# CHAPTER 2
# SOME IDEAL CONVERGENT SEQUENCE SPACES

# Some Ideal Convergent Sequence Spaces

Different classes of generalized difference convergent sequence spaces using sequences of modulus function have been studied by many authors (for instance see [30], [65], [93], [94], [126], [137], [138], [139] ).

In this chapter we have introduced some generalized difference paranormed sequence spaces $c^I(F, \Lambda, \Delta_m, p)$, $c_0^I(F, \Lambda, \Delta_m, p)$ and $\ell_\infty^I(F, \Lambda, \Delta_m, p)$ of real numbers associated with the multiplier sequence $\Lambda = (\lambda_k)$ determined by a sequence of modulli $F = (f_k)$. $p = (p_k)$ denotes the sequence of positive real numbers. Some of their properties like solidity, symmetricity, completeteness etc. and inclusion relations are studied.

Let $F = (f_k)$ be a sequence of modulli, then for a given multiplier sequence $\Lambda = (\lambda_k)$, we introduce the following sequence spaces:

$c^I(F, \Lambda, \Delta_m, p) = \{(x_k) \in \omega : \{n \in N : (f_k(| \lambda_k(\Delta_m x_k - L) |))^{p_k} \geq \varepsilon\} \in I,$ for some $L \in R\}$

$c_0^I(F, \Lambda, \Delta_m, p) = \{(x_k) \in \omega : \{n \in N : (f_k(| \lambda_k(\Delta_m x_k) |))^{p_k} \geq \varepsilon\} \in I\}$

$\ell_\infty^I(F, \Lambda, \Delta_m, p) = \{(x_k) \in \omega :$ there exists $M > 0$ such that $\{n \in N : (f_k(| \lambda_k(\Delta_m x_k - L) |))^{p_k} \geq M\} \in I\ \}$

When $f_k(x) = f(x)$, for all $k \in N$, then the above spaces are denoted by $c^I(f, \Lambda, \Delta_m, p), c_0^I(f, \Lambda, \Delta_m, p)$, $\ell_\infty^I(f, \Lambda, \Delta_m, p)$.

When $I = I_f$ and $f_k(x) = f(x)$, for all $k \in N$, then the above spaces become $c(f,\Lambda,\Delta_m,p)$, $c_0(f,\Lambda,\Delta_m,p)$, $\ell_\infty(f,\Lambda,\Delta_m,p)$, which was studied by Tripathy and Chandra [36].

When $\lambda_k = p_k = 1$, for all $k \in N$ and $m = 1$, then the above spaces are denoted by $c^I(F,\Delta)$, $c_0^I(F,\Delta)$, $\ell_\infty^I(F,\Delta)$, studied by Khan et. al.[138]

When $I = I_f$, $\lambda_k = p_k = 1$, for all $k \in N$ and $f_k(x) = x$, for all $k \in N$ then the above spaces are denoted by $c(\Delta_m)$, $c_0(\Delta_m)$, $\ell_\infty(\Delta_m)$, studied by Tripathy et. al.

When $I = I_f$, $\lambda_k = p_k = 1$, for all $k \in N$ and $f_k(x) = x$, for all $k \in N$ and $m = 1$, then the above spaces reduce to $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$, studied by Kizmaz [68].

**Theorem 2.1:** The classes of sequences $c^I(F,\Lambda,\Delta_m,p), c_0^I(F,\Lambda,\Delta_m,p)$, and $\ell_\infty^I(F,\Lambda,\Delta_m,p)$ are linear spaces.

**Proof:** We prove the theorem for the class of sequences $c_0^I(F,\Lambda,\Delta_m,p)$. The other cases can be proved similarly.

Let $(x_k),(y_k) \in c_0^I(F,\Lambda,\Delta_m,p)$, then

$A = \{k \in N : (f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([|\alpha|]+1)}\} \in I$

and $B = \{k \in N : (f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([|\beta|]+1)}\} \in I$

Our aim is to show that $(\alpha x_k + \beta y_k) \in c_0^I(F,\Lambda,\Delta_m,p)$, for scalars $\alpha,\beta$.

We have

$(f_k(\mid \lambda_k(\Delta_m(\alpha x_k + \beta y_k)) \mid))^{p_k}$

$\leq (f_k(\mid \alpha \mid\mid \lambda_k(\Delta_m x_k) \mid) + f_k(\mid \beta \mid) \mid)\lambda_k(\Delta_m x_k) \mid)^{p_k}$

$\leq D([|\alpha|]+1)(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} + D([|\beta|]+1)(f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k}$

Now, $C = \{k \in N : (f_k(\mid \lambda_k(\Delta_m(\alpha x_k + \beta y_k)) \mid))^{p_k} \geq \varepsilon\}$

$\subseteq \{k \in N : D([|\alpha|]+1)(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2}\} \cup \{k \in N : D([|\beta|]+1)(f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2}\}$

$= \{k \in N : (f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([|\alpha|]+1)}\} \cup \{k \in N : (f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([|\beta|]+1)}\}$

$= A \cup B$

i.e., $C \subseteq A \cup B$

But $A, B \in I$, hence $A \cup B \in I$, therefore $C \in I$.

**Theorem 2.2:** The classes of sequences $c^I(F, \Lambda, \Delta_m, p), c_0^I(F, \Lambda, \Delta_m, p)$, and $\ell_\infty^I(F, \Lambda, \Delta_m, p)$ are paranormed spaces paranormed by g,

$$g(x) = sup_k(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{\frac{p_k}{M}},$$

where $M = max\,(1,\, sup\, p_k)$.

**Proof:** Clearly $g(x) \geq 0, g(-x) = g(x), g(x+y) \leq g(x) + g(y)$.

Next we show the continuity of the product.

Let $\alpha$ be fixed and $g(x) \to 0$. Then it is obvious that $g(\alpha x) \to 0$.

Next let $\alpha \to 0$ and $x$ be fixed. Since $f_k$ are continuous, we have $f_k(| \alpha || \lambda_k \Delta_m x_k |) \to 0$, as $\alpha \to 0$.

Thus we have

$$sup_k(f_k(| \lambda_k(\Delta_m x_k) |))^{\frac{p_k}{M}} \to 0, \text{ as } \alpha \to 0.$$

Hence $g(\alpha x) \to 0$, as $\alpha \to 0$.

Therefore $g$ is a paranorm.

**Proposition 2.1:** $c_0^I(F, \Lambda, \Delta_m, p) \subset c^I(F, \Lambda, \Delta_m, p) \subset \ell_\infty^I(F, \Lambda, \Delta_m, p)$ and the inclusions are proper.

**Example 2.1:** Let $I = I_f,\ f_k(x_k) = x_k = (-1)^k, \lambda_k = p_k = 1, m = 1,$ then $(x_k) \in \ell_\infty^I(F, \Lambda, \Delta_m, p)$ but $(x_k) \notin c_0^I(F, \Lambda, \Delta_m, p)$ or $c^I(F, \Lambda, \Delta_m, p)$.

**Theorem 2.3:** The spaces $c^I(F, \Lambda, \Delta_m, p), c_0^I(F, \Lambda, \Delta_m, p)$ and $\ell_\infty^I(F, \Lambda, \Delta_m, p)$ are neither solid nor monotone in general, but the spaces $c_0^I(F, \Lambda, p)$ and $\ell_\infty^I(F, \Lambda, p)$ are solid and as such are monotone.

**Proof:** Let $(x_k)$ be a given sequence and $(\alpha_k)$ be a sequence of scalars such that $| \alpha_k |\leq 1$, for all $k \in N$.

Then we have

$$(f_k(| \lambda_k \alpha_k x_k |))^{p_k} \leq (f_k(| \lambda_k x_k |))^{p_k}, \text{ for all } k \in N.$$

The solidness of $c_0^I(F,\Lambda,p)$, and $\ell_\infty^I(F,\Lambda,p)$ follows from this inequality. Hence these spaces are monotone.

The first part of the proof follows from the following example.

**Example 2.2:** Let $I = I_f$, $f_k(x) = x$, for all $x \in [0,\infty]$, $m = 1, \lambda_k = 1$ for all $k \in N$, $p_k = 1$ for $k$ odd, $p_k = 3$ for $k$ even, $x_k = k$, for all $k \in N$ belongs to $c^I(\Delta,p)$ and $\ell_\infty^I(\Delta,p)$. For E, a sequence space, consider its step space $E_J$ defined by $(y_k) \in E_J$ implies $y_k = 0$ for all $k$ odd and $y_k = x_k$ for $k$ even. Then $(y_k)$ neither belongs to $(c^I(\Delta,p))_J$ nor to $\ell_\infty^I(\Delta,p)_J$. Hence the spaces are not monotone. Hence these spaces are not solid.

**Result 2.1:** The spaces $c^I(F,\Lambda,\Delta_m,p), c_0^I(F,\Lambda,\Delta_m,p)$ and $\ell_\infty^I(F,\Lambda,\Delta_m,p)$ are not symmetric in general.

**Proof:** The result follows from the following example.

**Example 2.3:** Let $I = I_f$, $f_k(x) = x$, for all $x \in [0,\infty]$, $m = 0, \lambda_k = k$ for all $k \in N$, $p_k = 1$ for $k$ odd, $p_k = 4$ for $k$ even, $x_k = k^{-2}$, for all $k \in N$. Then $(x_k)$ belongs to $c^I(F,\Lambda,p), c_0^I(F,\Lambda,p)$. Consider its rearrangement $(y_k)$ defined as follows:

$$(y_n) = (x_1, x_3, x_4, x_2, x_6, x_7, x_8, ....., x_{24}, x_5, x_{26}, x_{27}, ...., x_{624}, x_{25}, x_{626}, ...).$$

Then $(y_n)$ neither belongs to $c^I(F,\Lambda,p)$ nor to $c_0^I(F,\Lambda,p)$. Hence the spaces $c^I(F,\Lambda,\Delta_m,p), c_0^I(F,\Lambda,\Delta_m,p)$ and $\ell_\infty^I(F,\Lambda,\Delta_m,p)$ are not symmetric in general.

**Result 2.2:** The spaces $c^I(F,\Lambda,\Delta_m,p), c_0^I(F,\Lambda,\Delta_m,p)$, and $\ell_\infty^I(F,\Lambda,\Delta_m,p)$ are not convergence free.

**Example 2.4:** Let $I = I_f,\ f_k(x) = x$, for all $x \in [0, \infty)$, $m = 1, \lambda_k = 1$ for all $k \in N$, $p_k = 1$ for $k$ odd, $p_k = 2$ for $k$ even, consider the sequence $(x_k)$ defined by $x_k = k^{-1}$, for all $k \in N$, then $(x_k)$ belongs to each of $c^I(\Delta, p), c_0^I(\Delta, p)$, and $\ell_\infty^I(\Delta, p)$. Consider the sequence $(y_k)$ defined by $y_k = k^2$, for all $k \in N$. Then $(y_k)$ neither belongs to $c^I(\Delta, p)$ nor to $c_0^I(\Delta, p)$ nor to $\ell_\infty^I(\Delta, p)$. Hence the spaces are not convergence free.

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