l0} - y_{ln}| = |x_{l0} - \sum_{k=1}^{\infty} a_{lnk} x_{lk}|$

$= |\sum_{k=1}^{\infty} a_{lnk}\,(x_{l0} - x_{lk}) + x_{l0}\,(1 - \sum_{k=1}^{\infty} a_{lnk})|$

$\leq \left(\sum_{k=1}^{N} + \sum_{k=N+1}^{\infty}\right) |a_{lnk}||x_{l0} - x_{lk}| + |x_{l0}|\varepsilon$

$\leq \sum_{k=1}^{N} |a_{lnk}||x_{l0} - x_{lk}| + M\ \varepsilon + |x_{l0}|\varepsilon$

But, by (b) there is an integer $K^{'} > K$ such that $|a_{lnk}| < \frac{\varepsilon}{N}$ for $n > K^{'}$.

Hence, if $L = sup\ |x_{l0} - x_{lk}|$, we have for $n > K^{'}$

$|x_{l0} - y_{ln}| < N\ L\ \left(\frac{\varepsilon}{N}\right) + M\ \varepsilon + |x_{l0}|\varepsilon$

$= (L + M + |x_{l0}|)\,\varepsilon$

This implies that $y_{ln} \to x_{l0}$ as $n \to \infty$.

Similarly, we can prove that $y_{rn} \to x_{r0}$ as $n \to \infty$.

Hence, we get $\overline{y}_n \to \overline{x}_0$ as $n \to \infty$.

Conversely, let us suppose that $\overline{A} = (\overline{a}_{nk}) = ([a_{lnk}, a_{rnk}])$ is regular as $n \to \infty$.

(a) At first we shall show that the first condition is satisfied for (a_{lnk}).

It is obvious (see [63], Theorem 1.3.2, p. 5). Similar conclusion can be drawn for (a_{rnk}).

(b) For every $i > 1$ we define a sequence $\left(\overline{e}_k^{(i)}\right) = \begin{cases} [1,1] & (k = i); \\ [0,0] & (k \neq i); \end{cases}$

Then we have, $y_{ln}^{(i)} = \sum_{k=1}^{\infty} a_{lnk}\overline{e}_{lk}^{(i)} = a_{lni}$

and $y_{rn}^{(i)} = \sum_{k=1}^{\infty} a_{rnk} \overline{e}_{lk}^{(i)} = a_{rni}$.

As $\overline{A}$ is regular and for all i, $\overline{e}_{k}^{(i)} \to [0,0]$ as $k \to \infty$, it follows that $\overline{a}_{ni} \to \theta$ when $n \to \infty$ for all i.

(c) Consider the sequence $(\overline{x}_k) = \{[1,1], [1,1], [1,1], ..., [1,1], ...\}$. So, it is obvious that $y_{ln} = \sum_{k=1}^{\infty} a_{lnk} x_{lk} = \sum_{k=1}^{\infty} a_{lnk} \to 1$ and $y_{rn} = \sum_{k=1}^{\infty} a_{rnk} x_{lk} = \sum_{k=1}^{\infty} a_{rnk} \to 1$ as $n \to \infty$, since $x_{lk} \to 1$ and $x_{rk} \to 1$ as $k \to \infty$, which completes the proof.

Some sequence spaces on regular matrix of interval numbers based on Fibonacci numbers:

Now we define the Fibonacci matrix $\overline{F} = \left(\overline{f}_{nk}\right)_{n,k=1}^{\infty} = ([f_{lnk}, f_{rnk}])_{n,k=1}^{\infty}$, by

$$f_{lnk} = \begin{cases} \frac{f_k^2}{f_n f_{n+1}} & (1 \leq k \leq n); \\ 0 & (k > n) \end{cases}$$
$$f_{rnk} = \begin{cases} \frac{f_k}{f_{n+2} - 1} & (1 \leq k \leq n); \\ 0 & (k > n) \end{cases}$$

that is,

$$\begin{bmatrix} [1,1] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & ... \\ \left[\frac{1}{2},\frac{1}{2}\right] & \left[\frac{1}{2},\frac{1}{2}\right] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & ... \\ \left[\frac{1}{6},\frac{1}{4}\right] & \left[\frac{1}{6},\frac{1}{4}\right] & \left[\frac{4}{6},\frac{2}{4}\right] & [0,0] & [0,0] & [0,0] & [0,0] & ... \\ \left[\frac{1}{15},\frac{1}{7}\right] & \left[\frac{1}{15},\frac{1}{7}\right] & \left[\frac{4}{15},\frac{2}{7}\right] & \left[\frac{9}{15},\frac{3}{7}\right] & [0,0] & [0,0] & [0,0] & ... \\ \left[\frac{1}{40},\frac{1}{12}\right] & \left[\frac{1}{40},\frac{1}{12}\right] & \left[\frac{4}{40},\frac{2}{12}\right] & \left[\frac{9}{40},\frac{3}{12}\right] & \left[\frac{25}{40},\frac{5}{12}\right] & [0,0] & [0,0] & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \end{bmatrix}$$

It is obvious that the matrix $\overline{F}$ is a triangular matrix of interval numbers and by the theorem 3.2.1, $\overline{F}$ is regular.

We have introduced the following sequence spaces based on the infinite matrix $\overline{F}$:

$$c^i\left(\overline{F}\right) = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : \overline{F}\,\overline{x} \in c^i\right\}$$

$$c_0^i\left(\overline{F}\right) = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : \overline{F}\,\overline{x} \in c_0^i\right\}$$

$$\ell_\infty^i\left(\overline{F}\right) = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : \overline{F}\,\overline{x} \in \ell_\infty^i\right\}$$

where $\overline{F}\,\overline{x} = \left\{\left(\overline{F}_n\left(\overline{x}\right)\right)\right\}_{n=1}^{\infty}$ and $\overline{F}_n(\overline{x}) = [\sum_{k=1}^{\infty} f_{lnk}x_{lk}, \sum_{k=1}^{\infty} f_{rnk}x_{rk}]$ $(n \in N)$.

Theorem 3.2.2: $\left(c_0^i\left(\overline{F}\right), \overline{d}\right)$, $\left(c^i\left(\overline{F}\right), \overline{d}\right)$ are complete metric spaces with the metric defined by $\overline{d}\left(\overline{x}_k, \overline{y}_k\right) = sup_k\, max\left\{\left|x_{lk} - y_{lk}\right|, \left|x_{rk} - y_{rk}\right|\right\}$.

Proof: It can be established using standard technique.

Theorem 3.2.3: $c_0^i\left(\overline{F}\right)$ and $c^i\left(\overline{F}\right)$ are normed interval spaces with the norm

$$\|x\| = sup_k\, max\ \{|x_{lk},|\, |x_{rk}|\}$$

Proof: Let $\mu^i = c_0^i\left(\overline{F}\right)\ \left(or\ c^i\left(\overline{F}\right)\right)$ and $\overline{x}$, $\overline{y} \in \mu^i$

N_1. Since $\|\overline{x}\|_{\mu^i} = sup_k\, max \left\{|x_{lk}|\,,\,|x_{rk}|\right\}$, then we have $\|\ \overline{x}\ \|_{\mu^i} > 0\ \forall\ \overline{x} \in \mu^i$ - $\{\theta\}$.

N_2. $\|\ \overline{x}\ \|_{\mu^i} = 0 \Longleftrightarrow sup_k max \left\{|\ x_{lk}\ |, |\ x_{rk}\ |\right\} = 0 \Longleftrightarrow \overline{x} = \theta$, where $\theta = [0,\, 0]$.

N_3.$\|\ \overline{x} + \overline{y}\ \|_{\mu^i} = sup_k max \left\{|\ x_{lk} + y_{lk}\ |, |\ x_{rk} + y_{rk}\ |\right\}$

$\leq sup_k max \left\{|\ x_{lk}\ | + |\ y_{lk}\ |, |\ x_{rk}\ | + |\ y_{rk}\ |\right\}$

$\leq sup_k max \left\{|\ x_{lk}\ |, |\ x_{rk}\ |\right\} + sup_k max \left\{|\ y_{lk}\ |, |\ y_{rk}\ |\right\}$

$= \|\ \overline{x}\ \|_{\mu^i} + \|\ \overline{y}\ \|_{\mu^i}$

N_4. $\|\ \alpha\overline{x}\ \|_{\mu^i} = sup_k max \left\{|\ \alpha x_{lk}\ |, |\ \alpha x_{rk}\ |\right\}$

$=|\ \alpha\ |\ sup_k max \left\{|\ x_{lk}\ |, |\ x_{rk}\ |\right\}$

$= |\ \alpha\ |\|\ \overline{x}\ \|_{\mu^i}$

Hence, $\|\ \overline{x}\ \|_{\mu^i}$ is a norm on μ^i.

Theorem 3.2.4: The spaces $c_0^i\left(\overline{F}\right)$ and $c^i\left(\overline{F}\right)$ are solid.

Proof: We consider only $c_0^i\left(\overline{F}\right)$.

Now, let $\|\overline{y}_k\| \leq \|\overline{x}_k\|$, for all $k \in N$ and for some $\overline{x} \in c_0^i\left(\overline{F}\right)$. Then we have, $\overline{d}\left(\overline{y}_k, \theta\right) \leq \overline{d}\left(\overline{x}_k, \theta\right)$, that is $\{\mid y_{lk} - 0 \mid, \mid y_{rk} - 0 \mid\} \leq \{\mid x_{lk} - 0 \mid, \mid x_{rk} - 0 \mid\}$.

Thus we have $y_{lk} \leq x_{lk}$ and $y_{rk} \leq x_{rk}$, i.e., $\overline{y} \leq \overline{x}$.

So, clearly $\overline{y} \in c_0^i\left(\overline{F}\right)$. Hence $c_0^i\left(\overline{F}\right)$ is solid.

For the space $c^i\left(\overline{F}\right)$, the result can be proved similarly.

Theorem 3.2.5: The spaces $c_0^i\left(\overline{F}\right)$ and $c^i\left(\overline{F}\right)$ are sequence algebra.

Proof: We prove that $c_0^i\left(\overline{F}\right)$ is a sequence algebra.

Let $(\overline{x}_k)$, $(\overline{y}_k) \in c_0^i\left(\overline{F}\right)$.

Then, $lim_k \overline{x}_k = \theta$ and $lim_k \overline{y}_k = \theta$, where $\theta = [0,\ 0]$.

Then we have, $lim_k\left(\overline{x}_k \overline{y}_k\right) = \theta$.

Thus $(\overline{x}_k \overline{y}_k) \in c_0^i\left(\overline{F}\right)$. Hence $c_0^i\left(\overline{F}\right)$ is a sequence algebra.

For the space $c^i\left(\overline{F}\right)$, the result can be proved similarly.

Result 3.2.1: The spaces $c_0^i\left(\overline{F}\right)$and $c^i\left(\overline{F}\right)$ are not convergence free in general.

Proof: Here, we give a counter example.

Example 3.2.1: Let, $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be two sequences of interval numbers.

Now let, $\overline{x}_k = \left[\frac{1}{k}, \frac{(-1)^k}{k}\right]$

and $\overline{y}_k = \left[(-1)^k, 2 + \frac{1}{k}\right]$ for all $k \in N$.

Then $(\overline{x}_k) \in c^i\left(\overline{F}\right)$ and $c_0^i\left(\overline{F}\right)$ but $(\overline{y}_k) \notin c^i\left(\overline{F}\right)$ and $c_0^i\left(\overline{F}\right)$.

Hence the spaces $c^i\left(\overline{F}\right)$ and $c_0^i\left(\overline{F}\right)$ are not convergence free in general.

Theorem 3.2.6: The inclusions $c_0^i\left(\overline{F}\right) \subset c^i\left(\overline{F}\right) \subset \ell_\infty^i\left(\overline{F}\right)$ hold and are strict.

Proof: Clearly, the inclusion $c_0^i\left(\overline{F}\right) \subset c^i\left(\overline{F}\right) \subset \ell_\infty^i\left(\overline{F}\right)$ hold. The inclusions are strict from the following example.

Example 3.2.2: Consider the sequence $\overline{x} = (\overline{x}_k) = ([1,1])$, for all $k \in N$. Then we have

$$\overline{F}_n\,(\overline{x}) \to [1,1] \text{ as } n \to \infty.$$

This shows that $\overline{F}\,\overline{x} \in c^i\left(\overline{F}\right)$ but not in $c_0^i\left(\overline{F}\right)$. Hence the inclusion $c_0^i\left(\overline{F}\right) \subset c^i\left(\overline{F}\right)$ strictly holds.

Again, consider the sequence,

$$\overline{x} = (\overline{x}_k) = \left[(-1)^k \left(\frac{f_k f_{k+1} + f_k f_{k-1}}{f^2{}_k}\right),\ (-1)^k \left(\frac{f_{k+2} + f_{k+1} - 2}{f_k}\right)\right]$$

for all $k \in N$. Then we have for every n,

$$\overline{F}_n\,(\overline{x}) = [(-1)^{\,n},\,(-1)^n].$$

This shows that $\overline{x} = (\overline{x}_k) \in \ell_\infty^i\left(\overline{F}\right)$, but not in $c^i\left(\overline{F}\right)$. Hence the inclusion $c^i\left(\overline{F}\right) \subset \ell_\infty^i\left(\overline{F}\right)$ strictly holds.

Theorem 3.2.7: The inclusions $c_0^i \subset c_0^i\left(\overline{F}\right)$, $c^i \subset c^i\left(\overline{F}\right)$ and $\ell_\infty^i \subset \ell_\infty^i\left(\overline{F}\right)$ hold and are strict.

Proof: As $\overline{F}$ is a regular matrix, so the inclusion $c_0^i \subset {c^i}_0\left(\overline{F}\right)$ and $c^i \subset c^i\left(\overline{F}\right)$ are obvious.

Now let $\overline{x} = (\overline{x}_k) \in \ell_\infty^i$, then there is a constant $M > 0$ such that $|\overline{x}_k| < M$, i.e., $|x_{lk}| < M$ and $|x_{rk}| < M$, for all $k \in N$.

Thus for each $n \in N$,

$$\left|\overline{F}_n\,(\overline{x})\right| = |[\textstyle\sum_{k=1}^{\infty} f_{lnk}x_{lk},\, \sum f_{rnk}x_{rk}]| = max\,\{\sum_{k=1}^{\infty} f_{lnk}\,|x_{lk}|\,,\, \sum_{k=1}^{\infty} f_{rnk}\,|x_{rk}|\}$$

$\leq max\,\{\sum_{k=1}^{\infty} f_{lnk}.M,\, \sum_{k=1}^{\infty} f_{rnk}.M\} = M$, which shows that $\overline{F}\ \overline{x} \in \ell_\infty^i$, i.e.; $\overline{x} = (\overline{x}_k) \in \ell_\infty^i\left(\overline{F}\right)$. Thus we conclude that $\ell_\infty^i \subset \ell_\infty^i\left(\overline{F}\right)$.

Example 3.2.3: Consider the sequence $\overline{x} = (\overline{x}_k) = ([x_{lk},\, x_{rk}])$

$$= ([1,1]\,,\,[0,0]\,,\,[1,1]\,,\,[0,0]\,,\,.......)\,,$$

then we have for every $n \in N$,

$$\overline{F}_n(\overline{x}) = [\textstyle\sum_{k=1}^{\infty} f_{lnk}x_{lk},\, \sum_{k=1}^{\infty} f_{rnk}x_{rk}] = \left[\frac{1}{f_{n+2-1}}\sum_{k=1}^{n} f_k x_{lk},\, \frac{1}{f_n\cdot f_{n+1}}\sum_{k=1}^{n} f_k x_{lk}\right]$$

$= \left[\frac{1}{f_{n+2}-1} \left(f_1 + f_3 + f_5 + \ldots\ldots\ldots\ldots\ldots\ldots \right), \frac{1}{f_n \cdot f_{n+1}} \left(f_1^2 + f_3^2 + f_5^2 + \ldots\ldots\ldots \right) \right]$

which is convergent.

This shows that $\overline{F}\,\overline{x} \in c^i$ but $\overline{x}$ not in c^i. Thus the sequence $\overline{x}$ is in $c^i\left(\overline{F}\right)$. Hence the inclusion $c^i \subset c^i\left(\overline{F}\right)$ strictly holds.

Similarly, we can show the other inclusions are strict.

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